ON A CLASS OF KÖTHE SEQUENCE SPACES WITH NORMAL STRUCURE[1]

B. Zlatanov

Department of Mathematics and Informatics, Plovdiv University, 24 "Tzar Assen" str., Plovdiv 4000, Bulgaria, e-mail: bzlatanov@gmail.com

Abstract In this note we investigate the generalized modulus of convexity $\delta^{(\lambda)}$ and the generalized modulus smoothness $\rho^{(\lambda)}$. We find some estimates of these modulus for $X = \ell_p$. We obtain inequalities between WCS coefficient of a Köthe sequence space X and $\delta_X^{(\lambda)}$. We show that for a wide class of Köthe sequence spaces X if for some $\varepsilon \in (0, \frac{9}{10}]$ holds $\delta_X(\varepsilon) > \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon$, then X has normal structure.

Key words weakly convergent sequence coefficient; modulus of convexity; generalized modulus of convexity; normal structure.
2000 Mathematics Subject Classification 46B20; 46E30; 46B45; 46A45.

1 Introduction

The weakly convergent sequence coefficient $WCS(X)$ of a Banach space X was introduced by Bynum in [1]. The connections of the coefficient $WCS(X)$ with some geometric parameters is widely investigated. The notation of normal structure was introduced by Brodskii and Milman in [2]. A reflexive Banach space X with $WCS(X) > 1$ has normal structure [1].

Inequalities between the weakly convergent sequence coefficient $WCS(X)$ and the modulus of smoothness ρ_X and the constants $J(X)$, $C_{NJ}(X)$ are found by in [3]. These inequalities are used to show that if $J(X) < (1+\sqrt{5})/2$ or $C_{NJ}(X) < (1+\sqrt{3})/2$ then X has normal structure [4, 5]. It is known that if $\rho_X(\theta) < \frac{\theta}{2}$ holds for some $\theta \in (0, 1]$ then X has normal structure [6]. This result was sharpen in [3] by the statement if $\rho_X(\theta) < \frac{\theta - 2 + \sqrt{\theta^2 + 4}}{2}$ holds for some $\theta \in (0, 1]$, then X has normal structure.

The generalized modulus of convexity $\delta^{(\lambda)}$ was investigated by [7]. It was shown there that $\delta^{(\lambda)}$ shares the same properties as like as the Clrakson modulus of convexity δ.

Following the ideas in [7] we define a generalized modulus of smoothness $\rho^{(\lambda)}$ and we show that $\delta^{(\lambda)}$ and $\rho^{(\lambda)}$ are connected by the same formula as are δ and ρ. We obtain some upper and lower estimates for $\delta_{\ell_p}^{(\lambda)}$ and $\rho_{\ell_p}^{(\lambda)}$, for $p \geq 2$, which are sharp for $p = 2$. According to [8] if $\delta_X(\varepsilon) \geq \frac{1}{6}\varepsilon$ holds for some $\varepsilon \in (0, 3/2]$ then X has uniform normal structure. Following the ideas in [3] we show that if $\delta_X(\varepsilon) > \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right)\varepsilon$ holds for some $\varepsilon \in (0, \frac{9}{10}]$ then X has normal structure, provided X is a Köthe sequence spaces.

[1]Research is partially supported by National Fund for Scientific Research of the Bulgarian Ministry of Education and Science, Contract MM-1401/04.

2 Preliminaries

We use the standard Banach space terminology from [9]. Let X be a real Banach space, S_X be the unit sphere of X, $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the set of the real numbers. Let ℓ^0 stand for the space of all real sequences i.e. $x = \{x_i\}_{i=1}^{\infty} \in \ell^0$, $x_i \in \mathbb{R}$ for every $i \in \mathbb{N}$.

For a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of X, we use the following notation:

$$A(\{x^{(n)}\}) = \lim_{n\to\infty} \sup\{\|x^{(i)} - x^{(j)}\| : i, j \geq n, i \neq j\}.$$

Definition 2.1 *[1] The weakly convergent sequence coefficient of X, denoted by $WCS(X)$, is defined as follows:*

$$\begin{aligned} WCS(X) &= \sup\{k : \textit{for each weakly convergent sequence } \{x^{(n)}\}_{n=1}^{\infty}, \textit{ there exists} \\ &\quad \textit{some } y \in \operatorname{co}(\{x^{(n)}\}_{n=1}^{\infty}) \textit{ such that } k \limsup_{n\to\infty} \|x^{(n)} - y\| \leq A(\{x^{(n)}\})\}, \end{aligned}$$

where $\operatorname{co}(\{x^{(n)}\}_{n=1}^{\infty})$ denotes the convex hull of the elements of $\{x^{(n)}\}_{n=1}^{\infty}$.

It is easy to see that $1 \leq WCS(X) \leq 2$. A Banach space X is said to have weak uniform normal structure if $WCS(X) > 1$ [10].

Recall that a Banach space has Schur property if every weakly null sequence is norm null. We will assume in the sequel that the Banach spaces, we investigate are not Schur spaces. Thus there exists a weakly null sequence $\{x^{(n)}\}_{n=1}^{\infty} \in X$, which is not norm null. We will use the notation $x^{(n)} \xrightarrow{w} 0$ to indicate that $\{x^{(n)}\}_{n=1}^{\infty}$ converges weakly to zero.

It is known that Banach space with normal structure has the fixed point property [2, 1], and every reflexive Banach space X with $WCS(X) > 1$ has normal structure [1].

Definition 2.2 *A Banach space $(X, \|\cdot\|)$ is said to be Köthe sequence space if X is a subspace of ℓ^0 such that*

i) If $x \in \ell^0$, $y \in X$ and $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$ then $x \in X$ and $\|x\| \leq \|y\|$;

ii) There exists an element $x \in X$ such that $x_i > 0$ for all $i \in \mathbb{N}$.

A sequence $\{v_i\}_{i=1}^{\infty}$ in a Banach space X is called Schauder basis of X (or basis for short) if for each $x \in X$ there exists an unique sequence $\{a_i\}_{i=1}^{\infty}$ of scalars such that $x = \sum_{i=1}^{\infty} a_i v_i$. If $\{v_i\}_{i=1}^{\infty}$ is a basis in X such that the series $\sum_{i=1}^{\infty} a_i v_i$ converges whenever $\sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{n} a_i v_i\right\| < \infty$, then it is called a boundedly complete basis of X. A sequence of non zero vectors $\{x^{(n)}\}_{n=1}^{\infty}$ of the form $\sum_{i=p_n+1}^{p_{n+1}} a_i v_i$, with $\{a_i\}_{i=1}^{\infty}$ scalars and $0 = p_1 < p_2 < p_3 \ldots$ an increasing sequence of integers is called a block basic sequence or block basis of $\{v_i\}_{i=1}^{\infty}$ for short. By $\{e_i\}_{i=1}^{\infty}$ we denote the unit vectors.

The main tool in this note will be the next theorem:

Theorem 1 *[11] Let X be a Köthe sequence space with $\{e_i\}_{i=1}^{\infty}$–boundedly complete basis. Then*

$$WCS(X) = \inf\left\{ A(\{x^{(n)}\}) : x^{(n)} = \sum_{i=p_n+1}^{p_{n+1}} x_n(i)e_i \in S_X, x_n \xrightarrow{w} 0, 0 = p_1 < p_2 < p_3 \dots \right\}.$$

Modulus of convexity [12] is the function $\delta_X : [0,2] \to [0,1]$ given by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x-y\| \geq \varepsilon\right\}.$$

It is well known [13] that

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x-y\| = \varepsilon\right\}. \tag{1}$$

In [7] a generalized modulis of convexity $\delta_X^{(\lambda)}$, $\lambda \in (0,1)$ is defined by

$$\delta_X^{(\lambda)}(\varepsilon) = \inf\left\{1 - \|\lambda x + (1-\lambda)y\| : x, y \in S_X, \|x-y\| \geq \varepsilon\right\}$$

and is investigated. It is shown there that $\delta_X^{(\lambda)}$ shares the same properties as like as δ_X.

3 Main result

Theorem 2 *Let X be a Köthe sequence space with basis $\{e_i\}_{i=1}^{\infty}$ both shrinking and boundedly complete. If $\delta_X(\varepsilon) > \dfrac{1}{3}\left(1 - \dfrac{\sqrt{3}}{2}\right)\varepsilon$ for some $\varepsilon \in (0, 9/10]$, then X has normal structure.*

4 Generalized Modulus of Convexity and Smoothness

Let us recall that modulus of smoothness ρ_X is defined by the formula:

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}(\|x+\tau y\| + \|x - \tau y\| - 2) : x, y \in S_X\right\}.$$

Lindenshtrauss proved in [14] that

$$\rho_{X^*}(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : \varepsilon \in [0,2]\right\}$$

and

$$\rho_X(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : \varepsilon \in [0,2]\right\},$$

where ρ_{X^*} and δ_{X^*} are the modulus of smoothness and convexity in the dual space X^* with respect to the dual norm $\|\cdot\|^*$.

Following the same idea as in [14, 7] we define the generalized modulus of smoothness $\rho_X^{(\lambda)}$ by

$$\rho_X^{(\lambda)}(\tau) = \sup\left\{\frac{\|2\lambda x + \tau y\| + \|2(1-\lambda)x - \tau y\| - 2\|x\|}{2} : x, y \in S_X\right\},\ \lambda \in (0,1),\ \tau > 0.$$

Theorem 3 *Let X be a Banach space with a generalized modulus of convexity $\delta_X^{(\lambda)}$ and $\rho_{X^*}^{(\lambda)}$ be the generalized modulus of smoothness of the dual norm $\|\cdot\|^*$, then for any $\tau > 0$ holds*

$$\rho_{X^*}^{(\lambda)}(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_X^{(\lambda)}(\varepsilon) : \varepsilon \in [0,2]\right\}, \tag{2}$$

and let $\rho_X^{(\lambda)}$ be the generalized modulus of smoothness and $\delta_{X^*}^{(\lambda)}$ be the generalized modulus of convexity of the dual norm $\|\cdot\|^*$, then

$$\rho_X^{(\lambda)}(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*}^{(\lambda)}(\varepsilon) : \varepsilon \in [0,2]\right\}. \tag{3}$$

Proof: We will prove (2), the proof of (3) is similar.

We claim first that for any $\varepsilon \in [0,2]$ and any $\tau > 0$ the inequality

$$\rho_{X^*}^{(\lambda)}(\tau) + \delta_X^{(\lambda)}(\varepsilon) \geq \frac{\tau\varepsilon}{2} \tag{4}$$

holds. Indeed let $x, y \in S_X$ be such that $\|x - y\| \geq \varepsilon$. Choose $f, g \in S_{X^*}$ such that

$$f(\lambda x + (1-\lambda)y) = \|\lambda x + (1-\lambda)y\| \quad \text{and} \quad g(x-y) = \|x-y\|.$$

From the definition of $\rho_{X^*}^{(\lambda)}$ we have

$$\begin{aligned}
2\rho_{X^*}^{(\lambda)}(\tau) &\geq \|2\lambda f + \tau g\|^* + \|2(1-\lambda)f - \tau g\|^* - 2 \\
&\geq (2\lambda f + \tau g)(x) + (2(1-\lambda)f - \tau g)(y) - 2 \\
&= 2\lambda f(x) + 2(1-\lambda)f(y) + \tau g(x) - \tau g(y) - 2 \\
&= 2(f(\lambda x) + f((1-\lambda)y)) + \tau g(x-y) - 2 \\
&= 2f(\lambda x + (1-\lambda)y) + \tau g(x-y) - 2 \\
&= 2\|\lambda x + (1-\lambda)y\| + \tau\|x-y\| - 2 \\
&\geq 2\|\lambda x + (1-\lambda)y\| + \tau\varepsilon - 2.
\end{aligned}$$

Hence the inequality

$$1 - \|\lambda x + (1-\lambda)y\| \geq \frac{\tau\varepsilon}{2} - \rho_{X^*}^{(\lambda)}(\tau)$$

holds for every $\varepsilon \in [0,2]$, $\tau > 0$ and $x, y \in S_X$, such that $\|x - y\| \geq \varepsilon$. Thus from the definition of $\delta_X^{(\lambda)}$ we get (4) and consequently

$$\rho_{X^*}^{(\lambda)}(\tau) \geq \sup\left\{\frac{\tau\varepsilon}{2} - \delta_X^{(\lambda)}(\varepsilon) : \varepsilon \in [0,2]\right\}.$$

To prove the converse inequality, let $\tau > 0$ and $f, g \in S_{X^*}$. For any $\eta > 0$ there exist $x, y \in S_X$ such that

$$(2\lambda f + \tau g)(x) \geq \|2\lambda f + \tau g\|^* - \eta$$

and

$$(2(1-\lambda)f-\tau g)(y)\geq\|2(1-\lambda)f-\tau g\|^{*}-\eta.$$

Therefore

$$\begin{aligned}\|2\lambda f &+ \tau g\|^{*}+\|2(1-\lambda)f-\tau g\|^{*}\leq 2\lambda f(x)+2(1-\lambda)f(y)+\tau g(x)-\tau g(y)+2\eta\\ &= 2(f(\lambda x)+f((1-\lambda)y)+\tau g(x-y)+2\eta\\ &= 2(f(\lambda x+(1-\lambda)y)+\tau g(x-y)+2\eta\\ &\leq 2\|\lambda x+(1-\lambda)y\|+\tau g(x-y)+2\eta.\end{aligned}$$

Put $\varepsilon=\|x-y\|$, then $\|\lambda x+(1-\lambda)y\|\leq 1-\delta_X^{(\lambda)}(\varepsilon)$. Therefore

$$\|2\lambda f+\tau g\|^{*}+\|2(1-\lambda)f-\tau g\|^{*}\leq 2-2\delta_X^{(\lambda)}(\varepsilon)+\tau\varepsilon+2\eta$$

and consequently

$$\frac{\|2\lambda f+\tau g\|^{*}+\|2(1-\lambda)f-\tau g\|^{*}-2\|f\|}{2}\leq\frac{\tau\varepsilon}{2}-\delta_X^{(\lambda)}(\varepsilon)+\eta.$$

Thus

$$\frac{\|2\lambda f+\tau g\|^{*}+\|2(1-\lambda)f-\tau g\|^{*}-2\|f\|}{2}\leq\sup\left\{\frac{\tau\varepsilon}{2}-\delta_X^{(\lambda)}(\varepsilon):0\leq\varepsilon\leq 2\right\}+\eta.$$

Since $\eta>0$ is arbitrary chosen, we get

$$\frac{\|2\lambda f+\tau g\|^{*}+\|2(1-\lambda)f-\tau g\|^{*}-2\|f\|}{2}\leq\sup\left\{\frac{\tau\varepsilon}{2}-\delta_X^{(\lambda)}(\varepsilon):0\leq\varepsilon\leq 2\right\},$$

i.e.

$$\rho_{X^*}^{(\lambda)}(\tau)\leq\sup\left\{\frac{\tau\varepsilon}{2}-\delta_X^{(\lambda)}(\varepsilon):0\leq\varepsilon\leq 2\right\}.$$

The dual statement is obtained similarly. □

Proposition 4.1 *Let X be a Banach space with generalized modulus of convexity $\delta_X^{(\lambda)}$. Then*

$$\delta_X^{(\lambda)}(\varepsilon)=\inf\{1-\|\lambda x+(1-\lambda)y\|:x,y\in S_X,\|x-y\|=\varepsilon\}.$$

Proof: By a simple geometric argument we have

$$\delta_X^{(\lambda)}(\varepsilon)=\inf\left\{1-\|\lambda x-(1-\lambda)y\|:x,y\in B_X,\|x-y\|=\varepsilon\right\}.$$

Also, by multiplying x and y by the same number so that one of them reaches S_X we have:

$$\begin{aligned}\delta_X^{(\lambda)}(\varepsilon) &= \inf\left\{1-\|\lambda x-(1-\lambda)y\|:x,y\in B_X,\|x-y\|=\varepsilon\right\}\\ &= \inf\left\{1-\|\lambda x-(1-\lambda)y\|:x\in S_X,y\in B_X,\|x-y\|=\varepsilon\right\}.\end{aligned}$$

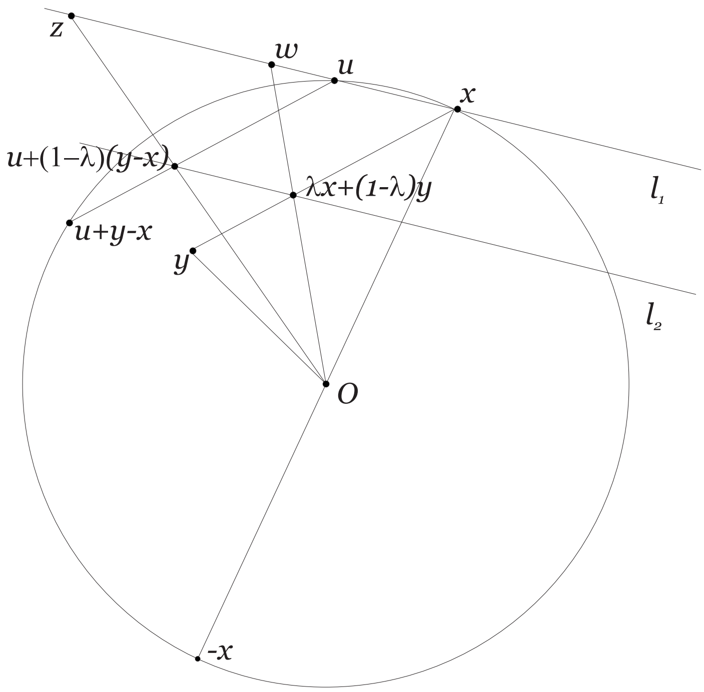
Consider the half plane H cut by the line through x and $-x$ in which y lies. Consider the family of pairs of points $\{u, u+y-x\}$, $u \in H \cap S_X$. If $u = x$, then $u + y - x = y \in \text{int } B_X$. If u is the intersection of the ray $x + t(y-x)$, $t \geq 0$ with S_X, then $u + y - x \notin B_X$. Therefore, by continuity there is $u \in S_X \cap H$, such that $u+y-x \in S_X$. Let ℓ_1 be the line through u, x and ℓ_2 be the line through $\lambda x + (1-\lambda)y$, $u + (1-\lambda)(y-x)$. By

$$(\lambda x + (1-\lambda)y) - (u + (1-\lambda)(y-x)) = x - y$$

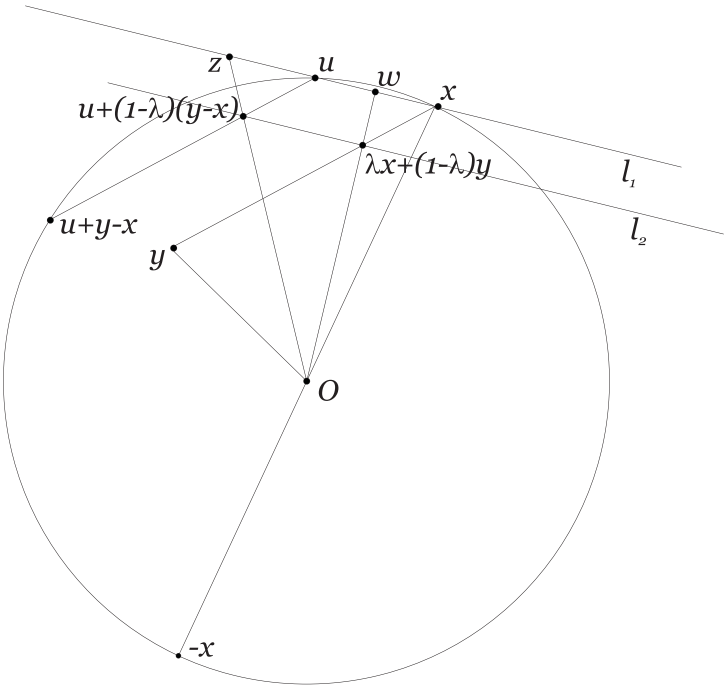
it follows that ℓ_1 and ℓ_2 are parallel. Let w be the intersection of ℓ_1 with the ray emanating form the origin and passing through $\lambda x + (1-\lambda)y$ and z be the intersection of ℓ_1 with the ray emanating form the origin and passing through $u + (1-\lambda)(y-x)$. By argument using similar triangles it follows that there is $\alpha > 0$, such that

$$\|z\| = \alpha\|u + (1-\lambda)(y-x)\|, \quad \|w\| = \alpha\|\lambda x + (1-\lambda)y\|.$$

There are two cases:
1) The points x, u, w, z are in this order in the ray emanating from x and going through u. By convexity of the norm function on this ray it follows that $\|z\| \geq \|w\| \geq \|u\| = \|x\| = 1$ and therefore $\|u + (1-\lambda)(y-x)\| \geq \|\lambda x + (1-\lambda)y\|$.



2) The points x, w, u, z are in this order in the ray emanating from x and going through u. By convexity of the norm function on this ray it follows that $\|z\| \geq \|u\| = 1 = \|x\| \geq \|w\|$ and therefore $\|u + (1-\lambda)(y-x)\| \geq \|\lambda x + (1-\lambda)y\|$.



Put in both cases $v = u + y - x$. Then $u, v \in S_X$, $\|u - v\| = \|x - y\|$ and $\|\lambda v + (1-\lambda)u\| = \|u + (1-\lambda)(y - x)\| \geq \|\lambda x + (1-\lambda)y\|$. $\square$

Let H be a Hilbert space. It is easy to check [7], that the equality $\delta_H^{(\lambda)}(\varepsilon) = 1 - \left(1 - \left(\varepsilon\sqrt{\lambda(1-\lambda)}\right)^2\right)^{1/2}$ holds.

Proposition 4.2 *Let X be a Banach space and H be a Hilbert space. Then the inequality*

$$\delta_X^{(\lambda)}(\varepsilon) \leq \delta_H^{(\lambda)}(\varepsilon)$$

holds for every $\varepsilon \in [0, 2]$ and any $\lambda \in (0, 1)$.

Proof: Since, by a well known theorem of Dvorezky [15], every infinite dimensional Banach space contains nearly isometric copies of ℓ_2^n for all n it follows that

$$\delta_X^{(\lambda)}(\varepsilon) \leq 1 - \left(1 - \left(\varepsilon\sqrt{\lambda(1-\lambda)}\right)^2\right)^{1/2} = \delta_H^{(\lambda)}(\varepsilon)$$

for every $\varepsilon \in [0, 2]$ and any $\lambda \in (0, 1)$. $\square$

There is a direct geometric proof of Proposition 4.2 for $\lambda = 1/2$ [16].

For the next several Lemmas we will need the following function:

$$\alpha^{(\lambda)}(t) = 1 - \delta_X^{(\lambda)}(t) + \left(1 - \sqrt{1 - \lambda(1-\lambda)}\right) t.$$

Lemma 4.1 *For any Banach space X with a generalized modulus of convexity $\delta_X^{(\lambda)}$ the inequality*

$$1 \leq \alpha^{(\lambda)}(t) \leq 1 + \left(1 - \sqrt{1 - \lambda(1-\lambda)}\right) t \tag{5}$$

holds for every $t \in [0, 1]$.

Proof: The righthandside inequality is obtained if taking into account that $\delta_X^{(\lambda)}(t) \geq 0$ for every $t \in (0,2]$ and $\lambda \in (0,1)$.

By Proposition 4.2 it follows that $\delta_X^{(\lambda)}(t) \leq \delta_H^{(\lambda)}(t) = 1 - \sqrt{1 - t^2\lambda(1-\lambda)}$ holds for every $t \in [0,2]$. It is easy to check, that for every $t \in [0,1]$ the inequality

$$\sqrt{1 - t^2\lambda(1-\lambda)} \geq 1 - \left(1 - \sqrt{1 - \lambda(1-\lambda)}\right) t$$

holds. Therefore for every $\lambda \in (0,1)$ and every $t \in [0,1]$ the inequalities

$$\begin{aligned} \alpha^{(\lambda)}(t) &\geq 1 - \delta_H^{(\lambda)}(t) + \left(1 - \sqrt{1 - \lambda(1-\lambda)}\right) t \\ &= \sqrt{1 - t^2\lambda(1-\lambda)} + \left(1 - \sqrt{1 - \lambda(1-\lambda)}\right) t \geq 1 \end{aligned}$$

hold for every $t \in [0,1]$. □

For simplicity of the notations let put: $WCS(X) = d$,
$f_{d+\varepsilon,\lambda}(t) = (d+\varepsilon)\left(1 - \sqrt{1 - \lambda(1-\lambda)}\right) t^2 + \left(d + \varepsilon - 1 + \sqrt{1 - \lambda(1-\lambda)}\right) t - 2 + \varepsilon$,
$D_{\varepsilon,\lambda} = \left(d + \varepsilon - 1 + \sqrt{1 - \lambda(1-\lambda)}\right)^2 + 4(2-\varepsilon)\left(1 - \sqrt{1 - \lambda(1-\lambda)}\right)(d+\varepsilon)$ and
$a(d+\varepsilon,\lambda) = \dfrac{1 - d - \varepsilon - \sqrt{1 - \lambda(1-\lambda)} + \sqrt{D_{\varepsilon,\lambda}}}{2(d+\varepsilon)\left(1 - \sqrt{1 - \lambda(1-\lambda)}\right)}$.

Lemma 4.2 *Let X be a Banach space and $\lambda \in (0,1)$. Then for any $t \in [0, a(d,\lambda))$ there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0,\varepsilon_0)$ the inequality*

$$\frac{1}{d+\varepsilon}\left(1 + \frac{1-\varepsilon}{\alpha^{(\lambda)}(t)}\right) \geq t$$

holds.

Proof: By (5) we have the inequality:

$$\frac{1}{d+\varepsilon}\left(1 + \frac{1-\varepsilon}{\alpha^{(\lambda)}(t)}\right) \geq \frac{1}{d+\varepsilon}\left(1 + \frac{1-\varepsilon}{1 + \left(1 - \sqrt{1-\lambda(1-\lambda)}\right) t}\right).$$

The inequality $\dfrac{1}{d+\varepsilon}\left(1 + \dfrac{1-\varepsilon}{1 + \left(1 - \sqrt{1-\lambda(1-\lambda)}\right) t}\right) \geq t$ is equivalent to the inequality

$$f_{d+\varepsilon,\lambda}(t) \leq 0. \tag{6}$$

For any $t \geq 0$, $\lambda \in (0,1)$, $0 \leq \varepsilon_1 < \varepsilon_2$ holds

$$f_{d+\varepsilon_1,\lambda}(t) < f_{d+\varepsilon_2,\lambda}(t), \tag{7}$$

and for any $\varepsilon \geq 0$, $\lambda \in (0,1)$, $0 \leq t_1 < t_2$ holds

$$f_{d+\varepsilon,\lambda}(t_1) < f_{d+\varepsilon,\lambda}(t_2). \tag{8}$$

By $f_{d+\varepsilon,\lambda}(0) = -2+\varepsilon < 0$, for $\varepsilon \in [0,2)$ it follows that for every $\varepsilon \in [0,2)$ and every $\lambda \in (0,1)$ the two roots t_1 and t_2 of the equation:

$$f_{d+\varepsilon,\lambda}(t) = 0 \tag{9}$$

have different signs.

Denote by $t_{\varepsilon,\lambda}$ the positive root of the equation (9). Then $t_{\varepsilon,\lambda} = a(d+\varepsilon,\lambda)$ and $t_{0,\lambda} = a(d,\lambda)$. Obviously $\lim_{\varepsilon\to 0} t_{\varepsilon,\lambda} = t_{0,\lambda}$ for any $\lambda \in (0,1)$.

By (7) for any $0 < \varepsilon_1 < \varepsilon_2 < 2$ we have the inequalities $0 = f_{d+\varepsilon_1,\lambda}(t_{\varepsilon_1,\lambda}) < f_{d+\varepsilon_2,\lambda}(t_{\varepsilon_1,\lambda})$ and $0 = f_{d,\lambda}(t_{0,\lambda}) < f_{d+\varepsilon_1,\lambda}(t_{0,\lambda})$ and therefore $t_{\varepsilon_2,\lambda} < t_{\varepsilon_1,\lambda} < t_{0,\lambda} = a(d,\lambda)$. Thus for any $t \in [0, a(d,\lambda))$ there is $\varepsilon_0 > 0$, such that $t < t_{\varepsilon_0,\lambda} < t_{0,\lambda} = a(d,\lambda)$ and for any $\varepsilon \in (0,\varepsilon_0]$ using (7) and (8) we get

$$f_{d+\varepsilon,\lambda}(t) \le f_{d+\varepsilon_0,\lambda}(t) < f_{d+\varepsilon_0,\lambda}(t_{\varepsilon_0,\lambda}) = 0,$$

which is (6). □

Theorem 4 *Let X be a Köthe sequence space with $\{e_i\}_{i=1}^{\infty}$–boundedly complete basis and with a generalized modulus of convexity $\delta_X^{(\lambda)}$. Then for any $t \in [0, a(d,\lambda))$ the inequality*

$$WCS(X) \ge \frac{1-\lambda\delta_X^{(\lambda)}(t) + \lambda\left(1-\sqrt{1-\lambda(1-\lambda)}\right)t}{(1-\delta_X^{(\lambda)}(t))(1-\delta_X^{(\lambda)}(t) + \left(1-\sqrt{1-\lambda(1-\lambda)}\right)t)} \tag{10}$$

holds.

Proof: If $\delta_X^{(\lambda)} \equiv 0$ then $\dfrac{1+\lambda\left(1-\sqrt{1-\lambda(1-\lambda)}\right)t}{1+\left(1-\sqrt{1-\lambda(1-\lambda)}\right)t} < 1$ and consequently (10) holds true.

If $\delta_X^{(\lambda)}(t) > 0$ for every $t \in (0,2]$ and every $\lambda \in (0,1)$ then X is reflexive, because according to [7]

$$2\min\{\lambda, 1-\lambda\}\delta(t) \le \delta^{(\lambda)}(t) \le 2\max\{\lambda, 1-\lambda\}\delta(t)$$

for $\lambda \in (0,1)$. Let $\{x_n\}_{n=1}^{\infty}$ be a weakly null, block basic sequence in S_X. Assume that $d = \lim_{\substack{n,m\to\infty \\ m\neq n}} \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}_{n=1}^{\infty}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X guarantees that there exists x^* such that $x_n^* \xrightarrow{w} x^*$. By Lemma 4.2 for every $t \in [0, a(d,\lambda))$ there exists ε_0, such that for every $\varepsilon \in (0,\varepsilon_0]$ the inequality

$$\frac{1}{d+\varepsilon}\left(1+\frac{1-\varepsilon}{1-\delta_X^{(\lambda)}(t) + \left(1-\sqrt{1-\lambda(1-\lambda)}\right)t}\right) \ge t$$

holds. Choose an arbitrary $\varepsilon \in (0, \varepsilon_0]$, then there exists $N \in \mathbb{N}$, large enough so that $|x^*(x_N)| < \varepsilon/2$ and

$$d - \varepsilon < \|x_N - x_m\| < d + \varepsilon$$

for all $m > N$. Note that

$$\lim_{\substack{n, m \to \infty \\ m \neq n}} \left\| \frac{x_n - x_m}{d + \varepsilon} \right\| \leq 1 \quad \text{and} \quad \left\| \frac{x_N}{d + \varepsilon} \right\| \leq 1.$$

By the assumption that X is a Köthe sequence space and that $\{x_n\}_{n=1}^{\infty}$ be a block basic sequence we have that $\|x_n - x_m\| = \|x_n + x_m\|$ holds for every $n, m \in \mathbb{N}$. Therefore we can choose $M > N$ large enough, such that the inequalities

$$\left\| \frac{x_N + x_M}{d + \varepsilon} \right\| = \left\| \frac{x_N - x_M}{d + \varepsilon} \right\| \leq 1 \leq 1 - \delta_X^{(\lambda)}(t) + \left(1 - \sqrt{1 - \lambda(1 - \lambda)}\right) t, \qquad (11)$$

$|(x_M^* - x^*)(x_N)| < \varepsilon/2$ and $|x_N^*(x_M)| < \varepsilon/2$ hold. Hence

$$|x_M^*(x_N)| \leq |(x_M^* - x^*)(x_N)| + |x^*(x_N)| < \varepsilon.$$

Put $x = \dfrac{X_N - X_M}{d + \varepsilon}$ and $y = \dfrac{X_N + X_M}{(d + \varepsilon)\alpha^{(\lambda)}(t)}$. By (5) it follows that $\alpha^{(\lambda)}(t) \geq 1$ and it is easy to see that $\alpha^{(\lambda)}(t) \leq 2$, therefore $x, y \in B_X$. We will need the next two inequalities

$$\begin{aligned} \|\lambda x + (1 - \lambda) y\| &= \frac{\|(\lambda\alpha^{(\lambda)}(t) + 1 - \lambda) x_N - (\lambda\alpha^{(\lambda)}(t) - 1 + \lambda) x_M\|}{(d + \varepsilon)\alpha^{(\lambda)}(t)} \\ &\geq \frac{(\lambda\alpha^{(\lambda)}(t) + 1 - \lambda) x_N^*(x_N) - (\lambda\alpha^{(\lambda)}(t) - 1 + \lambda) x_N^*(x_M)}{(d + \varepsilon)\alpha^{(\lambda)}(t)} \\ &\geq \frac{\lambda\alpha^{(\lambda)}(t) + 1 - \lambda - \varepsilon}{(d + \varepsilon)\alpha^{(\lambda)}(t)} \end{aligned}$$

and

$$\begin{aligned} \|x - y\| &= \frac{\|(\alpha^{(\lambda)}(t) + 1) x_M - (\alpha^{(\lambda)}(t) - 1) x_N\|}{(d + \varepsilon)\alpha^{(\lambda)}(t)} \\ &\geq \frac{((\alpha^{(\lambda)}(t) + 1) x_M^*(x_M) - (\alpha^{(\lambda)}(t) - 1) x_M^*(x_N))}{(d + \varepsilon)\alpha^{(\lambda)}(t)} \\ &\geq \frac{\alpha^{(\lambda)}(t) + 1 - \varepsilon}{(d + \varepsilon)\alpha^{(\lambda)}(t)}. \end{aligned}$$

By Lemma 4.2 we get the inequalities

$$\|x - y\| \geq \frac{\alpha^{(\lambda)}(t) + 1 - \varepsilon}{(d + \varepsilon)\alpha^{(\lambda)}(t)} \geq \frac{1}{d + \varepsilon} \left(1 + \frac{1 - \varepsilon}{\alpha^{(\lambda)}(t)}\right) \geq t.$$

By the definition of $\delta_X^{(\lambda)}$, we obtain

$$\delta_X^{(\lambda)}(t) \leq 1 - \|\lambda x + (1 - \lambda) y\| \leq 1 - \frac{\lambda\alpha^{(\lambda)}(t) + 1 - \lambda - \varepsilon}{d\alpha^{(\lambda)}(t)}.$$

Letting $\varepsilon \to 0$ we get

$$1 - \delta_X^{(\lambda)}(t) \geq \frac{\lambda\alpha^{(\lambda)}(t) + 1 - \lambda}{d\alpha^{(\lambda)}(t)}.$$

Consequently the inequality

$$d \geq \frac{\lambda\alpha^{(\lambda)}(t) + 1 - \lambda}{(1 - \delta_X^{(\lambda)}(t))\alpha^{(\lambda)}(t)} = \frac{1 - \lambda\delta_X^{(\lambda)}(t) + \lambda\left(1 - \sqrt{1 - \lambda(1-\lambda)}\right)t}{(1 - \delta_X^{(\lambda)}(t))(1 - \delta_X^{(\lambda)}(t) + \left(1 - \sqrt{1 - \lambda(1-\lambda)}\right)t)} \tag{12}$$

holds for any weakly null, block basic sequence $\{x_n\}_{n=1}^{\infty} \subset S_X$. Since $t \in (0, a(d, \lambda))$ was arbitrary choosen we get that (12) holds for every $t \in [0, a(d, \lambda))$ and therefore according to Theorem 1 we get (10). □

Corollary 4.1 *Let X be a Köthe sequence space with $\{e_i\}_{i=1}^{\infty}$–boundedly complete basis and with a modulus of convexity δ_X. Then for any $t \in (0, a(d, 1/2))$ the inequality*

$$WCS(X) \geq \frac{2 - \delta_X(t) + \left(1 - \frac{\sqrt{3}}{2}\right)t}{2(1 - \delta_X(t))(1 - \delta_X(t) + \left(1 - \frac{\sqrt{3}}{2}\right)t)} \tag{13}$$

holds.

Corollary 4.2 *Let X be a Köthe sequence space with $\{e_i\}_{i=1}^{\infty}$–boundedly complete basis and with a modulus of convexity δ_X. Then for any $t \in (0, a(2, 1/2))$ the inequality*

$$WCS(X) \geq \frac{2 - \delta_X(t) + \left(1 - \frac{\sqrt{3}}{2}\right)t}{2(1 - \delta_X(t))(1 - \delta_X(t) + \left(1 - \frac{\sqrt{3}}{2}\right)t)}$$

holds.

Proof: It is easy to check that $a(d, 1/2)$ is a decreasing function and therefore $[0, a(2, 1/2)) \subseteq [0, a(d, 1/2))$.

Corollary 4.3 *Let X be a Köthe sequence space with $\{e_i\}_{i=1}^{\infty}$ both shrinking and boundedly complete basis and with a modulus of convexity δ_X. If*

$$\delta_X(t) > \frac{3 + 2\left(1 - \dfrac{\sqrt{3}}{2}\right)t - \sqrt{9 + 4\left(1 - \dfrac{\sqrt{3}}{2}\right)t + 4\left(1 - \dfrac{\sqrt{3}}{2}\right)^2 t^2}}{4}$$

for some $t \in [0, a(2, 1/2))$ then X has normal structure.

Proof: If the inequality

$$\frac{2 - \delta_X(t) + \left(1 - \frac{\sqrt{3}}{2}\right)t}{2(1 - \delta_X(t))(1 - \delta_X(t) + \left(1 - \frac{\sqrt{3}}{2}\right)t)} > 1 \tag{14}$$

holds for some $t \in [0, a(2, 1/2))$ then by Theorem 4 it follows that $WCS(X) > 1$. The inequality (14) is equivalent to

$$2\delta_X^2(t) - \left(3 + 2\left(1 - \frac{\sqrt{3}}{2}\right)t\right)\delta_X(t) + \left(1 - \frac{\sqrt{3}}{2}\right)t < 0 \qquad (15)$$

i.e. if the inequalities

$$\delta_X(t) > \frac{3 + 2\left(1 - \frac{\sqrt{3}}{2}\right)t - \sqrt{9 + 4\left(1 - \frac{\sqrt{3}}{2}\right)t + 4\left(1 - \frac{\sqrt{3}}{2}\right)t^2}}{4} \qquad (16)$$

$$\delta_X(t) < \frac{3 + 2\left(1 - \frac{\sqrt{3}}{2}\right)t + \sqrt{9 + 4\left(1 - \frac{\sqrt{3}}{2}\right)t + 4\left(1 - \frac{\sqrt{3}}{2}\right)t^2}}{4}. \qquad (17)$$

hold for some $t \in [0, a(2, 1/2))$ then $WCS(X) > 1$. It is easy to check that the inequality (17) holds for every $t \in [0, 2]$. That's why if holds

$$\delta_X(t) > \frac{3 + 2\left(1 - \frac{\sqrt{3}}{2}\right)t - \sqrt{9 + 4\left(1 - \frac{\sqrt{3}}{2}\right)t + 4\left(1 - \frac{\sqrt{3}}{2}\right)t^2}}{4}$$

for some $t \in [0, a(2, 1/2))$ then holds (14) and thus $WCS(X) > 1$. According to [1] a reflexive Banach space X with $WCS(X) > 1$ has normal structure. To finish the proof we need to mention the well known fact that a Banach space X with a basis $\{e_i\}$ is reflexive iff $\{e_i\}$ is both shrinking and boundedly complete. □

5 Proof of the main result

By the inequalities

$$\frac{3 + 2\left(1 - \frac{\sqrt{3}}{2}\right)t - \sqrt{9 + 4\left(1 - \frac{\sqrt{3}}{2}\right)t + 4\left(1 - \frac{\sqrt{3}}{2}\right)t^2}}{4} \leq \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right)t,$$

for every $t \in [0, a(2, 1/2))$, $0.9 < a(2, 1/2)$ and Corollary 4.3 the proof follows. □

6 Some estimates of $\delta_{\ell_p}^{(\lambda)}$ and $\rho_{\ell_p}^{(\lambda)}$

Lemma 6.1 *For every $a, b \in \mathbb{R}$, $p \geq 2$ and $\lambda \in [0, 1]$ the inequality*

$$\begin{aligned} |\lambda \;\; + \;\; (1-\lambda)b|^p &+ \left|\sqrt[p]{\tfrac{\lambda(1-\lambda)}{2^{p-2}}}(a-b)\right|^p \\ &\leq \left(|\lambda a + (1-\lambda)b|^2 + \left|\sqrt[p]{\tfrac{\lambda(1-\lambda)}{2^{p-2}}}(a-b)\right|^2\right)^{p/2} \end{aligned} \qquad (18)$$

holds

Proof: The proof follows right away if taking into account that in $\mathbb{R}^2$ $\|x+y\|_p \le \|x+y\|_2$, where $x = \lambda a + (1-\lambda)b$ and $y = \sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}(a-b)$. □

Lemma 6.2 *For every $a, b \in \mathbb{R}$, $p \ge 2$ and $\lambda \in [0,1]$ the inequality*

$$\begin{aligned}(\lambda a + (1-\lambda)b)^2 \quad &+ \quad \left(\sqrt[p]{\tfrac{\lambda(1-\lambda)}{2^{p-2}}}(a-b)\right)^2 \\ &\le \quad a^2\left(\lambda^2 + \left(\sqrt[p]{\tfrac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right) + b^2\left((1-\lambda)^2 + \left(\sqrt[p]{\tfrac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right)\end{aligned} \tag{19}$$

holds

Proof: Obviously

$$\begin{aligned}(\lambda a \quad &+ \quad (1-\lambda)b)^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}(a-b)\right)^2 \\ &= \quad \lambda^2a^2 + 2\lambda(1-\lambda)ab + (1-\lambda)^2b^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2(a^2 - 2ab + b^2) \\ &= \quad a^2\left(\lambda^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right) + b^2\left((1-\lambda)^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right) \\ &- \quad 2ab\left(\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2 - \lambda(1-\lambda)\right) \\ &\le \quad a^2\left(\lambda^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right) + b^2\left((1-\lambda)^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right),\end{aligned}$$

because for any $\lambda \in (0,1)$ holds

$$\left(\frac{1}{2}\right)^{\frac{p-2}{p}} > \left(\frac{1}{2}\right)^{\frac{2(p-2)}{p}} \ge (\lambda(1-\lambda))^{\frac{p-2}{p}}$$

and thus $\sqrt[p]{\dfrac{\lambda(1-\lambda)}{2^{p-2}}} - \lambda(1-\lambda) > 0$. □

Lemma 6.3 *For every $a, b \in \mathbb{R}$, $p \ge 2$ and $\lambda \in (0,1)$ the inequality*

$$\begin{aligned}a^2 \quad &\left(\lambda^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right) + b^2\left((1-\lambda)^2 + \left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right) \\ &\le (|a|^p + |b|^p)^{2/p}\left(2^{\frac{p-2}{2}}(\lambda(1-\lambda))^{2/p} + \left(\lambda^{\frac{2p}{p-2}} + (1-\lambda)^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}}\right)\end{aligned} \tag{20}$$

holds

Proof: By Hölder inequality we get the inequality:

$$\begin{aligned}
a^2\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2 &+ b^2\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2 \\
&= (a^2+b^2)\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2 \\
&\leq \left(|a^2|^{p/2}+|b^2|^{p/2}\right)^{2/p}\left(1^{\frac{p}{p-2}}+1^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2 \qquad (21)\\
&= (|a|^p+|b|^p)^{2/p}\,2^{\frac{p-2}{p}}2^{-\frac{2(p-2)}{p}}\left(\sqrt[p]{\lambda(1-\lambda)}\right)^2 \\
&= (|a|^p+|b|^p)^{2/p}\,2^{-\frac{(p-2)}{p}}\left(\sqrt[p]{\lambda(1-\lambda)}\right)^2
\end{aligned}$$

and

$$a^2\lambda^2+b^2(1-\lambda)^2 \leq (|a|^p+|b|^p)^{2/p}\left((\lambda^2)^{\frac{p}{p-2}}+((1-\lambda)^2)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}. \qquad (22)$$

Now by (21) and (22) we get the proof. □

Theorem 5 *For $p \geq 2$ and any $\lambda \in (0,1)$ and $\varepsilon \in [0,2]$ holds:*

$$1-\left(1-\left(\varepsilon\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^p\right)^{1/p} \leq \delta_{\ell_p}^{(\lambda)}(\varepsilon) \leq 1-\left(1-\left(\varepsilon\sqrt[p]{\frac{p\lambda(1-\lambda)}{2^{p-1}}}\right)^p\right)^{1/p}.$$

Proof: For any $x, y \in S_{\ell_p}$ and $\lambda \in (0,1)$ by Lemmas 6.1, 6.2 and 6.3 we have

$$\begin{aligned}
\sum_{k=1}^{\infty}|\lambda x_k &+ (1-\lambda)y_k|^p + \sum_{k=1}^{\infty}\frac{\lambda(1-\lambda)}{2^{p-2}}|x_k-y_k|^p \\
&\leq \left(\sum_{k=1}^{\infty}|\lambda x_k+(1-\lambda)y_k|^2+\sum_{k=1}^{\infty}\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2|x_k-y_k|^2\right)^{p/2} \\
&\leq \left(\sum_{k=1}^{\infty}x_k^2\left(\lambda^2+\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right)+\sum_{k=1}^{\infty}y_k^2\left((1-\lambda)^2+\left(\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^2\right)\right)^{p/2} \\
&\leq \left[\left(\sum_{k=1}^{\infty}(|x_k|^p+|y_k|^p)\right)^{2/p}\left(2^{-\frac{p-2}{p}}(\lambda(1-\lambda))^{\frac{2}{p}}+\left(\lambda^{\frac{2p}{p-2}}+(1-\lambda)^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}}\right)\right]^{p/2} \\
&\leq \left[2^{2/p}\left(2^{-\frac{p-2}{p}}2^{-\frac{4}{p}}+\left(2.2^{-\frac{2p}{p-2}}\right)^{\frac{p-2}{p}}\right)\right]^{p/2} \\
&= \left[2^{\frac{2}{p}}\left(2^{\frac{-p-2}{p}}+2^{\frac{-p-2}{p}}\right)\right]^{p/2} = 1.
\end{aligned}$$

Hence $\|\lambda x+(1-\lambda)y\|^p+\left(\varepsilon\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^p \leq 1$ and therefore $\delta_{\ell_p}^{(\lambda)}(\varepsilon) \geq 1-\left(1-\left(\varepsilon\sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^p\right)^{1/p}$.

For the proof of the righthandside inequality let define:

$$x_1 = y_1 = \sqrt[p]{1 - \frac{\varepsilon^p \lambda(1-\lambda)}{(1-|2\lambda-1|^p)2^{p-2}}},$$

$$x_2 = -y_2 = \varepsilon \sqrt[p]{\frac{\lambda(1-\lambda)}{(1-|2\lambda-1|^p)2^{p-2}}}$$

and $x_i = y_i = 0$ for $i \geq 3$. Consider $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$. Obviously $\|x\|^p = \|y\|^p = 1 - \frac{\varepsilon^p \lambda(1-\lambda)}{(1-|2\lambda-1|^p)2^{p-2}} + \frac{\varepsilon^p \lambda(1-\lambda)}{(1-|2\lambda-1|^p)2^{p-2}} = 1$,

$$\begin{aligned} \|\lambda x + (1-\lambda)y\|^p &= |\lambda x_1 - \lambda y_1 + y_1|^p + |\lambda x_2 - \lambda y_2 + y_2|^p = |y_1|^p + |2\lambda y_2 - y_2|^p \\ &= 1 - \tfrac{\varepsilon^p \lambda(1-\lambda)}{(1-|2\lambda-1|^p)2^{p-2}} + \tfrac{\varepsilon^p \lambda(1-\lambda)}{(1-|2\lambda-1|^p)2^{p-2}}|2\lambda-1|^p \\ &= 1 - \left(\varepsilon \sqrt[p]{\tfrac{\lambda(1-\lambda)}{2^{p-2}}}\right)^p \end{aligned}$$

and

$$\|x-y\|^p = |x_2 - y_2|^p = 2^p \frac{\varepsilon^p \lambda(1-\lambda)}{(1-|2\lambda-1|^p)2^{p-2}} = 4\varepsilon^p \frac{\lambda(1-\lambda)}{1-|2\lambda-1|^p} \tag{23}$$

Claim 6.1 *For any $\lambda \in (0,1)$, and $p \geq 2$ holds the inequality:*

$$2p\lambda(1-\lambda) \geq 1 - |2\lambda-1|^p. \tag{24}$$

Proof: WLOG we may assume that $\lambda \in (0, 1/2]$. Consider the function

$$f_p(\lambda) = 2p\lambda(1-\lambda) - 1 + (1-2\lambda)^p.$$

By $f_p'(\lambda) = 2p - 4p\lambda - 2p(1-2\lambda)^{p-1} = 2p(1-2\lambda-(1-2\lambda)^{p-1}) \geq 0$ for any $\lambda \in (0,1/2]$, $p \geq 2$ it follows that $0 = f_p(0) \leq f_p(\lambda)$ for any $\lambda \in (0,1/2]$, $p \geq 2$ and the Claim is proved. □

Now by (23) and (24) we get that

$$\|x-y\| \geq \varepsilon \sqrt[p]{\frac{2}{p}}.$$

Therefore

$$\delta_{\ell_p}^{(\lambda)}\left(\varepsilon \sqrt[p]{2/p}\right) \leq 1 - \left(1 - \left(\varepsilon \sqrt[p]{\frac{\lambda(1-\lambda)}{2^{p-2}}}\right)^p\right)^{1/p},$$

i.e.

$$\delta_{\ell_p}^{(\lambda)}(\varepsilon) \leq 1 - \left(1 - \left(\varepsilon \sqrt[p]{\frac{p\lambda(1-\lambda)}{2^{p-1}}}\right)^p\right)^{1/p}.$$

□

Remark: For $p = 2$ the estimate obtained in Theorem 5 is exact and coincides with the result in [7] that

$$\delta_{\ell_2}^{(\alpha)}(\varepsilon) = 1 - \left(1 - \left(\varepsilon\sqrt{\alpha(1-\alpha)}\right)^2\right)^{1/2}.$$

Theorem 6 *For any Banach space X the inequality*

$$\rho_X^{(\lambda)}(\tau) \geq \sqrt{1+\frac{\tau^2}{4\lambda(1-\lambda)}}-1$$

holds for every $\tau>0$ and $\lambda \in (0,1)$.

Proof: By Theorem 3 we have

$$\begin{aligned}\rho_X^{(\lambda)}(\tau) &= \sup_{0\leq\varepsilon\leq 2}\left\{\tfrac{\tau\varepsilon}{2}-\delta_{X^*}^{(\lambda)}(\varepsilon)\right\} \geq \sup_{0\leq\varepsilon\leq 2}\left\{\tfrac{\tau\varepsilon}{2}-\delta_{\ell_2}^{(\lambda)}(\varepsilon)\right\} \\ &= \sup_{0\leq\varepsilon\leq 2}\left\{\tfrac{\tau\varepsilon}{2}-1+\sqrt{1-\varepsilon^2\lambda(1-\lambda)}\right\}.\end{aligned}$$

Put

$$f(\varepsilon)=\frac{\tau\varepsilon}{2}-1+\sqrt{1-\varepsilon^2\lambda(1-\lambda)}.$$

It is easy to check that $f(\varepsilon_\tau) \geq f(\varepsilon)$, where $\varepsilon_\tau = \dfrac{\tau}{\sqrt{4\lambda^2(1-\lambda)^2+\tau^2\lambda(1-\lambda)}}$ is the solution of the equation:

$$f'(\varepsilon_\tau)=\frac{\tau}{2}-\frac{\varepsilon_\tau\lambda(1-\lambda)}{\sqrt{1-\varepsilon_\tau^2\lambda(1-\lambda)}}=0,$$

Therefore

$$\begin{aligned}\rho_X^{(\lambda)}(\tau) &\geq f(\varepsilon_\tau) \\ &= \frac{\tau^2}{2\sqrt{4\lambda^2(1-\lambda)^2+\tau^2\lambda(1-\lambda)}}-1+\sqrt{1-\frac{\tau^2\lambda(1-\lambda)}{4\lambda^2(1-\lambda)^2+\tau^2\lambda(1-\lambda)}} \\ &= \frac{\tau^2+4\lambda(1-\lambda)}{2\sqrt{\lambda(1-\lambda)(4\lambda(1-\lambda)+\tau^2)}}-1=\frac{\sqrt{\tau^2+4\lambda(1-\lambda)}}{2\sqrt{\lambda(1-\lambda)}}-1 \\ &= \sqrt{1+\frac{\tau^2}{4\lambda(1-\lambda)}}-1.\end{aligned}$$

□

References

[1] W. Bynum. Normal structure coefficients for Banach spaces. Pacific J Math, 1980, **86**: 427–436.

[2] M. Brodskii, D. Milman. On the center of convex sets. Dokl Akad Nauk SSSR, 1948, **59**: 837-840.

[3] F. Wang, H. Cui. Some estimates on the weakly convergent sequence coefficient in Banach spaces. J Inequal Pure and Appl Math, 2006, **7**(5) Art. 161: 427–436.

[4] S. Dhompongsa, A. Kaewkhao, S. Tasena. On a generalized James constant. J Math Anal Appl, 2003, **285**: 419–435.

[5] S. Dhompongsa, A. Kaewkhao. A note on properties that imply the fixed point property. Abstr Appl Anal, 2006, **2006**: Article ID 34959.

[6] J. Gao. Normal structure ant smoothness in Banach spaces. J Nonlinear Functional Anal Appl, 2005, **10**: 103–115.

[7] Y. Changsen, W. Fenghui. On generalized modulus of convexity and uniform normal structure. Acta Math Scientia, 2007, **126**: 838–844.

[8] J. Gao, K. Lau. On two classes Banach spaces with uniform normal structure. Studia Math, 1991, **99**: 41–56.

[9] J. Lindenstrauss, L. Tzafriri. Classical Banach Spaces I, Sequence Spaces. Springer–Verlag: Berlin, 1977.

[10] F. Hiai. Representation of additive functional on vector–valued normed Köthe spaces. Kodai Math J, 1979, **2**: 300–313.

[11] Y. Cui. Weakly convergent sequence coefficient in Köthe sequence spaces. Proc Amer Math Soc, 1998, **126**: 195–201.

[12] J. Clarkson. Uniformly convex spaces. Trans Amer Math Soc, 1936 **40**: 394–414.

[13] P. Habala, P. Hajek, V. Zizler. Introduction to Banach Spaces. Charles University: Prague, 1996.

[14] J. Lindenstrauss. On the modulus of smoothness and divergent series in Banach spaces. Mich Math J, 1963 **10**: 241–252.

[15] A. Dvoretzky. Some results on convex bodies and Banach spaces. Proc Internat Sympos Linear Spaces. Jerusalem: Academic Press, 1961: 123-160.

[16] G. Nordlander. The modulus of convexity in normed linear spaces. Arkiv for Math, 1960, **4**: 167–170.