

# Optimization of the Courses in Geometry by the Usage of Dynamic Geometry Software Sam

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## Abstract

*We present a new approach to the teaching of Synthetic Geometry in schools and universities with the help of DGS. The introduction of new elements into DGS helps to optimize the teaching process. These new elements are infinite points in the extended Euclidean plane and the "Swap finite & infinite points" function. We give examples of the usage of these new elements in Projectivity, Homology, Conic Sections, Plane Sections of Polyhedra, and in the application of Pappus' Theorem and Desargues' Theorem. These new features increase the benefits of DGS in teaching and learning Geometry. We optimize the education process by saving time involved in drawing, generalizing large groups of problems, stimulating and helping investigations, and forming a creative style of thinking.*

## 1. Introduction

Dynamic Geometry Software (DGS) has been widely used for teaching and learning Euclidean geometry. Cinderella has made a significant step in the teaching of non-Euclidean Geometry (spherical and hyperbolic). Recent results in different fields of pure mathematical geometry of Banach spaces [2, 3], partial differential equations [9], topology [19] and number theory [22] were proven with the help of the fundamental theorems of Projective Geometry. Projective Geometry is used in applied science to prove results in mechanics [20], in composition analysis of building structure [21] and in coding theory and cryptography [4, 14]. All these show the need for a deeper study of Projective Geometry at university and at school. We hope that our specialized DGS Sam with its new "Swap finite & infinite points" function will optimize teaching and learning of geometry, and that it will enrich teachers' capabilities of replacing the old principle "learn and repeat" with a new "learn and create" principle.

An important goal for the mathematical teacher is to present challenging problems to the student so that the student can show his ideas to deal with tasks. The use of technological tools seems to offer a good solution to this problem. Significant attempts were made over the last twenty years to implement computer algebra systems in the teaching process of mathematics in schools

and in universities. There are several types of computer algebra system that are currently used in classrooms. The first type of system is general, which includes as many tools as possible for solving mathematical problems. We would like to mention Mathematica, Maple, MatLab. A second type are Dynamic Geometry Systems. This type of system has been developed primarily for the needs of algebra and geometry classes. We would like to mention Cinderella, GeoGebra, GeoNext, Sketchpad. These are very powerful systems for modeling problems and suggesting solutions. Recent features of GeoGebra also make it possible to present geometric and numeric visualizations of formulas in algebra and calculus. Another type of system is the family of specialized systems that have been developed to help the teaching process of specific mathematical courses. For example, the Interactive Platform for Learning Calculus (PIAC) [1] is a tool designed for the needs of a specific subject.

## 2. Learning Environment

The Dynamic Geometric Software Sam is specialized educational software that was created for the needs of a course in Synthetic Geometry at Plovdiv University “Paisii Hilendarski”. It is written in C # using the .NET Framework 4 environment. That is why **the user has to install .NET Framework 4 before running the Sam program**. The first version was developed for the interactive training of students in “Mutual intersection of polyhedra” [10, 11]. With the help of this software and our classification of the pierce points of surrounding edges we were able turn training into a creative process [11]. Observing the increased interest of students in this traditionally difficult area, we extended the software so that it could be applied to a more comprehensive range of areas in Synthetic Geometry. In particular, we implemented two new features; infinite points and the “Swap finite & infinite points” function. The software has helped us to overcome the usual difficulties associated with learning Synthetic Geometry.

### 2.1. Infinite Elements and the “Swap finite & infinite points” Function.

#### 2.1.1. Infinite Elements

The abstract axiomatic construction of projective spaces can be realized in different models. The widespread model is the Euclidean one, where most of the sketches are drawn. This model is created by extending Euclidean space with infinite elements. We would like to mention just a few facts about extended Euclidean space.

Following [6, p.8] we accept that *it is convenient to use the name range for the set of all points on a line, and pencil for the set of all lines that lie in a plane and pass through a point*.

The set of all lines, which are parallel to a fixed line  $g$  and the line  $g$  are called **an infinite point**, which is denoted by  $G_\infty$  or  $U_g$ . Any Euclidean line is incident with exactly one infinite point. So with every pencil of parallel lines an infinite point is associated, which signals their direction.

The set of all planes, which are parallel to a fixed plane  $\alpha$  and the plane  $\alpha$  are called **an infinite line**. Any Euclidean plane is incident with exactly one infinite line and is denoted by  $u_\alpha$ . All infinite points, which are incident with the plane  $\alpha$  and only they, are incident with the infinite line  $u_\alpha$ . As all the investigations in this work are in the Euclidean model of the projective plane we will denote its infinite line with  $\omega$ . For the precise definition of the projective space one can see [4, 5, 6].

It is well known that the infinite elements (points and line) play a crucial role in the Euclidean model of the projective space and all the sketches of the problems are made in the extended Euclidean plane. This is what inspired us to introduce new objects for DGS - an arbitrary infinite point and the infinite point of a given line. They are included as instruments in DGS Sam.

#### 2.1.2. DGS Sam

Each drawn object appears in the list (tree diagram) on the left of the drawing with a brief description, including the label (or <no name>, if there is not any) and a list of its parents.

Free objects have no parents, that is, they are not dependent on any object. Their labels are not preceded by any sign in the tree diagram.

Dependent objects are defined on the basis of other objects (their parents). Their labels are preceded by “+” sign in the tree diagram. By clicking on the sign “+” the sub-list with their parents appears.

There are objects with limited motion (an arbitrary point on a given line, an arbitrary line through a given point).

**Example:** The points  $R$  and  $Q$  are free (Figure 1). The line  $g=RQ$  is dependent and its parents are the points  $R$  and  $Q$ . The line  $x$  has only one parent, the point  $R$ , and that is why one can rotate  $x$  around  $R$  using the right button of the mouse.

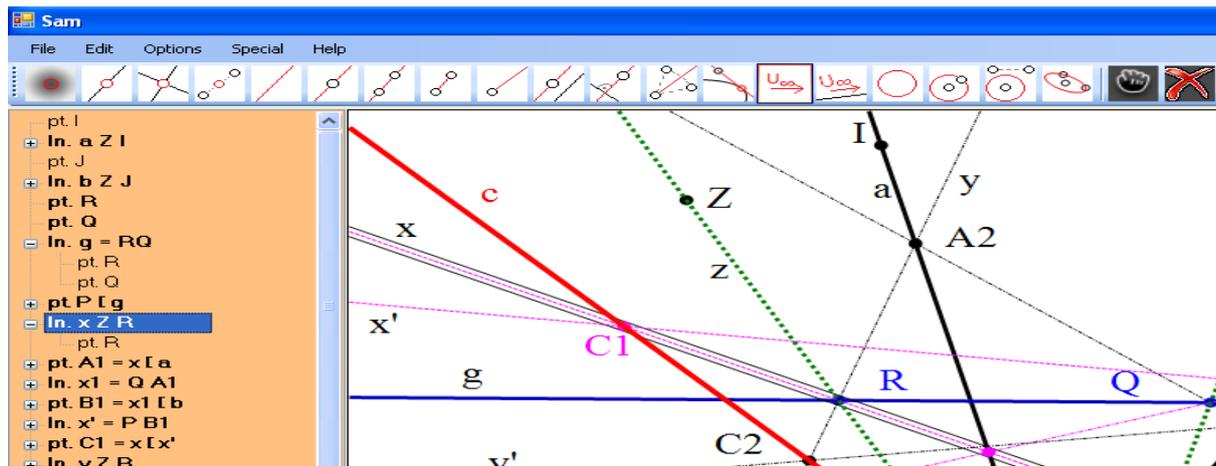


Figure 1 “Free objects”

The user can change the direction of the free infinite point by rotation. Rotation is effected by clicking the right button of the mouse and rotating the vector to the desired position.

It is well known how to draw a line through two points: choose the “line through two points” instrument and select both of the points. The order of point selection is not important. It does not matter which of them is first. The procedure is the same whether the two points are finite or one of them is infinite.

Let us note that in the present version of DGS Sam we cannot construct a line through two infinite points.

One of the advantages of DGS Sam is the possibility of presenting the solution “step by step” or in stages (chosen by the user) with accompanying comments, prepared by the user. This facility may be accessed through the presentation panel at the bottom left part of the screen (an enlarged screenshot is shown on Figure 2.2)

To prepare the presentation of a dynamic sketch, the teacher marks a phase by selecting a stage from the list (tree diagram) and by checking the “The chosen object is a phase” button. The teacher also decides how much detail should be presented to the audience. The teacher may also write comments for every phase via the “View and edit phases” window which is accessed with the “Phases” button. The presentation is controlled by play buttons. These buttons allow the teacher to move one step forward or backward through the construction steps. This tool is very useful because it helps the teacher focus the students’ attention on particular constructions.

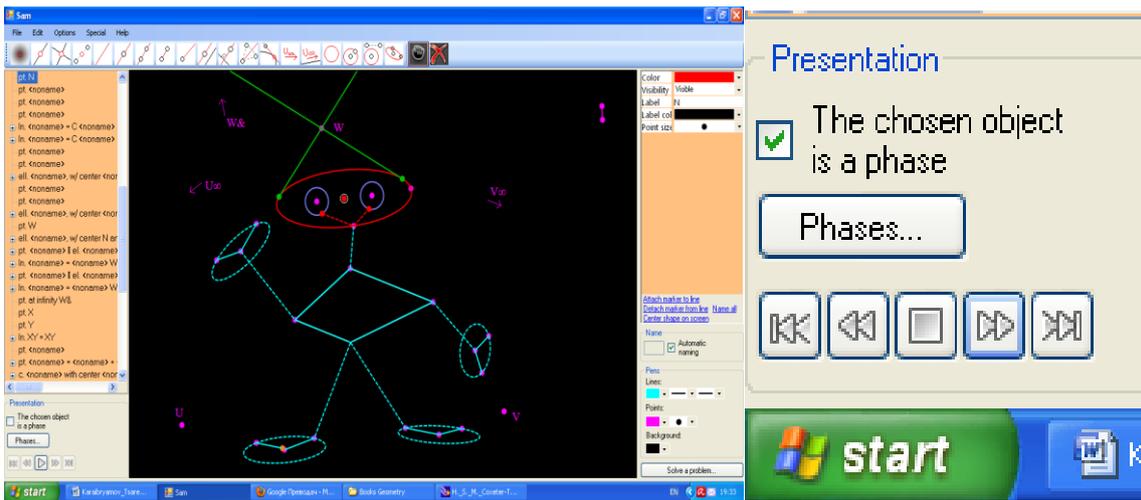


Figure 2.1 "DGS Sam"

Figure 2.2 "Presentation panel"

### 2.1.3. Swap Finite and Infinite Points

The next new feature of the Dynamic Geometry Systems that we have introduced is the **"Swap finite & infinite points"** function, which we will refer to in the text by **SFIP**. This function is accessed via the "Special" menu item. A window with two lists of all the free finite points and all the free infinite points appears (Figure 3.1). By double clicking or by using the buttons ">>" and "<<" the user can select which two points (one finite and one infinite) are to be swapped. The swapping is effected by clicking on the button "OK".

For example, Figure 3.1 presents the free finite points  $U$ ,  $A$ ,  $B$ , the free infinite point  $U_\infty$  and the lines  $g=AB$ ,  $a=AU_\infty$ ,  $b=BU_\infty$ ,  $c=CU_\infty$ , where  $C$  is an arbitrary point on the line  $g$ . After applying SFIP function to the points  $U_\infty$  and  $U$  from Figure 3.1, all objects dependent on  $U_\infty$  will be redefined so as to be dependent on  $U$  instead. Thus, we obtain Figure 3.2 where one can see how the lines  $g=AB$ ,  $a=AU_\infty$ ,  $b=BU_\infty$ ,  $c=CU_\infty$  are converted to lines  $a=AU$ ,  $b=BU$ ,  $c=CU$ , respectively. If SFIP is applied a second time to the same pair of points, then the original figure is obtained.

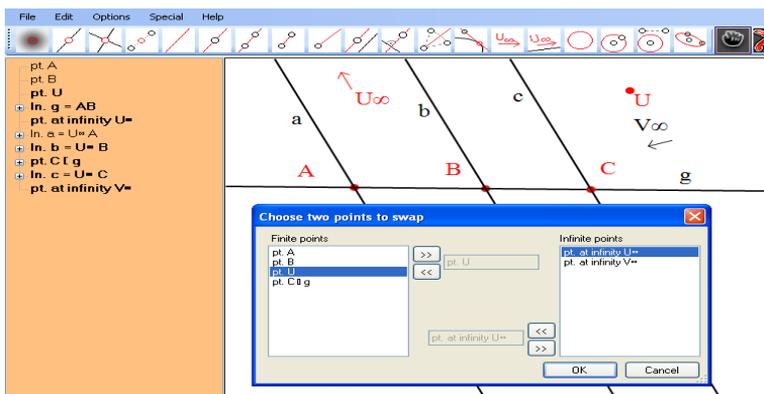


Figure 3.1

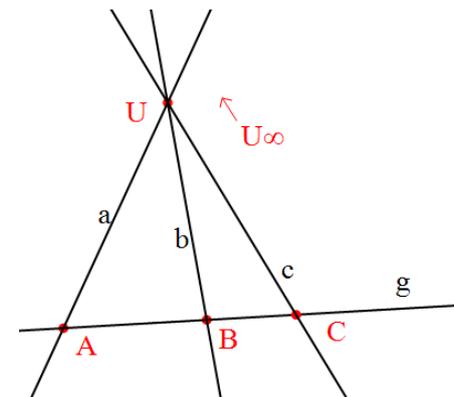


Figure 3.2

This special function gives the opportunity to generate automatically new problems with their solutions. Thus the new function optimizes a lot of the drawing work. On the other hand, by discovering invariant relations between the investigated objects and generalizing of some problems and their solutions, the SFIP function provokes and facilitates a research style of thinking.

**Definition 1:** The problems, which are generated from a problem  $A$  with the help of the SFIP function, will be called satellites of problem  $A$  and be denoted in the paper by  $A^*$ ,  $A^{**}$ , etc.

Note that a pencil of parallel lines can be constructed by procedures which are standard for all DGS: the user has chosen to construct the lines with the “a line parallel to another line” instrument. The benefit of the first construction is that the user can use the function “Swap finite & infinite points”.

**Remark 1:** For the user’s convenience we will note that sometimes after using the SFIP function this may happen:

- 1.1 The lines of a pencil overlap. Then it is necessary to separate these lines by hand. This is achieved by clicking on the lines and activating translation (by the left button of the mouse if the center of the pencil is an infinite point) or rotation (by the right button of the mouse if the center of the pencil is a finite point);
- 1.2 Some of the points may appear close together. Then it is necessary to separate these points by hand. This is achieved by: a) clicking on any free point of this set using the list (tree diagram) on the left side of the screen; b) dragging the selected point.

**Remark 2:** After applying the SFIP function it is possible that some objects will not be visible on the screen. This depends on the location of the free objects. In this case it is enough to select the missing object in the “tree diagram” and to choose from the right side of the screen the “Center shape on the screen” function (Figure 2.1). It is convenient to mark a free object, because the user can move it so that the whole sketch fits on the screen.

## 2.2. Connected Figures

**Definition 2:** We will call the figures that are obtained by applying the “Swap finite & infinite points” function **connected figures**.

By Definition 2 it follows that the pencils  $U_\infty$  and  $U$  from the Figure 3.1 and Figure 3.2 are connected figures.

Consider the parallelogram from Figure 4.1. The free points in its construction are:  $A$ ,  $U_\infty$ ,  $V_\infty$ . The points  $B$  and  $D$  are arbitrary points from the lines  $a=AU_\infty$  and  $b=AV_\infty$ , respectively.  $C$  is the intersection point of the lines  $d=DU_\infty$  and  $c=BV_\infty$ . At the end the sides are constructed as segments. The sequence of the constructions can be traced through the presentation panel.

**Definition 3:** The parallelogram  $ABCD$  (Figure 4.1) will be called a **universal parallelogram**.

If  $U$ ,  $V$  are arbitrary free finite points then SFIP can transform a universal parallelogram into a trapezium (by swapping the points  $V_\infty$  and  $V$ ; see Figure 4.2) or into a quadrilateral (by swapping the points  $U_\infty$  and  $U$ ; see Figure 4.3). Thus, the *parallelogram, the trapezium and the quadrilateral are connected figures*. Each of these figures can be transformed into the other ones by SFIP.

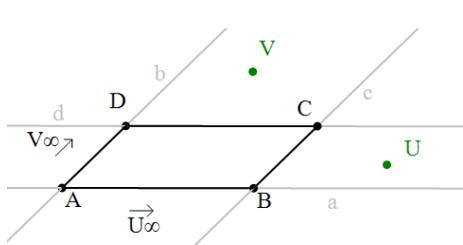


Figure 4.1

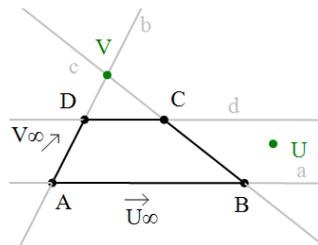


Figure 4.2

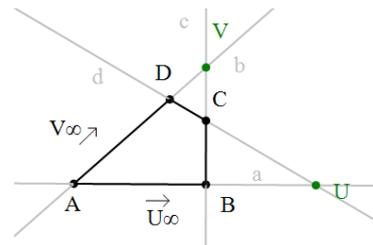
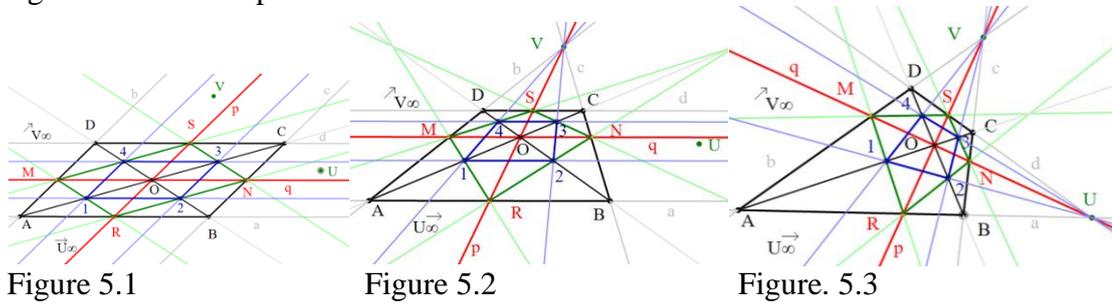


Figure 4.3

The constructions that were made for one of the connected figures are carried over to the new ones (Figures 5.1, 5.2, 5.3). This fact is important as it has the following benefits:

- 1) it saves a lot of drawing work;
- 2) it helps to identify invariant relations;
- 3) it generalizes some problems and their solutions.



Let  $ABCD A_1B_1C_1D_1$  be an arbitrary parallelepiped. Its edges pass through three different infinite points  $U_\infty, V_\infty, W_\infty$ . It is naturally to expect that the consecutive swapping of the points  $U_\infty, V_\infty, W_\infty$  with the free finite points  $U, V, W$  will transform the parallelepiped into a prism, a truncated pyramid, a pyramid with a base, a parallelogram, trapezium, quadrilateral or triangle.

How can one construct a parallelepiped so that to apply the SFIP function?

Let the base  $ABCD$  be a universal parallelogram with infinite points on its sides  $U_\infty, V_\infty$  (Figure 6.1). The surrounding edges lie on the lines  $A W_\infty, B W_\infty, C W_\infty, D W_\infty$ . Thus the free points are:  $A, U_\infty, V_\infty, W_\infty$ . The vertex  $A_1$  is an arbitrary point on the line  $A W_\infty$ .

There are two possibilities for the construction of the second base, each with its applications:

**First construction:** The “a line through a point and parallel to a line” instrument is used for the construction of the second base  $A_1B_1C_1D_1$ , i.e. : 1) The lines  $a_9$  passes through the point  $A_1$  and it is parallel to the line  $a_1 = AB$  ; 2) The lines  $a_{10}$  passes through the point  $A_1$  and it is parallel to the line  $a_2 = AD$  ; 3) The points  $B_1 = a_9 \cap B W_\infty, D_1 = a_{10} \cap D W_\infty$  ; 4) The lines  $a_{11}$  passes through the point  $B_1$  and it is parallel to the line  $a_5 = BC$  ; 5) The lines  $a_{12}$  passes through the point  $D_1$  and it is parallel to the line  $a_4 = CD$  ; 6) The point  $C_1 = a_{11} \cap a_{12}$  (Figure 6.1).

At the end the edges are constructed again as segments.

The sequence of the constructions can be traced through the presentation panel.

The rotation of the infinite points  $U_\infty, V_\infty, W_\infty$ , which is activated by the right button of the mouse, can be used to change the angles between the edges. Moving the points  $B, D, A_1$  changes the length of the edges.

**Definition 4:** The parallelepiped  $ABCD A_1B_1C_1D_1$  (Figure 6.1) will be called a **universal parallelepiped**.

Let  $U, V, W$  be arbitrary free finite points. Now we can apply the SFIP function.

Applying the SFIP function the parallelepiped transforms instantly into a prism, a truncated pyramid, or a pyramid; with a parallelogram, trapezium, quadrilateral or triangular base, with a special positioning of a chosen vertex (Figures 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6).

Several examples are shown in Figures 6.1 to 6.6. Thus all these *parallelepipeds, prisms, truncated pyramids, pyramids* are connected figures.

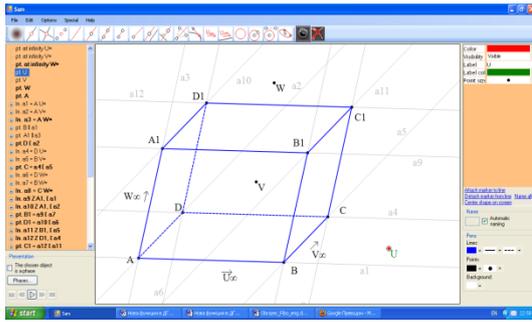


Figure 6.1

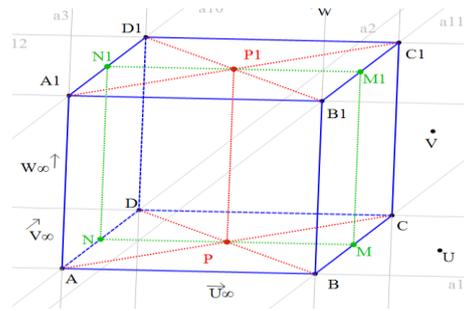


Figure 6.2

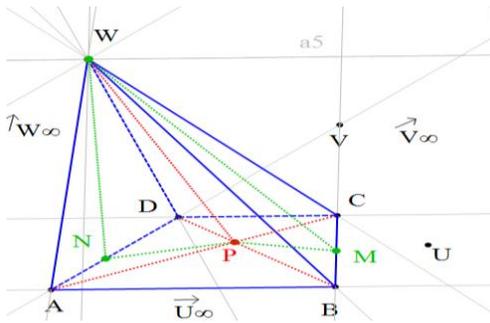


Figure 6.3

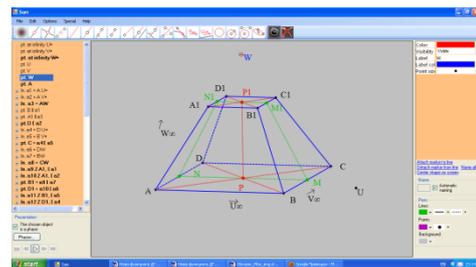


Figure 6.4

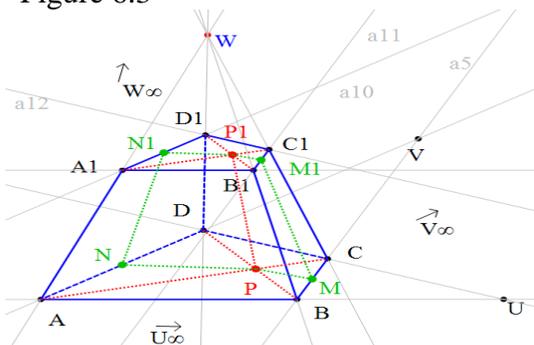


Figure 6.5

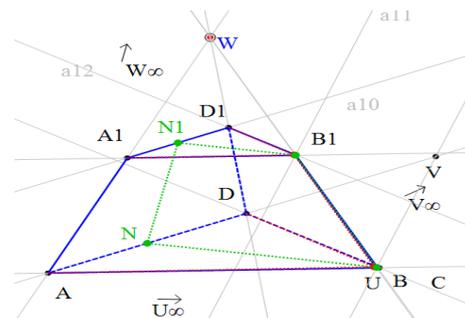


Figure 6.6

This way of showing school students different types of prisms and pyramids holds their attention and creates interest in the topic.

**Second construction:** Now let the second base  $A_1B_1C_1D_1$  of the parallelepiped  $ABCD A_1B_1C_1D_1$  be constructed by using the instrument “line through two points”, i.e. : 1) The lines  $A_1U_\infty$  and  $A_1V_\infty$  are drawn; 2) The points  $B_1 = A_1U_\infty \cap BW_\infty$  and  $D_1 = A_1V_\infty \cap DW_\infty$  are found; 3) The point  $C_1 = B_1V_\infty \cap D_1U_\infty$  is determinate (Figure 6.7).

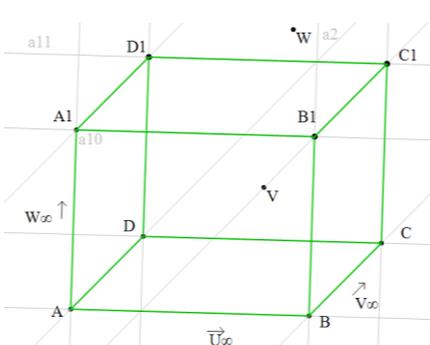


Figure 6.7

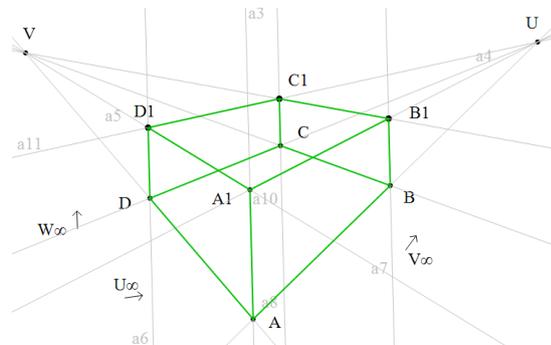


Figure 6.8

Let  $U, V, W$  be arbitrary finite points. Let us swap by the SFIP function  $U_\infty$  with  $U$ , and  $V_\infty$  with  $V$ . As a result we obtain the Figure 6.8 . It is not difficult to see that these connected figures present the parallelepiped  $ABCD A_1B_1C_1D_1$  in an arbitrary axonometry (Figure 6.7 ) and in perspective (Figure 6.8) . Another pair of connected figures are shown in Figures 6.9, and 6.10.

Moving the free points  $U_\infty$  and  $V_\infty$  in Figures 6.7, 6.9 changes the parameters of the axonometry. Moving the free points  $U$  and  $V$  in Figures 6.8, 6.10 changes the observer's position.

This application could be very useful for lecturers and students, because it presents the transition from axonometry in perspective and back in the shortest time, and at the same time it explains the relationship between them.

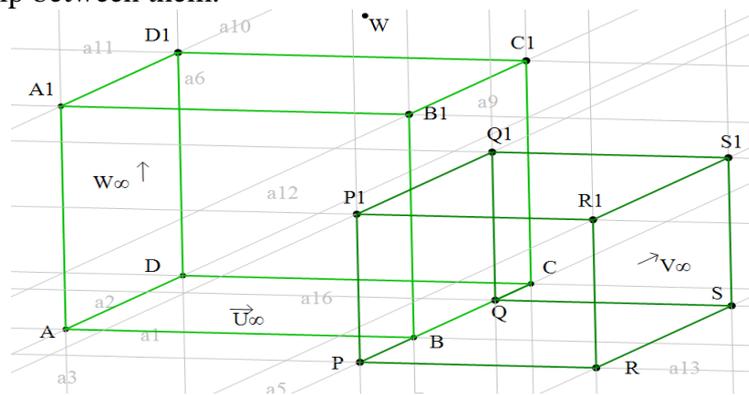


Figure 6.9

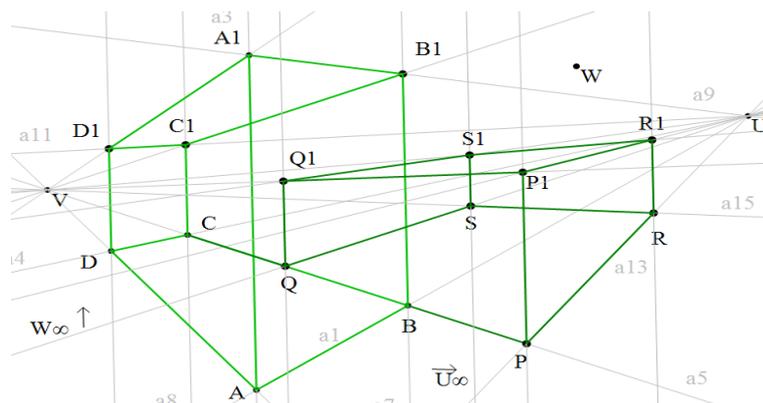


Figure 6.10

The first construction also refers to any auxiliary sketches that are needed for the solutions of the problems in 3-dimensional geometry for cylinders, truncated conics, conics, for inscribed and circumscribed polyhedrons. The bases of the circle conics and cylinders in the sketches are ellipses that are created with a couple of conjugated semi-diameters. They are drawn with the “ellipse with center and conjugated semi-diameters” instrument by pointing at the center and the end points of the conjugated semi-diameters. The movement of any of these three points changes the ellipse. Some examples can be seen in Figures 7.1, 7.2 and 7.3 showing that the “Swap finite & infinite points” function can be applied to points for which tangential lines are constructed towards ellipses or circles. Here the length of  $AA_0$  is determined from distance by the segment  $XY$ , because  $A_0$  is the intersecting point of the line  $A U_\infty$  and the circle  $k(A, r=XY)$ .

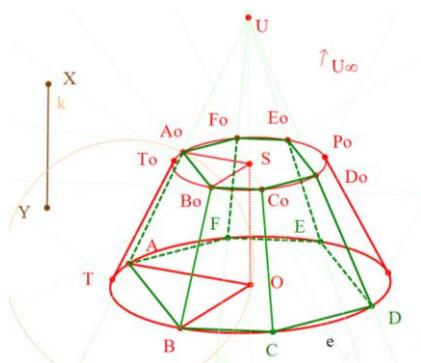


Figure 7.1

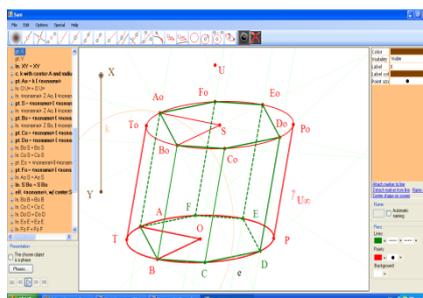


Figure 7.2

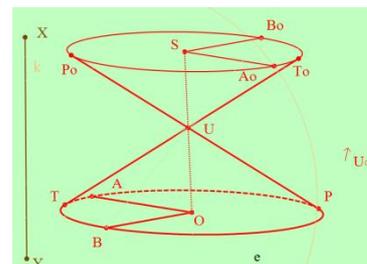


Figure 7.3

### 3. Main Benefits of the Use of the Infinite Elements and the SFIP Function

The main benefits of using Dynamic Geometry Software Sam are:

- Its first benefit is the one common to all Dynamic Geometry Systems; the sketch can be presented in the most suitable way on-screen or a sheet of paper without the need of being drawn again. Dynamic transformations are possible throughout the presentation, showing students, which properties are preserved and which are not;
- Its interface is free of special drawing macros, and thus stimulates student to learn and to remember basic constructions in geometry;
- Its new “infinite points” elements allow students to reach beyond the level of abstraction easily and naturally;
- Its special SFIP function, that optimizes drawing, saves a lot of time in the classroom and at home. The user can construct one case of the problem and then look at the connected figures and follow their construction step by step;
- Its special SFIP function helps to immediately show common properties between the connected figures and any differences that appear;
- Its special SFIP function helps create new problems, stimulate the investigative curiosity of students, and generalize these problems;
- DGS Sam with its new features allows a joint discussion of Euclidean and Projective Geometries. Thus it connects studying geometry at university and at secondary school.

The evidence for these benefits will be presented by examples in the next part of the paper.

### 4. Applications of the Infinite Elements and the SFIP Function in the Classes in Geometry

The purpose of this section is to demonstrate, through selected examples from our course on Synthetic Geometry for universities and secondary schools, benefits of the introduction of new elements into DGS. The fact that affinity is collineation which preserves the infinite line is essentially used in the sequel.

We used our software in our classes on Synthetic Geometry at Plovdiv University “Paisii Hilendarski” and its affiliated college at Smolyan for students of Mathematics and Informatics Pedagogy, Physics and Mathematics Pedagogy and Mathematics. Over years we have observed that using this software students have increased their interest in the area.

Our experience has confirmed that students with excellent mathematics are confident in using computers, and have a positive attitude to learning mathematics with computers. It is interesting to point out that it is possible to create a positive attitude to learning mathematics in students with

negative attitudes toward mathematics and a low mathematical achievement through learning mathematics with computers.

We have made several demonstrations in secondary schools.

#### 4.1. Applications in the University Classes

We would like to share our experience with the application of the “Swap finite & infinite points” function in classes on Synthetic Geometry.

##### 4.1.1. Applications in Projectivity

**The Fundamental Theorem of Projective Geometry:** If  $A, B, C$  are three distinct objects of the one-dimensional figure  $f$  and  $A', B', C'$  are three distinct objects of the one-dimensional figure  $f'$ , then there exists exactly one projectivity  $\varphi: f \rightarrow f'$  such that  $\varphi(ABC) = A'B'C'$ .

The figures  $f$  and  $f'$  can be a range of points or a pencil of lines. They can be different or coincident. The students have to know how to find the objects  $X' = \varphi(X)$  and  $Y = \varphi^{-1}(Y')$ , where  $X \in f$  and  $Y' \in f'$ .

**Pappus' Theorem:** Let be given two lines  $g$  and  $g'$ . If  $A, B, C \in g$  and  $A', B', C' \in g'$ , then the points  $P = BC' \cap CB'$ ,  $Q = AC' \cap CA'$ ,  $R = AB' \cap BA'$  are collinear.

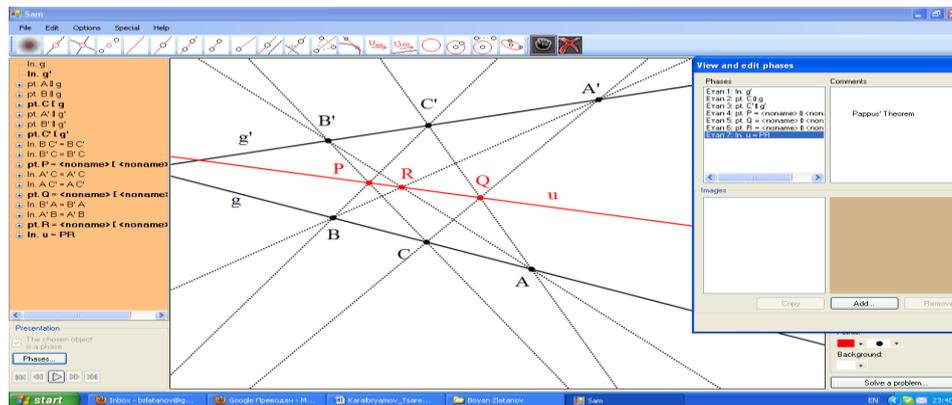


Figure 8 “Pappus’ Theorem”

**Problem 1.** Let  $\varphi$  be a projectivity of the range  $g$  onto the range  $g'$ , so that  $\varphi(A B C) = A' B' C'$ , where  $A, B, C$  are points on the range  $g$  and  $A', B', C'$  are points on the range  $g'$ .

- i) Find the points  $X' = \varphi(X)$  and  $Y = \varphi^{-1}(Y')$ , where  $X \in g$  and  $Y' \in g'$  are arbitrary points.
- ii) Describe the curves of the second class, generated by the projectivity  $\varphi: g \rightarrow g'$ , when  $X$  is variable.

**Solution:** It is well known that Pappus’ Theorem and its dual theorem present the most symmetric constructions for finding the points  $X'$  and  $Y$ . That is why we will present a solution, where the construction will be based on Pappus’ Theorem. The main element in this construction is the axis of projectivity  $u = (AB' \cap A'B)(BC' \cap B'C) = I-2$  (Figure 9.1). Since  $\varphi = \pi_2 \circ \pi_1$ , where  $\pi_1: g \xrightarrow{B'} \wedge u$  and

$\pi_2: u \xrightarrow{B} \wedge g'$  are perspectivities, then  $\varphi(X) = \pi_2 \circ \pi_1(X) = \pi_2(3) = X'$  or  $X' = 3B \cap g'$ , where  $3 = XB' \cap u$ ;

$\varphi^{-1}(Y') = (\pi_1)^{-1} \circ (\pi_2)^{-1}(Y') = (\pi_1)^{-1}(4) = Y$  or  $Y = 4B' \cap g$ , where  $4 = BY' \cap u$ . We have found also the points  $U' = \varphi(U_\infty) = 6A \cap g'$ , where  $6 = U_\infty A' \cap u$  and  $V = \varphi^{-1}(V'_\infty) = 7C' \cap g$ , where  $7 = V'_\infty C \cap u$ . The constructions can be traced by the presentation of Figure 9.1. Moving the point  $X$  along the

line  $g$  (the point  $Y'$  along the line  $g'$ ) the students observe finding the image  $X'$  (the preimage  $Y$ ) for an arbitrary point. □

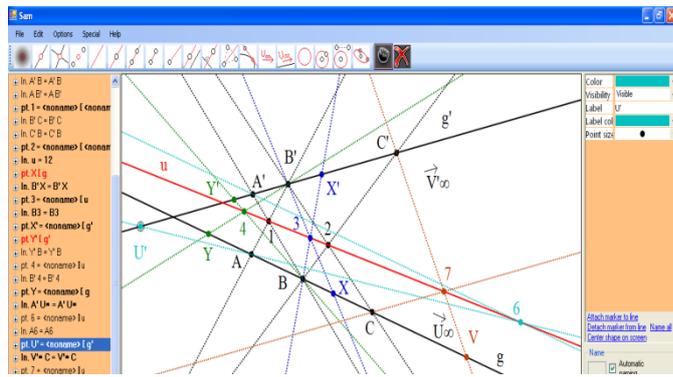


Figure 9.1

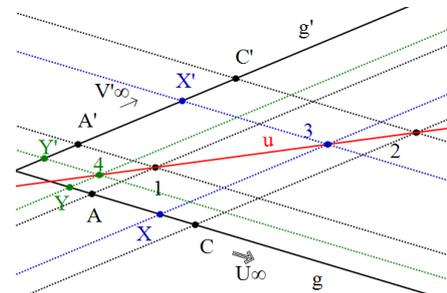


Figure 9.2

By swapping of the finite point  $B$  with the infinite point  $U_\infty$  on the line  $g$  and the finite point  $B'$  with the infinite point  $V'_\infty$  on the line  $g'$  we obtain the following satellites of Problem 1 with their solutions:

**Problem 1\*** Let  $\varphi$  be a projectivity of the range  $g$  onto the range  $g'$ , so that  $\varphi$  is defined in one of the following ways:

- a)  $\varphi(A U_\infty C) = A' B' C'$ ;
- b)  $\varphi(A B C) = A' V'_\infty C'$ ;
- c)  $\varphi(A U_\infty C) = A' V'_\infty C'$  (Figure 9.2).

Find the points  $X' = \varphi(X)$  and  $Y = \varphi^{-1}(Y')$ , where  $X \in g$  and  $Y' \in g'$  are arbitrary points.

Describe the curves of the second class, generated by the projectivity  $\varphi: g \rightarrow g'$ , when  $X$  is variable.

The teacher (using a projector) and the students (everyone uses a computer) solve Problem 1 together. At the end the students present the solution applying dynamics to the free points  $X$  or  $Y'$  and comment on the strategy of the choice of free objects. They produce satellite tasks with the SFIP function and they just follow the presentation and explain the constructions that were done. They move some of the free objects  $g, g', A, B, C, A', B', C'$  and obtain a different view of the sketches, but the construction is the same.

Before introducing the SFIP function it was necessary to draw each of the satellite tasks, starting from a blank file.

During the debates on the talk “Interactive training on Synthetic Geometry in Dynamic Environment” [24], where we presented Problem 2, Prof. P. Rusev suggested a visualization of the curves of the second class, generated by the projectivity  $\varphi: g \rightarrow g'$ , to be made. We accepted and developed this idea because it is believed that the visualization of non-degenerate and degenerate curves of the second class is missing from the classroom. Their visualization would help in the understanding of the concept a curve of the second class in terms of Steiner’s definition and would increase the interest towards it.

In this way the second part is related to the application of the projectivities in the theory of conic sections. In order to describe the curves of the second class it is enough to sketch the connecting lines of the pairs of corresponding points for the projectivity  $\varphi$ . According to Steiner’s definition these lines exactly describe the second class curve  $K(g, g'; \varphi)$  when the point  $X$  describes the range  $g$  (Figure 10.1). The projectivity  $\varphi$  is a similarity in the case c) and the second class curve  $K$  is a set of tangents to a parabola (Figure 10.2). Naturally, the next interesting thing to teach students is the process for generating degenerated curves of the second class. We illustrate cases where the projectivity  $\varphi: g \rightarrow g'$  from the Figures 10.1 and 10.2 is transformed into a perspectivity in Figures

10.3 and 10.4, respectively. It is enough to choose  $A=A' = g \cap g' = O$ . The “Swap finite & infinite points” function of DGS Sam offers the possibility for fast and complete investigation and visualization of the curves of the second class.

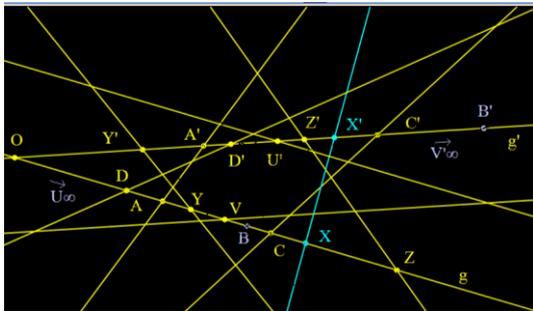


Figure 10.1

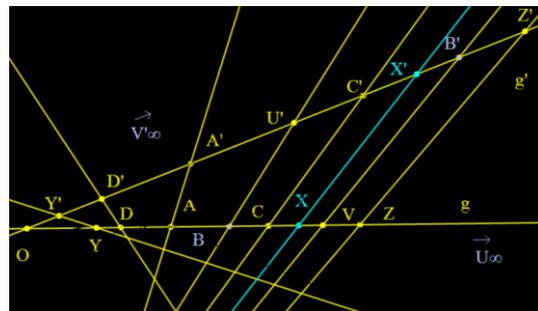


Figure 10.2

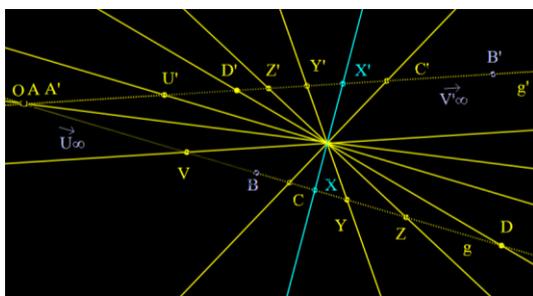


Figure 10.3

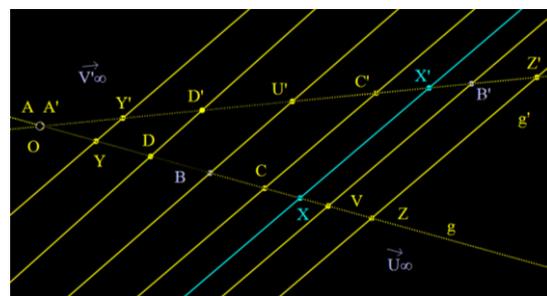


Figure 10.4

It is possible to hide parts of the construction with an appropriate choice of colors of lines and background so that only a curve of the second class remains on screen. We suggest some of these solutions (Figures 10.1, 10.2, 10.3 and 10.4).

The second part of Problem 1 considers studying Conic sections.

**Problem 2.** Let  $\varphi$  be a projectivity of the pencil  $O$  onto the pencil  $O'$ , such that  $\varphi(a b c) = (a' b' c')$ , where the lines  $a, b, c$  are from pencil  $O$  and the lines  $a', b', c'$  are from pencil  $O'$ .

i) Find the lines  $x' = \varphi(x)$  and  $y = \varphi^{-1}(y')$ , where  $x \in O$  and  $y' \in O'$  are arbitrary lines.

ii) Describe the curves of the second power, generated by the projectivity  $\varphi: O \rightarrow O'$ , when  $x$  is variable.

The teacher conducts a discussion of a solution to Problem 2 and he writes on the blackboard the theoretical background to the constructions.

**Solution:** The pair of pencils is intersected with two different ranges  $g$  and  $g'$  (Figure 11.1). Then the projectivity  $\varphi: O \rightarrow O'$  can be presented in the following way:  $\varphi = (\pi_2)^{-1} \circ \psi \circ \pi_1$ , where

$\pi_1: O \xrightarrow{\wedge} g$  and  $\pi_2: O' \xrightarrow{\wedge} g'$  are perspectivities,  $\psi: g \rightarrow g'$  is the projectivity, generated by  $\varphi$ , such

that  $\psi(A B C) = A' B' C'$ . Let  $u = l - 2$  be the axis of  $\psi$ . Hence  $\psi = \pi_{40} \pi_3$ , where  $\pi_3: g \xrightarrow{\wedge} u$  and

$\pi_4: u \xrightarrow{\wedge} g'$  are perspectivities. Then  $\varphi(x) = (\pi_2)^{-1} \circ \psi \circ \pi_1(x) = (\pi_2)^{-1} \circ \pi_{40} \pi_3 \circ \pi_1(x) = (\pi_2)^{-1} \circ \pi_{40} \pi_3(X)$

$= (\pi_2)^{-1} \circ \pi_4(\pi_3(X)) = (\pi_2)^{-1}(X') = x'$  and  $\varphi^{-1}(y') = y$ . □

Since Problem 1 is a component of this solution, students work alone, but the teacher presents on-screen the main fragments of the solution, which he has prepared in advance.

Rotating the line  $x$  around  $O$  or the line  $y'$  around  $O'$ , the students observe finding the image  $x'$  or the preimage  $y$ , respectively of an arbitrary line.  $\square$

Moving some of the free objects  $O, O', a, b, c, a', b', c', g, g'$  students change the viewpoint of the sketch, but the constructions are preserved. This moment is a test of a correct solution.

Taking into account the fact that the centers  $O, O'$  can be finite or infinite, different or coincident the teacher explains to students that they can produce solutions to the next five cases from Figure 11.1. alone. One case is shown in Figure 11.2. Here, the points  $O$  and  $O_\infty$  are swapped, the points  $O'$  and  $O'_\infty$  are swapped, and the point  $O'_\infty$  is rotated until it coincides with the point  $O_\infty$ .

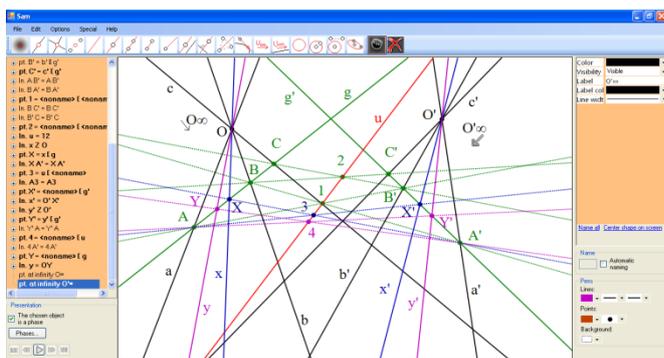


Figure 11.1

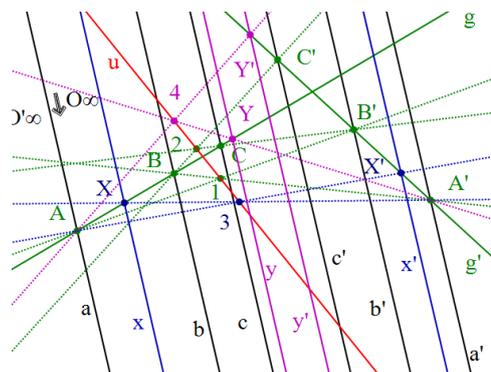


Figure 11.2

The remaining cases (Figures 11.3-11.6) may be accessed through <http://fmi-plovdiv.org/GetResource?id=1186>). Thus the preparation and presentation of a large group of problems could be made in a very short time.

Please, if it is necessary after swapping, refer to Remark 1.1.

We can describe, similarly to Problem 1, the curves of the second power that are generated by the projectivity  $\varphi: O \rightarrow O'$  for all of the six cases. It is enough to sketch the intersection points of the couples of corresponding lines for the projectivity  $\varphi$ .

Of course, students might prefer to use Pappus' Dual Theorem. Then the lines  $O_\infty U = \varphi^{-1}(\omega)$  and  $O'_\infty U = \varphi(\omega)$ , where  $U$  is the center of projectivity  $\varphi$ , will be exactly the asymptotes of the hyperbola  $k(O_\infty, O'_\infty; \varphi)$ .

Another groups of problems relating to this theme are connected with cases when  $f$  is a range of points and  $f'$  is a pencil of lines or when  $f=f'$ . In the last case the projectivity  $\varphi$  can hold invariant elements or  $\varphi$  can be an involution. The teacher can present all varieties of the problems on this theme in a short time with the help of the "Swap finite & infinite points" function or groups of students can prepare them.

The time, which has been saved by the special SFIP function can be used, to show applications of Pappus' Theorem to Euclidean geometry. Some of these applications are Problems 9, 11, 12 and their satellites. These applications are useful for students when they become teachers at school. The teacher can offer problems for home-work to students. Some students will prefer to work alone, others to work in groups.

#### 4.1.2. An Etude on Well Known Pappus' Proposition

For the sake of completeness let us recall Pappus' Proposition:

**Proposition (Pappus):** *If the three sides of a variable triangle are passing through three fixed, collinear points and two of its vertices are moving along two fixed lines, then the third vertex is moving along a line, which is passing through the intersection point of the first two lines.*

This is Pappus' porism, which was the inspiration of much of Maclaurin's work on loci, beginning in 1772 [5, p 40].

We will start with the following problem:

**Problem 3:** Let  $P, Q, R$  be non-collinear points, and let  $a, b, c$  be non-concurrent lines that do not pass through the points  $P, Q, R$ . Construct a triangle with vertices that lie on the lines  $a, b, c$  and sides that pass through the points  $P, Q, R$ .

**Solution:** Let us denote the triangle that will be constructed by  $ABC$  (Figure 12.1). Without loss of generality we can assume that  $A \in a, B \in b, C \in c, P \in BC, Q \in AB, R \in AC$ . The side  $AC$  is an invariant line for the projectivity  $\varphi = \pi_3 \circ \pi_2 \circ \pi_1: R \rightarrow R$ , where  $\pi_1: R \xrightarrow{a} Q, \pi_2: Q \xrightarrow{b} P, \pi_3: P \xrightarrow{c} R$ . Therefore

we need to produce three couples of corresponding lines for  $\varphi$ , i.e.  $\varphi(x, y, z) = x', y', z'$  and to solve the problem of finding the invariant lines of  $\varphi$ . According to Steiner's method we use an arbitrary circle  $k$  that passes through the center  $R$  of the pencil of lines.  $\square$

The reader can trace the solution in detail with the help of the presentation panel. He can apply dynamics to the free points  $P, Q, R$  and to the free lines  $a, b, c$  to change the conditions of the problem, or to the lines  $x, y, z$  (by rotation) to replace the couple of corresponding lines of the projectivity  $\varphi$ .

The "Swap finite & infinite points" function allows us to state and to present solutions to two more problems, where the points  $P, Q$  are replaced with infinite points  $P_\infty, Q_\infty$ . We present in Figure 12.2 a solution when the points  $P$  and  $Q$  were swapped with the infinite points  $P_\infty$  and  $Q_\infty$ , respectively.

An attempt to swap the third point  $R$  with  $R_\infty$  will turn out to be unsuccessful. A student will face a difficult research problem, for two reasons:

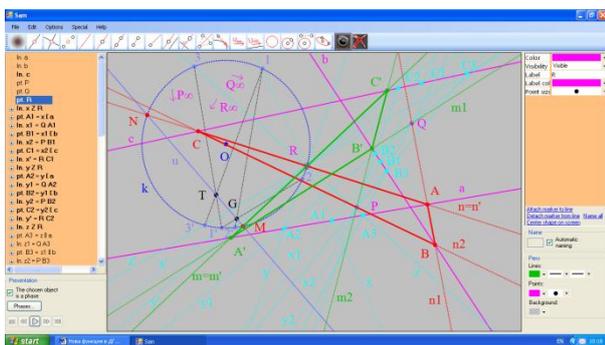


Figure 12.1

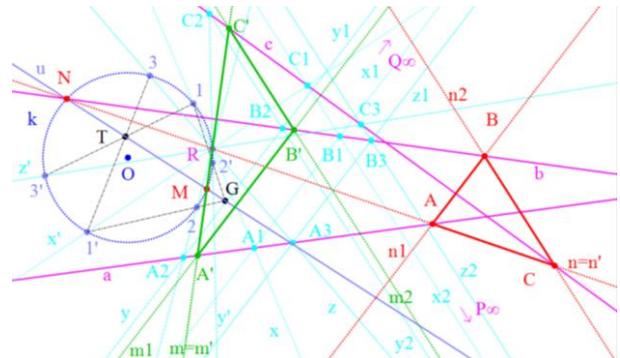


Figure 12.2

1) The point  $R$  is not free any more, because it lies on the circle  $k$ . This is an occasion for a useful discussion; is there another solution of the task, such that the three points  $P, Q, R$  are equivalent? Indeed there is such a construction. In our opinion finding a new solution and describing all of the variants is a good course project.

2) A student has to realize that any three infinite points are collinear. Thus essential changes should be made to the condition of Problem 3. A student can establish by experiment that the line  $c$  cannot be an arbitrary line when the points  $P, Q, R$  are collinear. Proof of this observation is not difficult to see. The projectivity  $\psi = \pi_2 \circ \pi_1: R \rightarrow P$  is a perspectivity, because for the common line  $s$  of both pencils holds  $\psi(s = RP) = s$ . Thus the third vertex  $C$  of the triangle will move along a line that passes through the intersection point of the lines  $a, b$ . By this investigation a student will reach the next problem.

**Problem 4:** Let the three sides of a triangle  $ABC$  pass through three fixed, collinear points  $P, Q, R$  and let two of its vertices move along two fixed lines  $a, b$  ( $Q \in AB, R \in AC, P \in BC, A \in a, B \in b$ ).

Prove that the third vertex  $C$  of the triangle moves along a line  $c$ , which passes through the intersection point of the lines  $a, b$ . Find the line  $c$ .

It is easy to see that this is Pappus' proposition.

**Solution:** (Figure 13.1). Let  $x, y$  be arbitrary lines, belonging to the pencil  $R$  and let  $\psi(x) = x', \psi(y) = y'$ . Then the points  $C_1 = x \cap x'$  and  $C_2 = y \cap y'$  (or  $C_1$  and  $O = a \cap b$ ) define the line  $c$ . Since  $x, y$  are free lines, they can be moved by rotation. Now let  $Z$  be an arbitrary point and let the line  $z$  pass through  $Z$ . If  $\psi(z) = z'$  let us denote the point  $C = z \cap z'$ . Now, after rotation of the line  $z$  with the help of the free point  $Z$ , a student can see how the point  $C$  describes the line  $c = C_1 C_2$ .  $\square$

We included two different ways for dynamics of the free lines  $x$  ( $y$ ) and  $z$  in this problem in order to develop students' skills in the construction of dynamic drawings.

The special function SFIP of DGS Sam (applied to  $P$  and the infinite point  $P_\infty$  of the line  $g$ ) allows us to generate a statement of the next problem and a solution to it.

**Problem 4\*:** Let the sides of the triangle  $ABC$  pass through two fixed points  $Q$  and  $R$  and let its third side be parallel to the line  $g = QR$ . Let two of the vertices of the triangle move along two fixed lines  $a, b$  ( $Q \in AB, R \in AC, QR \parallel BC, A \in a, B \in b$ ). Find the geometric place of the points, that the vertex  $C$  describes (Figure 13.2).

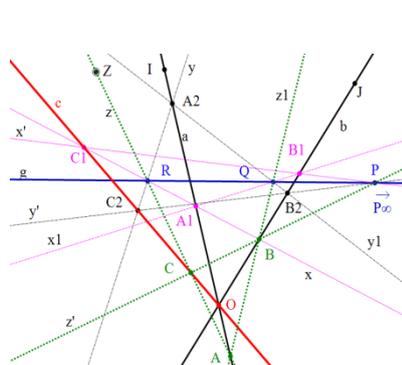


Figure. 13.1

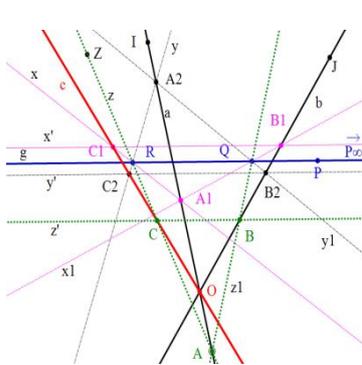


Figure 13.2

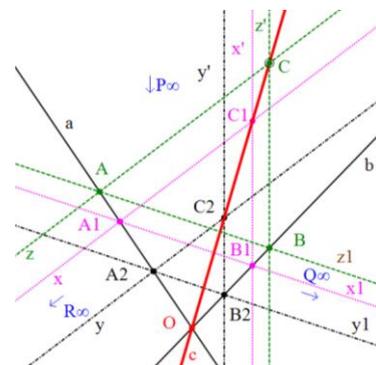


Figure. 13.3

The SFIP function is not “heal-all”. There is not a possibility to swap consequently the three collinear points  $P, Q, R$  with infinite ones, because we will face an impossible case  $P_\infty, Q_\infty, R$  to be collinear. That is why the sketch of the next problem could not be generated by Sam.

**Problem 4\*\*:** Let the sides of the triangle  $ABC$  pass through three fixed infinite points  $P_\infty, Q_\infty, R_\infty$ . Let two of the vertices of the triangle move along two fixed lines  $a, b$  ( $P_\infty \in BC, Q_\infty \in AB, R_\infty \in AC, A \in a, B \in b$ ). Find the geometric place of the points, that the vertex  $C$  describes (Figure 13.3).

After translation of the line  $z$ , with the help of the left button of the mouse, student can observe how the point  $C$  describes the line  $c = C_1 C_2$ .

Note that the order of consideration of the main Problems 3 and 4 can be reversed. The tendency in recent years to reduce the number of class hours (which we do not support) prevents the opportunity to study the issue of finding the invariant points of any projectivity. This fact puts Problem 3 out of the classroom. That is why the etude, suggested in 4.1.2 is suitable as a core of a bachelor thesis.

#### 4.1.3. Application in Homology

Some powerful DGS such as Geogebra offer just the images of an arbitrary figure for the classical transformation (axis symmetry, homothety, central symmetry, translation, rotation, inversion, etc.). The two-dimensional projectivities (collineation and correlation) are included in the Curriculum Course of Synthetic Geometry. We share the view that students should know and remember the basic constructions that are needed for basic transformations.

The “Swap finite & infinite points” function allows a new approach to the teaching of the transformations: axis symmetry, homothety, central symmetry and translation. All these transformations are affine homologies. The first three (axis symmetry, homothety, and central symmetry) are general affine homologies (their center and axis are not incidental) and the fourth (translation) is a special affine homology (the axis is the infinite line  $\omega$  and its center lies on its axis). This approach to the problem helps students understand the common features and the differences between these transformations. This style, applied by using the SFIP function from DGS Sam, will give students (the future teachers) a deeper knowledge of this theme and will save much time for drawing.

We will illustrate this approach with two examples.

**Problem 5:** Let  $\Phi (O, o; A \rightarrow A')$  be a homology with center  $O$ , axis  $o$  and a couple of corresponding points  $A, A'$ . Find the image of the parallelogram  $ABCD$ .

**Solution:** By using the properties of the homology (the connecting line of any two corresponding points of  $\Phi$  passes through the center of  $\Phi$ ; the intersection points of any two corresponding lines of  $\Phi$  lies on the axis of  $\Phi$ ) the students find the quadrilateral  $A'B'C'D'$  (Figure 14.1).  $\square$

Students can apply dynamics to the free points  $O, I, J, A, A'$  to change the homology, or to the free points  $A, U_\infty, V_\infty, B, D, U, V$  to change the shape of the universal parallelogram  $ABCD$  and its connected figures.

**Problem 5\*:** Let  $\Phi (o; A \rightarrow A')$  be an affine homology with center  $O_\infty = U_{AA'}$ , axis  $o$  and a couple of corresponding points  $A, A'$ . Find the image of the parallelogram  $ABCD$ .

**Solution:** It is enough only to swap the point  $O$  with an arbitrary infinite point  $O_\infty$ . Thus we convert the homology  $\Phi$  into an affine homology and then automatically the quadrilateral  $A'B'C'D'$  transforms into a parallelogram (Figure 14.2).  $\square$

This is a fine presentation of the basic property of the affine transformations to preserve the parallel lines.

If it turns out that the image  $A'B'C'D'$  is off-screen, refer to Remark 2 and apply the instructions to the free point  $A'$ .

**Problem 5\*\*:** Let  $\Phi (o)$  be axis symmetry with axis  $o$ . Find the image of the trapezium  $ABCD$ .

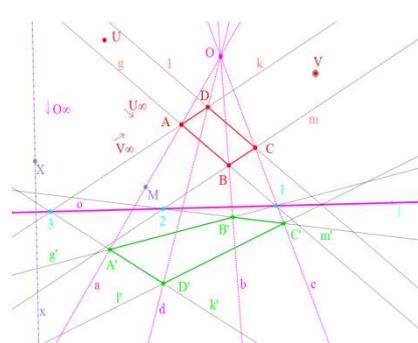


Figure 14.1

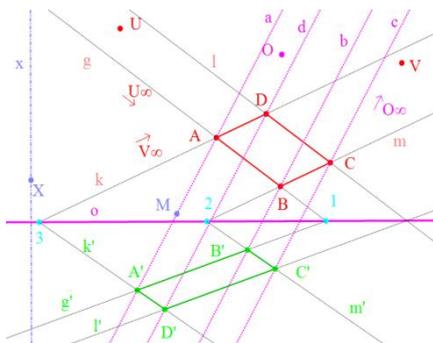


Figure 14.2

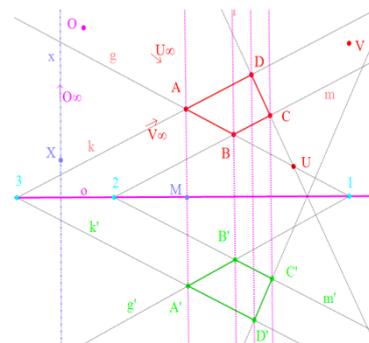


Figure 14.3

**Solution:** By swapping the points  $U_\infty$  with  $U$  the students obtain the trapezium  $ABCD$  on the computer screen straight away. This is enough for students to know the definition of axis symmetry and to use the basic construction from the Figure 14.1 or the Figure 14.2.

Let  $x$  be a line orthogonal to the axis  $o$  and let  $M$  be the middle of the segment  $AA'$  (the point  $M$  is obtained with the “scaled point” instrument). Let us choose  $O_\infty$  to be incidental with the line  $x$ . We translate the point  $A'$  along the line  $a=OA$  until  $M$  lies on the axis  $o$  (Figure 14.3). In this way the general affine homology  $\Phi$  is changed into axis symmetry.  $\square$

Again there is no need to draw anything more. Students can experiment in this example by moving the free points  $B, D, U$  or  $V_\infty$  that the trapeziums  $ABCD$  and  $A'B'C'D'$  are congruent. They can exchange the trapezium  $ABCD$  with some of its connected figures, using the SFIP function and to see that again the figures are congruent.

**Problem 6:** Let  $ABCD$  be a parallelogram and let  $\Phi(O; A \rightarrow A')$  be a homothety with a center  $O$  and a couple of corresponding points  $A, A'$ . Find the image of  $ABCD$ .

**Solution:** The homothety is an affine homology with an axis the infinite line  $\omega$  and thus the corresponding lines will be parallel to each other (Figure 14.4). By using this property it is easy to construct the image  $A'B'C'D' = \Phi(ABCD)$ .  $\square$

The students can apply dynamics to the free points  $O, A, A'$  to change the homothety, or to the free points  $A, U_\infty, V_\infty, B, D, U, V$  to change the shape of the universal parallelogram  $ABCD$  and its connected figures.

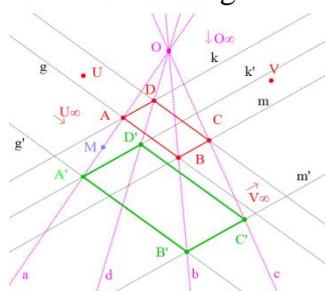


Figure 14.4

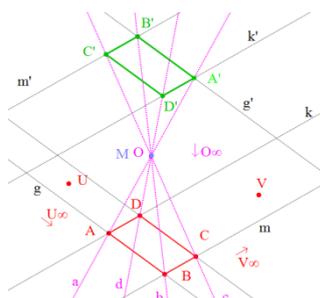


Figure. 14.5

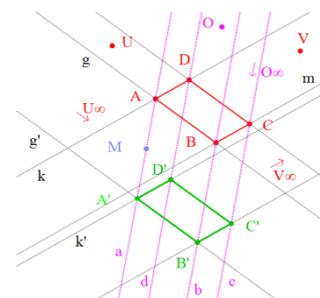


Figure 14.6

**Problem 6\*:** Let  $ABCD$  be a parallelogram and let  $\Phi(O)$  be a central symmetry with a center  $O$ . Find the image of  $ABCD$ .

**Solution:** Again it is enough for students to know the definition of central symmetry and to use the basic construction from the Figure 14.4. If they choose the point  $A'$  so that the point  $M$  (the middle of the segment  $AA'$ ) coincides with the center  $O$  of  $\Phi$ , then the homothety  $\Phi$  transforms into a central symmetry and the corresponding parallelograms  $ABCD$  and  $A'B'C'D'$  will be congruent (Figure 14.5).  $\square$

**Problem 6\*\*:** Let  $ABCD$  be a parallelogram and let  $\Phi$  be an arbitrary translation. Find the image of  $ABCD$ .

**Solution:** By swapping the center of the homothety  $O$  with an arbitrary infinite point  $O_\infty$  the homothety transforms into a translation and we have a sketch of the image of the figure without needing to draw it again (Figure 14.6).  $\square$

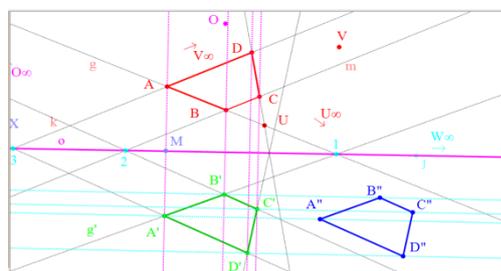


Figure 14.7

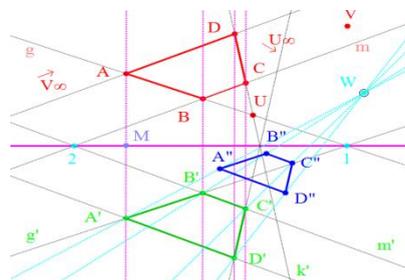


Figure 14.8

The examples showed that a user can construct a solution in the general case and DGS Sam with its special SFIP function presents a solution in specific cases. Of course, a parallelogram is only one of the possible examples.

This approach not only saves time, but it also unifies different transformations and explains the reasons for their similarities and differences. This way of presenting transformations stimulates students to analyze them. Our approach and the fast visualization by the SFIP function are conditions which encourage students to set and solve new problems. For example Figure 14.7 represents translated axis symmetry. After swapping points  $W_\infty$  with  $W$ , which means a substitution of the translation with a homothety, students obtain Figure 14.8, and they can investigate this transformation. Another example is connected with rotation. As is well known, rotation can be considered a composition of two axis symmetries (Figure 14.9). What will happen when one or both axis symmetries will be substituted by other affine homologies (Figure 14.10)? A natural question arises to investigate relations between the compositions which consist of a couple of affine transformations.

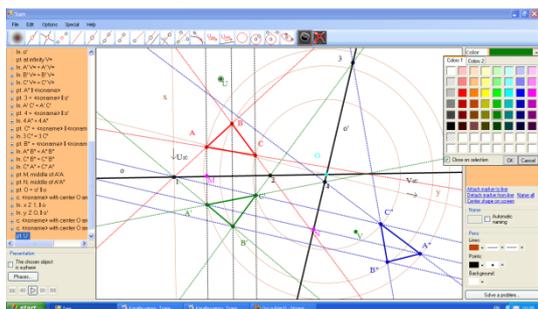


Figure 14.9

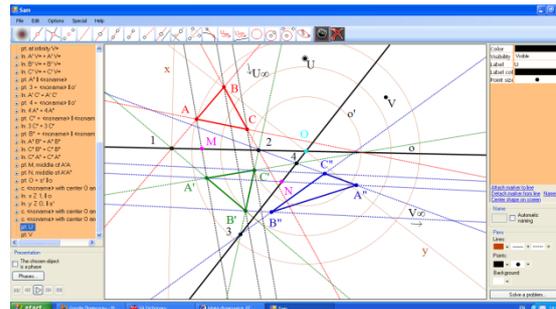


Figure 14.10

We think that if students do not study homology, then discovering of its properties as a result of investigation of relations between images and pre-images (for the homothety, central and axis symmetries, and translation) is an interesting moment in training connected with transformations. It enriches the knowledge of future teachers in basic transformations, which he (she) will teach at school.

#### 4.1.4. Application to Conic Sections

The teaching of conic sections in schools [23] and the investigation of problems that are connected with conic sections in journals that deal with school mathematics [8, 15] has posed a problem about modernizing the way that they are taught in school and university.

The teaching of central curves of the second power can be optimized with the “Swap finite & infinite points” function, too.

Consider the problem:

**Problem 7:** A curve of the second power  $\kappa$  passes through the points  $A, B, C, D$  and line  $a$  is tangential to the curve at point  $A$ . Construct the lines  $c$  and  $d$ , such that they are tangential to the curve  $k$  at points  $C$  and  $D$ , respectively. Choose an arbitrary point  $X$  from  $k$ . Prove that the points  $O=c \cap d, I=BC \cap DX, 2=CX \cap BD$  are collinear.

**Solution:** (Figure 15.1) The tangential lines  $c$  and  $d$  through the points  $C$  and  $D$ , respectively are constructed by applying Pascal’s Theorem to the points  $AABCCD$  and  $AABCDD$ , respectively. The second intersection point  $X$  of an arbitrary line  $x$  through  $B$  describes the curve, when  $x$  spins around the point  $B$ . The collinearity of the points  $O, I, 2$  follows immediately from Pascal’s Theorem, applied to the points  $CCBDDX$ .  $\square$

By swapping the point  $C$  with  $C_\infty$  and  $D$  with  $D_\infty$  we transform the curve  $k$  into a hyperbola with asymptotes  $c, d$  and a center  $O=c \cap d$  (Figure 15.2). It is interesting for students to see how after rotation of line  $x$  around  $B$ , point  $X$  describes the two branches of the hyperbola and how it

passes through the infinite points. We have added only the axes  $o_1$  and  $o_2$  of the hyperbola. It is well known that axes  $o_1$  and  $o_2$  are bisectors of angles between the asymptotes.

Students will observe that the collinearity of the points  $O, I, 2$  is preserved. Repeating swapping points  $C_\infty$  with  $C$  and  $D_\infty$  with  $D$ , students will return to the original drawing and the natural question is: Whether the lines  $o_1$  and  $o_2$  are special ones for the curve?

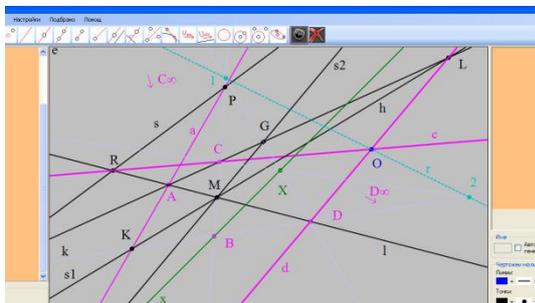


Figure 15.1

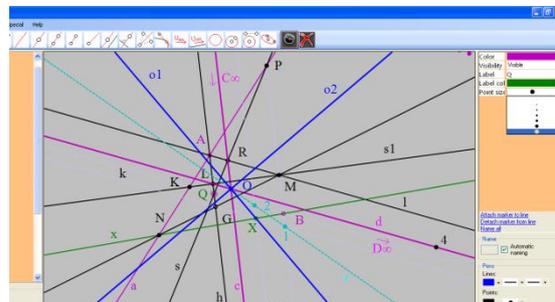


Figure 15.2

They are only a couple of corresponding lines of the involution of the conjugated lines induced in the point  $O$  from the curve  $k$ . The next question arises: Could  $o_1$  be a diameter? If the line  $o_1$  passes through the middle of the segment  $CD$  then it will be a diameter. This is possible to see with dynamics, applied to some of the free objects  $A, B, C, D$  or  $a$  (Figure 15.3 <http://fmi-plovdiv.org/GetResource?id=1186>). Thus the SFIP function of DGS Sam should stimulate inquisitive students to deepen their knowledge of the subject.

#### 4.1.5. Application in Perspective Triangles

**Desargues' Theorem for perspective triangles:** *The connecting lines of the couples of corresponding vertices of two triangles  $ABC$  and  $A'B'C'$  intersect at a point  $S$  if and only if the intersection points of the couples of corresponding sides  $P=BC \cap B'C'$ ,  $Q=AC \cap A'C'$ ,  $R=AB \cap A'B'$  lie on a line  $s$  (Figures 16.1, 16.2).*

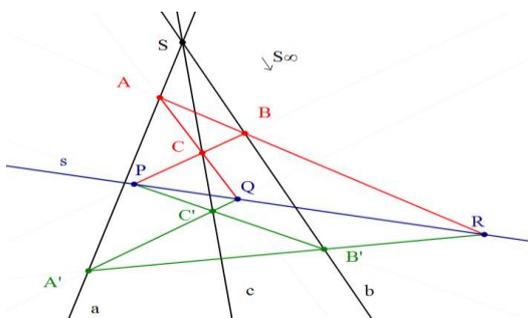


Figure 16.1

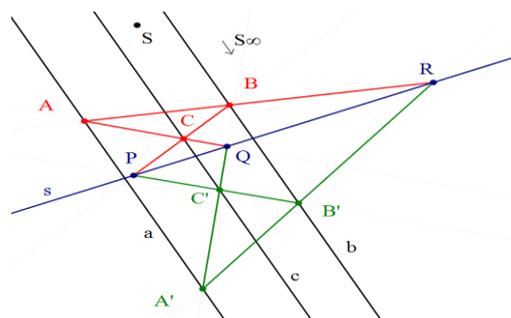


Figure 16.2

**Definition 5:** *Two triangles that satisfy the conditions of Desargues' Theorem for perspective triangles are called **perspective**. The point  $S$  is called a **perspective center** and the line  $s$  is called a **perspective axis**.*

The perspective center and axis can be either finite or infinite objects.

Let us mention that Desargues' Theorem about perspective triangles does not restrict us to choose the vertices of the two perspective triangles to lie on two lines. Thus we obtain another well known problem, which has interesting applications not only in university courses on Geometry, but also in school classes [12, 13].

**Problem 8:** Let  $g$  and  $g'$  be two lines. Denote  $T=g \cap g'$  and let us choose points  $A, B, C \in g$  and  $A', B', C' \in g'$ . Prove that if the lines  $AA', BB', CC'$  intersect at a point  $S$ , then the points  $P=AB' \cap A'B, Q=BC' \cap B'C, T$  lie on a line  $s$ .

**Solution:** The point  $S$  is a perspective center of the pair of triangles  $ABC$  and  $A'B'C'$  and according to Desargues' Theorem the intersection points  $P, Q, T$  of the couples of corresponding sides are collinear.

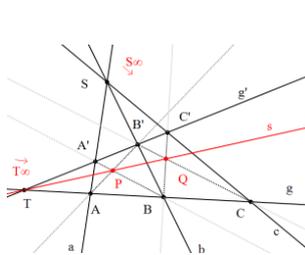


Figure 17.1

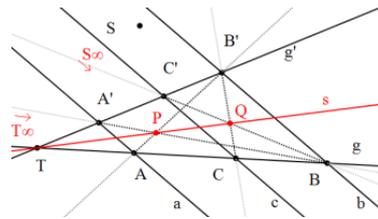


Figure 17.2

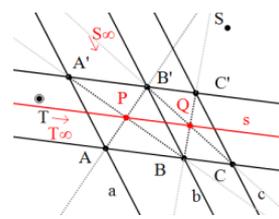


Figure 17.3

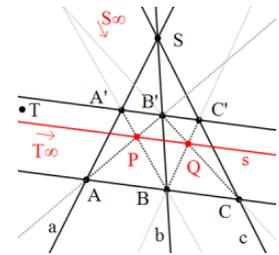


Figure 17.4

The statement of Problem 8 holds true in four different cases (Figure 17). These cases are sketched by swapping the finite points  $S$  and  $T$  with the infinite points  $S_\infty$  and  $T_\infty$ , respectively. Please, if it is necessary after swapping, refer to Remark 1.1.

#### 4.2. Application in School

From our experience of teaching Synthetic Geometry we have reached the idea that the time, when the fundamental theorems of Projective Geometry (namely the theorems of Pappus, Desargues, Pascal) are to be included in the curricula of the secondary schools, has come. This idea is not a new one. It dates from a century ago and was proposed by Lehman (1917). His idea was more radical. DGS Sam facilitates this in secondary schools. This will be a small step away from school Euclidean Geometry.

Let us mention Whitehead's words delivered in his presidential address to the Mathematical Association of England in 1916: "Let the main ideas which are introduced into a child's education be few and important, and let them be thrown into every combination possible. The child should make them his own, and should understand their application here and now in the circumstances of his actual life." (The Aims of Education)

In our paper "Vertical integration of the training in the university and the secondary school by projective methods in dynamical environment", published in the Bulgarian journal "Mathematics Plus" [12] we have shown that large groups of school geometry problems can be solved by these theorems rather quickly and clearly. We illustrate this with a series of examples on how finding the common projective root of a large group of tasks will help school students to obtain better knowledge, to investigate this and to create this with the new SFIP function of DGS Sam.

We performed an experiment with several seven-year old children, with basic computer literacy. Our experience with children of this age, is very small, episodic and not in a classroom setting. We presented them with the tools of Sam and the picture called "DGS Sam", Figure 2.1. They began to draw. They transformed the hat, the body from a trapezium into a quadrilateral or a parallelogram, and changed the shapes of the head, eyes, hands, and feet. Then they started to draw sketches in a blank file, and used the infinite elements and the SFIP function. It was very interesting that they did not ask why there are finite and infinite points, and they just used them in their drawings. They applied the Swap function and enjoyed the results

An intuitive understanding of infinite elements, discovering of relationships between figures, peeking into the world of the dynamic drawings are good initial conditions for developing an interest in mathematics.

#### 4.2.1. Pappus' Theorem out of the Context of Projectivity

Analogy is one of the methods that can be, with computer support, used for development of ability to generalize [25]. The process of analyzing parts of certain geometric configurations represents a challenge for students, allowing them to observe and ... to reconstruct or discover new theorems or relationships [23].

**Problem 9:** *Through the vertices of the triangle ABC are constructed three couples of lines  $(a, a')$ ,  $(b, b')$  and  $(c, c')$ , such that  $a \parallel b \parallel c$  and  $a' \parallel b' \parallel c'$ . Among the parallelograms that are generated by these lines there are three such that the lines AB, BC, CA are diagonals. Prove that the second diagonals of these three parallelograms intersect at one point.*

**Solution:** Using the notation in Figure 18.1 the parallelograms for which AB, BC, CA are diagonals are AQB<sub>P</sub>, BRCS, ATCK. Let us denote with  $U_\infty$  the intersection point of the parallel lines  $a, b, c$ , and with  $V_\infty$  the intersection point of the parallel lines  $a', b', c'$ .

The shortest solution of this problem is to apply Pappus' Theorem to the triads of collinear points  $P, T, V_\infty$  and  $K, Q, U_\infty$ . Immediately we get that the points  $O=TK \cap PQ$ ,  $S=TU_\infty \cap QV_\infty$ ,  $R=KV_\infty \cap PU_\infty$  are collinear, i.e. the lines  $TK, PQ, SR$  are concurrent.  $\square$

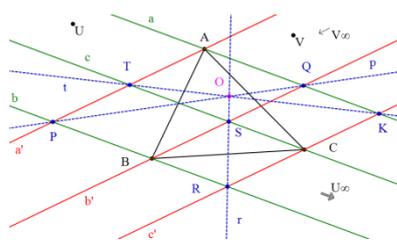


Figure 18.1

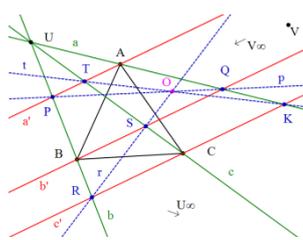


Figure 18.2

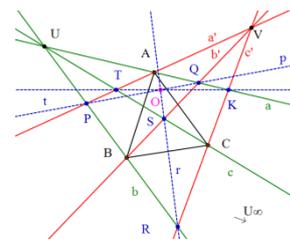


Figure 18.3

The “Swap finite & infinite points” function provokes us not only to formulate two new problems by swapping  $U_\infty$  with  $U$  and  $V_\infty$  with  $V$ , but also suggests to us right away, their sketches.

**Problem 9\*:** *Through the vertices of the triangle ABC are constructed three couples of lines  $(a, a')$ ,  $(b, b')$  and  $(c, c')$ , such that  $a, b, c$  are incident with an arbitrary point  $U$  and  $a' \parallel b' \parallel c'$ . Among the trapeziums that are generated by these lines there are three such that the lines AB, BC, CA are diagonals. Prove that the second diagonals of these three trapeziums intersect at one point (Figure 18.2).*

The proof is literally the same. We have just to apply Pappus' Theorem to the triads of collinear points  $P, T, V_\infty$  and  $K, Q, U$ .

**Problem 9\*\*:** *Through the vertices of the triangle ABC are constructed three couples of lines  $(a, a')$ ,  $(b, b')$  and  $(c, c')$ , such that  $a, b, c$  are incident with an arbitrary point  $U$  and  $a', b', c'$  are incident with an arbitrary point  $V$ . Among the quadrilaterals that are generated by these lines there are three, such that the lines AB, BC, CA are diagonals. Prove that the second diagonals of these three quadrilaterals intersect at one point (Figure 18.3).*

The proof is the same as we have to apply Pappus' Theorem to the triads of collinear points  $P, T, V$  and  $K, Q, U$ .

The school students can apply dynamics to all free points  $(A, B, C, U_\infty, V_\infty, U, V)$ . The problems 8 and 9 and their satellites provide a link between training in universities and secondary schools.

### 4.2.2. Application to Plane Sections of Polyhedra

Teaching the Platonic solids is done with the use of Cabri3D [26]. Non-trivial problems of constructing the common parts of overlapping solids are presented in [18].

The theme of three dimensional geometric shapes is traditionally difficult for pre-university students. We propose an approach that uses the common features of plane sections of prisms and pyramids.

The Dynamic Geometry Software Sam suggests a type of universal solution to the problem of plane sections of polyhedrons for one kind of connected figures (see Definition 2).

**Problem 10:** Construct the section of the universal parallelepiped  $ABCD A' B' C' D'$  with the plane  $\alpha$ , defined by three points  $P \in (ABC)$ ,  $Q \in DD'$  and  $R \in (BCB')$ .

**Solution:** The plane section is presented on Figure 19.1. The truncation  $o = (ABC) \cap (\alpha)$  is found by the method of the projections with the help of the point  $R_1 = RW_\infty \cap (ABC) = RW_\infty \cap BC$ . Since the point  $Z = QR \cap DR_1$  lies on the plane  $(ABC)$ ,  $o = PZ$ . Then we construct all the intersection points of  $o$  with the edges of the parallelepiped that lie in the plane  $(ABC)$ :  $S = o \cap AB$ ,  $T = o \cap BC$ ,  $L = o \cap CD$ ,  $K = o \cap AD$ . We continue with the truncation  $KQ = (ADD_1) \cap (\alpha)$ . Again we construct all the intersection points of the line  $KQ$  with the edges of the parallelepiped that lie in the plane  $(A D D_1)$ :  $KQ \cap AA_1 = A'$ ;  $KQ \cap A_1 D_1 = M$ . Similarly we proceed with the truncation of  $\alpha$  with all the other faces of the universal parallelepiped: Since  $RT = (BCC_1) \cap (\alpha)$ , then  $RT \cap CC_1 = C'$ ,  $RT \cap BB_1 = B'$ ,  $RT \cap B_1 C_1 = N$ ;  $LQ = (CDD_1) \cap (\alpha)$ , then  $LQ \cap D_1 C_1 = J$ ;  $MJ = (A_1 B_1 C_1) \cap (\alpha)$ , then  $MJ \cap A_1 B_1 = F$ .

The user can trace all constructions by presentation of Figure 19.1. There the plain section is  $STC'JMA'$ . The constructed plane section differs when the points  $P$ ,  $Q$  or  $R$  are moving.

Let  $U, V, W$  be three arbitrarily chosen free points. Let us transform the parallelepiped into a truncated pyramid or pyramid by swapping the points  $W$  and  $W_\infty$  (Figures 19.2, 19.3, 19.4). All of the constructions above are preserved. Plane sections are obtained automatically. The user can follow all the constructions “step by step” for any of the cases via the presentation panel, and he may produce other examples.

When the user has applied dynamics to the points  $P, Q, R$  then the plane sections, constructed by Sam, differ. If s/he wishes to change the type of the bases it is enough to transform the universal parallelogram  $ABCD$  into one of its connected figures by swapping the points  $U_\infty$  with  $U$  and  $V_\infty$  with  $V$ . Some cases are presented in Figure 19.2 where  $V_\infty$  is swapped with  $V$ , and in Figure 19.3 where  $U_\infty$  is swapped with  $U$  and  $V_\infty$  is swapped with  $V$ . □

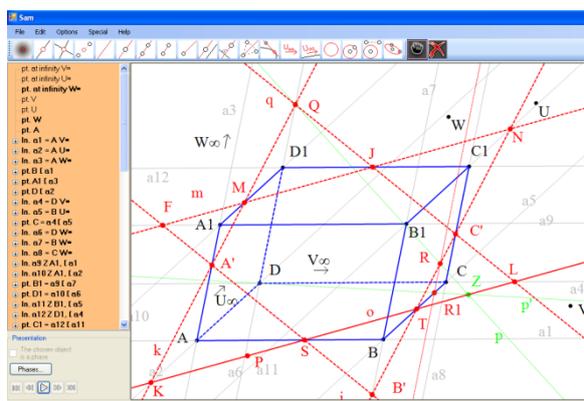


Figure 19.1

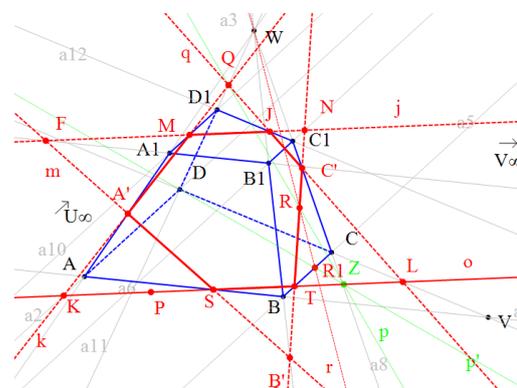


Figure 19.2

The method for finding of the plane section is the same if the points, which define the plane  $\alpha$ , are situated on arbitrary faces or edges of the parallelepiped. For example the user can put the

point  $P \in (ABC)$  on some edge of the face  $ABCD$ , or  $P$  can coincide with some special point of this face, if the conditions of the problem require this.

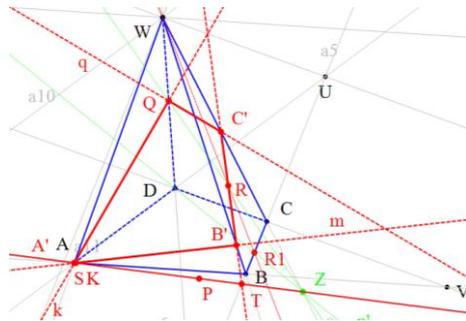


Figure 19.3

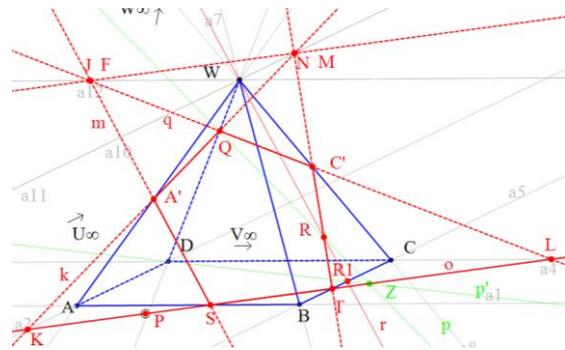


Figure 19.4

This approach and DGS Sam optimize the drawing work that is required for the construction of plane sections, and at the same time one can follow the construction of plane sections in details in each case via the presentation panel. This approach and DGS Sam unify the method of projections used for prisms and pyramids.

The pupils, to whom we have made demonstrations, attended closely and were very interested in this theme that has been traditionally difficult for them.

#### 4.2.3. Generation of Creative Thinking in Geometry Classes

Basic experiments in Sketchpad with circles and a few lines, or triangles inscribed in a circle, lead to “discoveries” of many interesting properties. These properties usually are well known, and their proofs are published in geometry textbooks, but students can discover them on their own, check them and later prove them or search a textbook for the proof. This builds some special interest in elementary geometry [7]. A crucial aspect that emerged in students’ problem solving instruction is that with the use of dynamic software they have the opportunity to engage in a way of thinking that goes beyond reaching a particular solution or response to a particular problem [23]. The “Swap finite & infinite points” function can extend the role of Dynamic Geometry Software, pointed out in [7, 23], by helping students to think creatively, replacing the old principle “learn and repeat” with “learn and create”. We will illustrate this with the following problems. Let us start with this school problem:

**Problem 11:** If  $ABCD$  and  $ARKM$  are parallelograms, where  $R \in AB$ ,  $M \in AD$ , then the points  $D$ ,  $K$  and  $T = AN \cap CR$  are collinear (Figure. 20.1).

**Solution:** First proof The classical solution given in textbooks [4, p.48, problem 83] goes as follows: Denote  $q = MK$ ,  $p = RK$ ,  $L = q \cap CR$ ,  $F = p \cap AN$  and let us consider a homothety  $h$  with a center  $T$  and  $h(L) = C$ . From the condition that  $q$  is parallel to  $CD$  it follows that  $h(q) = CD$ . Applying Thales’ Theorem to the lines  $AN$  and  $RC$ , taking into account the conditions that  $FR$  is parallel to  $CN$ , and  $AR$  is parallel to  $LN$ , we get  $TF/TR = TN/TC$  and  $TN/TL = TA/TR$ . From these equalities it follows that  $TA/TF = TC/TL$ , which ensures that  $h(F) = A$ . Consequently  $h(p) = AD$  and  $h(K = p \cap q) = AD \cap CD = D$ , which means the line  $KD$  passes through the center  $T$  of the homothety  $h$ .  $\square$

Let us note for the reader’s convenience that in Figure 20.1 the arbitrary lines  $p$  and  $q$  can be moved to the points  $M$  and  $R$ , respectively.

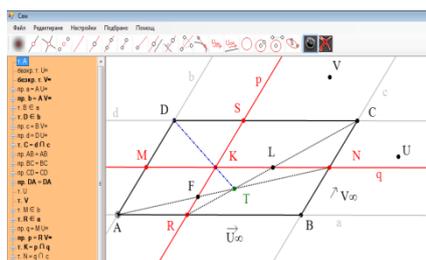


Figure 20.1

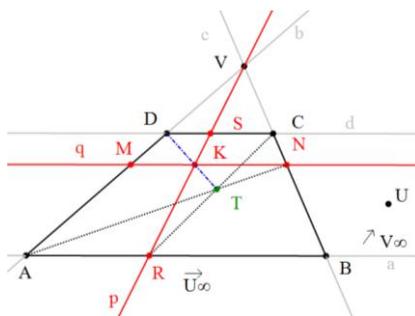


Figure 20.2

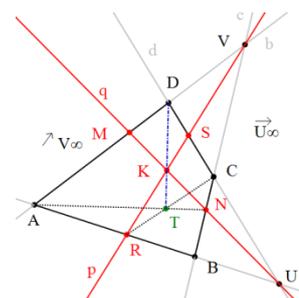


Figure 20.3

Students apply the SFIP function and the sketches show them right away that the collinearity of the point  $D, K, T$  is preserved in cases where  $ABCD$  and  $ARKM$  are trapeziums (Figure 20.2) or  $ABCD$  and  $ARKM$  are quadrilaterals (Figure 20.3). Three questions appear: How should one formulate these new statements? Are the sketches proofs? How can one prove these new statements? They are convincing that the proof given in textbooks for the case of parallelograms is not appropriate for the connected figures. The teacher can direct students to Pappus' Theorem, and the second proof shown.

*Second proof)* Consider the couple of the triads collinear points  $(A, R, U_\infty)$  and  $(C, N, V_\infty)$ , where  $U_\infty = AB \cap CD$  and  $V_\infty = AD \cap BC$ . According to Pappus' Theorem it follows that the points  $T = AN \cap CR, K = RV_\infty \cap NU_\infty, D = AV_\infty \cap CU_\infty$  are collinear.  $\square$

As far as we know the second proof is new. This proof is not only the shortest one, but its projective nature makes it applicable to figures connected with the universal parallelogram.

Really, Figure 20.2 is obtained from Figure 20.1 after applying the SFIP function to infinite point  $V_\infty$  and an arbitrary finite point  $V$ . Now it is enough to apply Pappus' Theorem to the triads of collinear points  $(A, R, U_\infty)$  and  $(C, N, V)$  to prove that the points  $T, K, D$  are collinear. Figure 20.3 is obtained from Figure 20.2 after applying the SFIP function to the infinite point  $U_\infty$  and an arbitrary finite point  $U$ . Now it is enough to apply Pappus' Theorem to the triads of collinear points  $(A, R, U)$  and  $(C, N, V)$  to prove that the points  $T, K, D$  are collinear.

The benefit of this approach with using DGS Sam continues. It allows us to compose a sequence of similar problems. A teacher can start a discussion with the following question: Are there any other triads of collinear points, associated with the connected figures (parallelogram, trapezium and quadrilateral)? Students make their propositions and with the teacher's help the next problem is formulated.

**Problem 12:** Let  $ABCD$  be a parallelogram and  $K$  be a point in its interior. Let us construct the lines  $p$  and  $q$  through the point  $K$ , such that  $p \parallel AD$  and  $q \parallel AB$ . Let us denote:  $R = p \cap AB, S = p \cap CD, M = q \cap AD, N = q \cap BC, T = AN \cap CR, Q = BS \cap DN, G = CM \cap AS, P = DR \cap BM$ . Prove that the following triads of points  $(P, K, C); (T, K, D); (Q, K, A); (G, K, B)$  are collinear.

**Solution:** (Figure 21.1) Let us note for the reader's convenience that dynamics is possible for the free points associated with the universal parallelogram and the point  $K$ .

By applying Pappus' Theorem consecutively to the pairs of collinear points:  $(B, R, U_\infty)$  and  $(D, M, V_\infty); (A, R, U_\infty)$  and  $(C, N, V_\infty); (D, S, U_\infty)$  and  $(B, N, V_\infty); (C, S, U_\infty)$  and  $(A, M, V_\infty)$  we obtain the collinearity of the triads of points  $(P, K, C); (T, K, D)$  (problem 11);  $(Q, K, A); (G, K, B)$ , respectively.  $\square$

It is enough to consider one case in the class and the rest can be set as homework. Students generate the solutions of the two satellite problems for trapeziums and complete quadrilaterals instantly with the SFIP function[12, 13]:

**Problem 12\*:** Let  $ABCD$  be a trapezium such that  $AD \cap BC = V$  and  $K$  be a point in its interior. Let us construct the lines  $p = KV$  and  $q$  through the point  $K$ , such that  $q \parallel AB$ . Let us denote:  $R = p \cap AB,$

$S=p \cap CD, M=q \cap AD, N=q \cap BC, T=AN \cap CR, Q=BS \cap DN, G=CM \cap AS, P=DR \cap BM$ . Prove that the following triads of points  $(P, K, C); (T, K, D); (Q, K, A); (G, K, B)$  are collinear.

**Problem 12\*\*:** Let  $ABCD$  be a complete quadrilateral with diagonal points  $U=AB \cap CD, V=AD \cap BC$  and  $K$  be an arbitrary point. Let us construct the lines  $p=KV, q=KU$  and denote:  $R=p \cap AB, S=p \cap CD, M=q \cap AD, N=q \cap BC, T=AN \cap CR, Q=BS \cap DN, G=CM \cap AS, P=DR \cap BM$ . Prove that the following triads of points  $(P, K, C); (T, K, D); (Q, K, A); (G, K, B)$  are collinear (Figure 21.2).

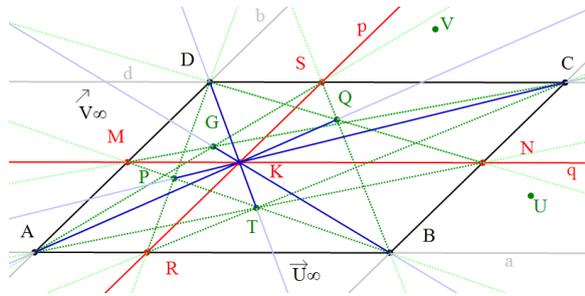


Figure 21.1

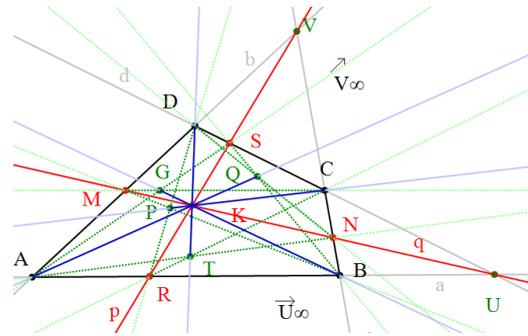


Figure 21.2

Please, if it is necessary after swapping, refer to Remark 1.2 (use the point  $B$  or  $U$ ).

From the fact that the points  $K$  lies on the lines  $DT, AQ, CP, BG, RS$  and  $MN$ , we can see that there are a number of perspective triangles from a point  $K$  and we can propose a number of new problems for proving of collinearity of three points. For instance by the perspectivity of the triangles  $MRA$  and  $NSQ$  it follows that the points  $E=MR \cap NS, D=MA \cap NQ, B=RA \cap SQ$  are collinear ([16] p. 71, Problem 28). Thus the teacher offers another idea for student's creativity. The description of all the possible pairs of perspective triangles, and thereafter of all triads of collinear points is a good combinatorial problem. Investigation of the axis of perspectivity of the pairs of triangles is eased by the dynamic environment of the software. Coxeter's famous problem ([5], p. 80) raises another interesting research problem, which is connected with all these pairs of perspective triangles.

**Coxeter's Problem** Let  $ABC$  and  $A'B'C'$  be two perspective triangles. Then the six intersection points of the non-corresponding sides of the triangles are lying along a curve of the second power.

A difficult research problem is to describe the conic sections that are generated by the perspective triangles in Problems 12, 12\*, 12\*\*. We have not yet received a solution to the combinatorial tasks although they were put to a Ph.D. student at the university, who is preparing a thesis on Combinatory problems in secondary school. Thus the problem is open.

A natural question concerning Problem 12 is whether the condition that the lines  $p$  and  $q$  are parallel to the sides of the parallelogram, or, equivalently that they are passing through the diagonal points  $U$  and  $V$  of the complete quadrilateral is necessary. The answer is given in the next problem.

**Problem 13:** Let  $ABCD$  be an arbitrary parallelogram and  $e_1, e_2$  be an arbitrary lines. Let us denote  $E=e_1 \cap e_2, M=e_1 \cap AD, R=e_1 \cap AB, S=e_2 \cap CD, N=e_2 \cap BC, F=MS \cap NR, RS=p, MN=q; K=MN \cap RS, T=AN \cap CR, Q=BS \cap DN, G=CM \cap AS, P=BM \cap DR$ . Prove that the following statements are equivalent:

- |  |  |
|--|--|
| 1) the points $B, D, E$ are collinear; | 5) the points $B, K, G$ are collinear; |
| 2) the points $A, C, F$ are collinear; | 6) the points $D, K, T$ are collinear; |
| 3) the points $A, K, Q$ are collinear; | 7) the points $T, G, E$ are collinear; |
| 4) the points $C, K, P$ are collinear; | 8) the points $P, Q, F$ are collinear. |

**Solution:** (Figure 22.1) Let us note for the reader's convenience that dynamics of the sketch depends on the free points associated with the universal parallelogram, the point  $E$  and the lines  $e_1, e_2$ .

We will prove the problem in several steps:

i) that 1) is equivalent to 2) and 1) is equivalent to 8).

The condition 1) that the points  $B, D, E$  be collinear can be stated in the following way: “the lines  $MR, NS, BD$  are concurrent”.

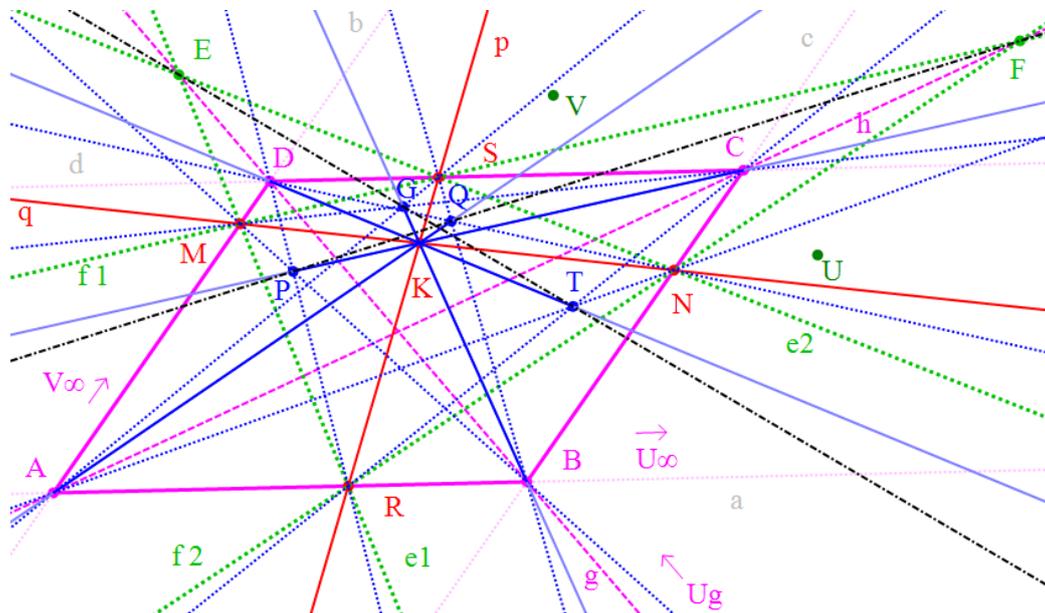


Figure 22.1

According to Desargues' Theorem, applied to  $\Delta MSD$  and  $\Delta RNB$ , the lines  $MR, SN, DB$  are concurrent if and only if the points  $F=MS \cap RN, A=MD \cap RB, C=SD \cap NB$  are collinear.

According to Desargues' Theorem, applied to  $\Delta MSB$  and  $\Delta RND$ , the lines  $MR, SN, DB$  are concurrent if and only if the points  $F=MS \cap RN, P=MB \cap RD, Q=SB \cap ND$  are collinear.

The last statements can be also formulated in the following way:

The points  $B, D, E$  are collinear if and only if the points  $F, A, C$  are collinear;

The points  $B, D, E$  are collinear if and only if the points  $F, P, Q$  are collinear.

ii) We will show that 1) is equivalent to 3) and 1) is equivalent to 4).

Let us consider the two pairs of triangles  $\Delta MRA, \Delta NSQ$  and  $\Delta MRP, \Delta NSC$ .

According to Desargues' Theorem, applied to  $\Delta MRA$  and  $\Delta NSQ$ , the points  $E=MR \cap NS, B=RA \cap SQ, D=MA \cap NQ$  are collinear if and only if the lines  $MN, RS, AQ$  are concurrent.

According to Desargues' Theorem, applied to  $\Delta MRP$  and  $\Delta NSC$ , the points  $E=MR \cap NS, D=RP \cap SC, B=MP \cap NC$  are collinear if and only if the lines  $MN, RS, PC$  are concurrent.

Taking into account that  $MN \cap RS = K$  we can write the last statements in the following way:

The points  $B, D, E$  are collinear if and only if the points  $K, A, Q$  are collinear.

The points  $B, D, E$  are collinear if and only if the points  $K, P, C$  are collinear.

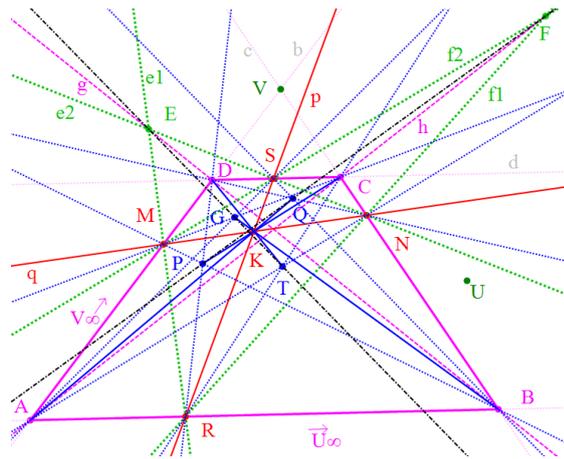


Figure 22.2

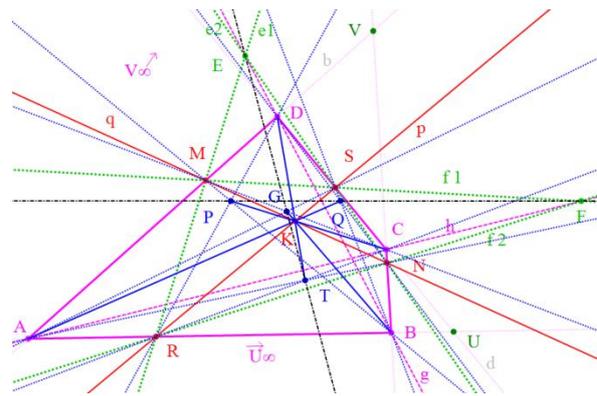


Figure 22.3

iii) We will show that 2) is equivalent to 7). The condition 2) that the points  $A, C, F$  are collinear can be stated in the following way: “the lines  $MS, NR, AC$  are concurrent”.

According to Desargues’ Theorem, applied to  $\Delta MRC$  and  $\Delta SNA$ , the lines  $MS, RN, CA$  are concurrent if and only if the points  $E=MR \cap SN, T=RC \cap NA, G=MC \cap SA$  are collinear.

The last statement can be also formulated in the following way:

The points  $A, C, F$  are collinear if and only if the points  $E, T, G$  are collinear;

iv) We will show that 2) is equivalent to 5) and 2) is equivalent to 6).

Let us consider the two pairs of triangles  $MSG, NRB$  and  $MSD, NRT$ .

According to Desargues’ Theorem, applied to  $\Delta MSG$  and  $\Delta NRB$ , the points:  $MS \cap NR = F, SG \cap RB = A, MG \cap NB = C$  are collinear if and only if the lines  $MN, SR, GB$  are concurrent.

According to Desargues’ Theorem, applied to  $\Delta MSD$  and  $\Delta NRT$ , the points:  $MS \cap NR = F, SD \cap RT = C, MD \cap NT = A$  are collinear if and only if the lines  $MN, SR, DT$  are concurrent.

Taking into account that  $MN \cap SR = K$  we can write the last statements in the following way:

The points  $A, C, F$  are collinear if and only if the points  $K, G, B$  are collinear;

The points  $A, C, F$  are collinear if and only if the points  $K, D, T$  are collinear.  $\square$

This problem shows that if we ensure the collinearity of just one of the eight triads of points, then the other seven triads of points will be collinear points, too.

The statement holds true if we swap the finite point  $E$  with the infinite point of the line  $BD$  (Figures 22.4 and 22.5).

Please, if it is necessary after swapping, refer to Remark 1.2 (use the points  $B$  or  $U$ ).

The choice of the projective method for the proof of Problem 13 makes its statement true, and after applying the SFIP function, i.e. for the connected figures, trapezium or quadrilateral (Figures 22.2 and 22.3). Every one now can formulate the satellite problems 13\*, 13\*\*.

A new challenge appears for the case when  $E=U_{BD}$  and  $ABCD$  is a parallelogram (Figure 22.4). The points  $A, C, F, K, P, Q$  are collinear. It is enough to prove that  $K$  lies on  $AC$ . Indeed the triangles  $MRA$  and  $NSC$  are perspective with a perspective axis the infinite line  $\omega$ , from where it follows  $MN, RS, AC$  are concurrent lines, i.e.  $K \in AC$ .

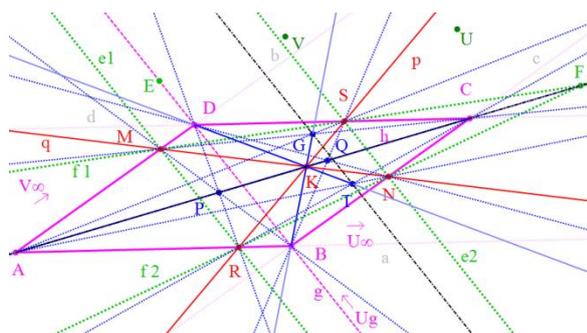


Figure 22.4

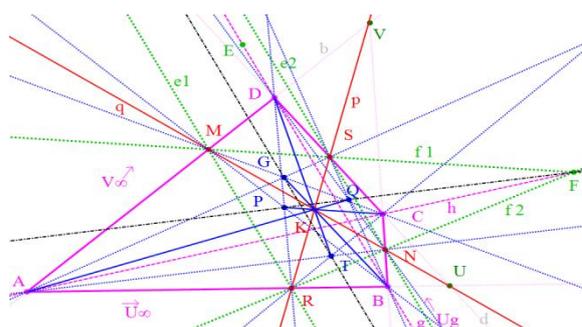


Figure 22.5

We have found other problems in traditional Bulgarian school textbooks [16, 17] that are stated with conditions that involve parallel lines, but which are true in more general cases. We give these problems to students as homework; to read the solutions; to apply the SFIP function; to generate the satellites of the problem (if they exist) and to prove them (usually with the fundamental theorems of Projective Geometry).

## 5. Conclusions

Dynamic Geometry Software gives the possibility for a deep and creative study of Geometry. The introduction of infinite points and the new SFIP function (“Swap finite & infinite points”), increases the benefit of DGS in teaching and learning Geometry. Through the range of the examples in Section 4, we have shown that the new features in DGS offered by Sam, optimize the education process by saving a lot of time for drawing, generalizing large groups of problems, stimulating and helping investigations, forming a creative style of thinking. They give the opportunity of preparing interesting course projects and carrying into effect the integration of teaching at university with that at secondary school.

Currently, much of the training in geometry in secondary school is in the framework of Euclidean geometry. Instead, we propose the study of the extended Euclidean plane in secondary school. DGS Sam, with its new features favors this development. DGS Cinderella has made very important steps towards teaching Non-Euclidean geometry (spherical and hyperbolic). Further, anecdotal evidence from children shows that exposing them to the projective plane will help them gain an intuitive understanding of concepts such as infinite points, dynamic drawings, quadrilaterals, circles, ellipses and the relationships between them. We feel and hope that such an experiment and investigation will confirm our prediction that early introduction of infinite elements in school facilitates intuitive understanding.

Deeper study of Projective Geometry in the university and in schools is also warranted, given that many recent results in various mathematical areas were proven with the help of the fundamental theorems of Projective Geometry. We hope that our study stimulates the integration of infinite elements in more technological tools.

**Supplemental Electronic Materials** The software “DGS - Sam” and the sketches are free for any type use and can be downloaded from: <http://fmi-plovdiv.org/GetResource?id=1186>.

The user has to install .NET Framework 4 before running the program Sam.

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