On Some Expressions for the Riesz Angle of Weighted Orlicz Sequence Spaces

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Abstract

We obtain an expression for computation of the Riesz angle in weighted Orlicz sequence spaces. We use this expression to find some estimates of the Riesz angle for large classes of weighted Orlicz sequence spaces.

1 Introduction

In order to generalize the technique in [14] for c_0 to a larger class of Banach lattices, J. Borwein and B. Sims introduced in [3] the notion of a weakly orthogonal Banach lattice and Riesz angle $a(X)$. A Banach lattice is weakly orthogonal if $\lim_{n\to\infty} \| \, |x_n| \wedge |x| \, \| = 0$ for all $x \in X$, whenever $\{x_n\}_{n=1}^{\infty}$ is a weakly null sequence, where $|x| \wedge |y| = \min(|x|, |y|)$. The Riesz angle $a(X)$ of a Banach lattice $(X, \| \cdot \|)$ is

$$a(X) = \sup\{\|(|x| \vee |y|)\| : \|x\| \leq 1, \|y\| \leq 1\},$$

where $|x| \vee |y| = \max(|x|, |y|)$. Clearly $1 \leq a(X) \leq 2$. If there exists a weakly orthogonal Banach lattice Y such that $d(X, Y).a(Y) < 2$, where $d(X, Y)$ is the Banach-Mazur distance between the Banach spaces X and Y, and $a(Y)$ is the Riesz angle of Y, then X has the weak fixed point property [3]. The coefficient $R(X)$ for a Banach space X is defined and a connection between $a(X)$ and $R(X)$ is found in [8]. If there exists a Banach space Y with weak Opial condition, such that $d(X, Y).R(Y) < 2$, then X has the fixed point property [8]. A N–dimensional Riesz angle for a Banach lattice is introduced and studied in [2]. A fixed point theorem is proved involving the N–dimensional Riesz angle of the space [2].

The Riesz angle is an important geometric coefficient in Banach lattices [1], [4], [5], therefore the finding of formulas for its calculation or estimation is an interesting problem. Estimations of the Riesz angles in Orlicz function spaces are found in [10]. Some estimations of the Riesz angles in Orlicz sequence spaces equipped with Luxemburg norm and Orlicz norm were found in [21]. Later a formula for computing the Riesz angle in Orlicz spaces is obtained in [22]. We refine the technique in [22] to obtain a formula for the Riesz angle in a wide class of weighted Orlicz sequence spaces. We apply this formula to find the Riesz angle in some classes of weighted Orlicz sequence spaces.
An open problem is to find formula for the computation of the N–dimensional Riesz angle, defined recently in [2], in Orlicz spaces.

2 Preliminaries

We use the standard Banach space terminology from [12]. Let X be a real Banach space, S_X be the unit sphere of X. Let ℓ^0 stand for the space of all real sequences i.e. $x = \{x_i\}_{i=1}^{\infty} \in \ell^0$, $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ is the set of the real numbers.

Definition 1. *A Banach space $(X, \|\cdot\|)$ is said to be Köthe sequence space if X is a subspace of ℓ^0 such that*
i) If $x \in \ell^0$, $y \in X$ and $|x_i| \le |y_i|$ for all $i \in \mathbb{N}$ then $x \in X$ and $\|x\| \le \|y\|$;
ii) There exists an element $x \in X$ such that $x_i > 0$ for all $i \in \mathbb{N}$.

We recall that M is an Orlicz function if M is even, convex, $M(0) = 0$, $M(t) > 0$ for $t > 0$. The Orlicz function $M(t)$ is said to have the property Δ_2 if there exists a constant c such that $M(2t) \le cM(t)$ for every $t \in \mathbb{R}$. A weight sequence $w = \{w_i\}_{i=1}^{\infty}$ is a sequence of positive reals. Following [9] we say that $w = \{w_i\}_{i=1}^{\infty}$ is from the class Λ if there exists a subsequence $w = \{w_{i_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} w_{i_k} = 0$ and $\sum_{k=1}^{\infty} w_{i_k} = \infty$. A weighted Orlicz sequence space $\ell_M(w)$ generated by an Orlicz function M and a weight sequence w is the set of all sequences $x \in \ell^0$, such that the inequality $\widetilde{M}(x/\lambda) = \sum_{i=1}^{\infty} w_i M(x_i/\lambda) < \infty$ holds for some $\lambda < \infty$.
It is well known that the space $\ell_M(w)$ is a Banach space if endowed with the Luxemburg's norm
$\|x\|_M = \inf \{r > 0 : \sum_{i=1}^{\infty} w_i M(x_i/r) \le 1\}$ or with the Amemiya's norm
$|||x|||_M = \inf \left\{\frac{1}{k}\left(1 + \sum_{i=1}^{\infty} w_i M(kx_i)\right) : k > 0\right\}$.
We will write $\ell_M(w, \|\cdot\|_M)$ and $\ell_M(w, |||\cdot|||_M)$ for the weighted Orlicz sequence spaces equipped with the Luxemburg's and Amemiya's norm respectively. Luxemburg's and Amemiya's norms are connected by the

inequalities:

$$\|\cdot\|_M \le |||\cdot|||_M \le 2\|\cdot\|_M.$$

We will write $\ell_M(w)$, when the statement holds for the weighted Orlicz sequence space equipped with both norms – Luxemburg and Amemiya. The space $\ell_M(w)$, endowed with the Luxemburg's or Amemiya's norm is a Köthe sequence space. In [18] C. Ruiz proved, that the weighted Orlicz sequence spaces $\ell_M(w)$ are all mutually isomorphic for the weight sequences $w = \{w_n\}_{n=1}^{\infty} \in \Lambda$. Sharp estimates are found in [13] for the cotype of $\ell_M(w)$, which depends only of the generating Orlicz function, when the weight sequence verifies the condition $w = \{w_n\}_{n=1}^{\infty} \in \Lambda$. It is proved in [23] that $\ell_M(w)$, endowed with the Luxemburg's or Amemiya's norm, has weak uniform normal structure iff $M \in \Delta_2$ at zero, when weight sequence verifies the condition $w = \{w_n\}_{n=1}^{\infty} \in \Lambda$. Weighted Orlicz sequence spaces were investigated for example in [7], [15], [11]. Let us mention that if the weight sequence is from the class Λ, then a lot of the properties of the space $\ell_M(w)$ depend only on the generating Orlicz function M [13], [23], which is in contrast with the results when $w \notin \Lambda$ [9], [18], [23]. All these inspired us to find the Riesz angle in wide class of weighted Orlicz sequence spaces.

3 Main results

Theorem 1. *Let M be an Orlicz function with the Δ_2–condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then the Riesz angle of $X = (\ell_M(w), \|\cdot\|)$ can be expressed as:*

$$a(X) = \sup\left\{k_x : \widetilde{M}_w\left(\frac{x}{k_x}\right) = \frac{1}{2}, x \in S_{\ell_M(w)}\right\}.$$

Theorem 2. *Let M be an Orlicz function with the Δ_2–condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then the Riesz angle of $X = (\ell_M(w), |||\cdot|||)$ can be expressed as:*

$$d(X) \le a(X) \le \frac{3}{2}\, d(X),$$

where

$$d(X) = \sup_{|||x|||=1} \inf_{k>1}\left\{d_{x,k} : \widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}\right\}.$$

4 Auxiliary results

Lemma 1. *Let $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$ and $v = \{v_i\}_{i=1}^{\infty} \in \Lambda$ be an arbitrary subsequence of w. Then there exist sequences of naturals $\{m_i^{(s)}\}_{i=1}^{\infty}$, $\{k_i^{(s)}\}_{i=1}^{\infty}$, $s \in \mathbb{N}$, such that*

$$1 \le m_1^{(1)} \le k_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)}, \quad m_i^{(s)} \le k_i^{(s)}, \quad k_i^{(s)} < m_{i+1}^{(s-1)}$$

$$for \;\; n, i, s \in \mathbb{N}, n \ge 2, i+s = n+1$$

and for every $i \in \mathbb{N}$ there holds the equality

$$\sum_{s=1}^{\infty} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i.$$

Proof By $v \in \Lambda$ it follows that there is a subsequence $\{v_{i_j}\}_{j=1}^{\infty}$, such that $\lim_{j\to\infty} v_{i_j} = 0$ and $\sum_{j=1}^{\infty} v_{i_j} = \infty$. For the simplicity of the notations let denote the subsequence $\{v_{i_j}\}_{j=1}^{\infty}$ by $\{v_j\}_{j=1}^{\infty}$. We will prove the Lemma by induction on n:

I) Let $n = 1$. We can choose $m_1^{(1)}, k_1^{(1)}$, so that

$$m_1^{(1)} \le k_1^{(1)}$$

and

$$w_1 - \frac{w_1}{2} \le \sum_{j=m_1^{(1)}}^{k_1^{(1)}} v_j < w_1.$$

Let us use the notation

$$f(i,p) = \sum_{s=1}^{p} \sum_{j=m_i^s}^{k_i^s} v_j.$$

II) Let $n = 2$. We will show that we can choose $m_i^{(s)}, k_i^{(s)}$, $i + s = 3$, so that

$$k_1^{(1)} < m_1^{(2)} \le k_1^{(2)} < m_2^{(1)} \le k_2^{(1)}$$

and

$$w_i - \frac{w_i}{2^{3-i}} \le \sum_{s=1}^{3-i} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j < w_i, \quad \text{for} \;\; i = 1, 2.$$

Indeed let choose first $m_1^{(2)}, k_1^{(2)} \in \mathbb{N}$: $k_1^{(1)} < m_1^{(2)} \le k_1^{(2)}$, such that

$$w_1 - \frac{w_1}{2^2} \le f(1,1) + \sum_{j=m_1^{(2)}}^{k_1^{(2)}} v_j < w_1.$$

Then we choose $m_2^{(1)}, k_2^{(1)} \in \mathbb{N}$: $k_1^{(2)} < m_2^{(1)} \le k_2^{(1)}$, such that

$$w_2 - \frac{w_2}{2} \le \sum_{j=m_2^{(1)}}^{k_2^{(1)}} v_j < w_2.$$

III) Suppose that for $n = p$ we have chosen $\{m_i^{(s)}\}_{i=1}^{\infty}$, $\{k_i^{(s)}\}_{i=1}^{\infty}$, $i + s = p + 1$ with the properties:

$$k_{p-1}^{(1)} < m_1^{(p)}, m_i^{(s)} \leq k_i^{(s)}, k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $i + s = p + 1$ and

$$w_i - \frac{w_i}{2^{p+1-i}} \leq \sum_{s=1}^{p+1-i} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j < w_i, \quad \text{for} \quad i \leq p.$$

IV) Let $n = p + 1$. We will show that we can choose $\{m_i^{(s)}\}_{i=1}^{\infty}$, $\{k_i^{(s)}\}_{i=1}^{\infty}$, $i + s = p + 2$, so that

$$k_p^{(1)} < m_1^{(p+1)}, m_i^{(s)} \leq k_i^{(s)}, k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $i + s = p + 2$ and

$$w_i - \frac{w_i}{2^{p+1-i}} \leq \sum_{s=1}^{p+1-i} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j < w_i, \tag{1}$$

for $i \leq p + 1$.

Indeed let choose first $m_1^{(p+1)}, k_1^{(p+1)} \in \mathbb{N}$: $k_p^{(1)} < m_1^{(p+1)} \leq k_1^{(p+1)}$, such that

$$w_1 - \frac{w_1}{2^{p+1}} \leq f(1, p) + \sum_{j=m_1^{(p+1)}}^{k_1^{(p+1)}} v_j < w_1.$$

Then we choose $m_2^{(p)}, k_2^{(p)} \in \mathbb{N}$: $k_1^{(p+1)} < m_2^{(p)} \leq k_2^{(p)}$, such that

$$w_2 - \frac{w_2}{2^p} \leq f(2, p - 1) + \sum_{j=m_2^{(p)}}^{k_2^{(p)}} v_j < w_2.$$

If for $i_0 \leq p - 1$ we have chosen $m_{i_0}^{(p-i_0+2)}, k_{i_0}^{(p-i_0+2)} \in \mathbb{N}$ to satisfy the inequalities $k_{i_0}^{(p-i_0+1)} < m_{i_0}^{(p-i_0+2)} \leq k_{i_0}^{(p-i_0+2)}$, such that

$$w_{i_0} - \frac{w_{i_0}}{2^{p-i_0+2}} \leq f(i_0, p - i_0 + 1) + \sum_{j=m_{i_0}^{(p-i_0+2)}}^{k_{i_0}^{(p-i_0+2)}} v_j < w_{i_0},$$

then for $i_0 + 1 \leq p$ we choose $m_{i_0+1}^{(p-i_0+1)}, k_{i_0+1}^{(p-i_0+1)} \in \mathbb{N}$: $k_{i_0+1}^{(p-i_0)} < m_{i_0+1}^{(p-i_0+1)} \leq k_{i_0+1}^{(p-i_0+1)}$, such that

$$\begin{aligned} w_{i_0+1} \quad - \quad & \frac{w_{i_0+1}}{2^{p-i_0+1}} \leq f(i_0 + 1, p - i_0) \\ & + \sum_{j=m_{i_0+1}^{(p-i_0+1)}}^{k_{i_0+1}^{(p-i_0+1)}} v_j < w_{i_0+1}. \end{aligned}$$

On the last step for $i = p+1$ we choose $m^{(1)}_{p+1}, k^{(1)}_{p+1} \in \mathbb{N}$: $k^{(2)}_p < m^{(1)}_{p+1} \le k^{(1)}_{p+1}$, such that

$$w_{p+1} - \frac{w_{p+1}}{2} \le \sum_{j=m^{(1)}_{p+1}}^{k^{(1)}_{p+1}} v_j = f(p+1,1) < w_{p+1}.$$

By (1) it follows that $\lim_{n\to\infty} \sum_{s=1}^{n} \sum_{j=m^{(s)}_i}^{k^{(s)}_i} v_j = w_i$ holds for every $i \in \mathbb{N}$. $\square$

Theorem 3. *[6] Let be given the iterated series $\sum_{n=1}^{\infty}\sum_{s=1}^{\infty} a^s_n$. If the series $\sum_{n=1}^{\infty}\sum_{s=1}^{\infty} |a^s_n|$ is convergent, then for any permutations $\pi, \sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n,s} a^{\sigma(s)}_{\pi(n)}$ is convergent and $\sum_{n,s} a^{\sigma(s)}_{\pi(n)} = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} a^s_n$.*

Lemma 2. *Let M be an Orlicz function with the Δ_2–condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Let $\ell_M(w)$ be equipped with Luxemburg's or Amemiya's norm. Then*

1) For every $x \in \ell_M(w)$, such that $\widetilde{M}_w(\lambda x) < \infty$, for every $\lambda > 0$, there are $y, z \in \ell_M(w)$, such that $|y| \wedge |z| = 0$, $\widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z) = \widetilde{M}_w(\lambda x)$, for any $\lambda > 0$;

2) For every $x \in S_{(\ell_M(w), \|\cdot\|)}$, there are y, z, such that $|y| \wedge |z| = 0$, $y, z \in S_{(\ell_M(w), \|\cdot\|)}$;

3) For every $x \in S_{(\ell_M(w), |||\cdot|||)}$, there are y, z, such that $|y| \wedge |z| = 0$, $y, z \in S_{(\ell_M(w), |||\cdot|||)}$.

Proof 1) Let $x = \{x_n\}_{n=1}^{\infty} \in \ell_M(w)$ be arbitrary chosen. By $M \in \Delta_2$ it follows that $\widetilde{M}_w(\lambda x) < \infty$ for every $\lambda > 0$. By $w \in \Lambda$ it follows that we can choose two subsequences $v = \{v_i\}_{i=1}^{\infty}$, $u = \{u_i\}_{i=1}^{\infty}$ of w, such that $v, u \in \Lambda$, $v \cap u = \emptyset$, $v \cup u = w$.

By Lemma 1 there are sequences of naturals $\{m^{(s)}_i\}_{i=1}^{\infty}$, $\{k^{(s)}_i\}_{i=1}^{\infty}$, $\{\alpha^{(s)}_i\}_{i=1}^{\infty}$, $\{\beta^{(s)}_i\}_{i=1}^{\infty}$, $s \in \mathbb{N}$, such that

$$1 \le m^{(1)}_1 \le k^{(1)}_1, \quad 1 \le \alpha^{(1)}_1 \le \beta^{(1)}_1$$

$$k^{(1)}_{n-1} < m^{(n)}_1, \quad m^{(s)}_i \le k^{(s)}_i, \quad k^{(s)}_i < m^{(s-1)}_{i+1}$$

$$\text{for } \ n, i, s \in \mathbb{N}, n \ge 2, i+s = n+1$$

$$\beta^{(1)}_{n-1} < \alpha^{(n)}_1, \alpha^{(s)}_i \le \beta^{(s)}_i \beta^{(s)}_i < \alpha^{(s-1)}_{i+1}$$

$$\text{for } \ n, i, s \in \mathbb{N}, n \ge 2, i+s = n+1$$

and there hold the equalities

$$\sum_{s=1}^{\infty} \sum_{j=m^{(s)}_i}^{k^{(s)}_i} v_j = w_i = \sum_{s=1}^{\infty} \sum_{j=\alpha^{(s)}_i}^{\beta^{(s)}_i} u_j,$$

for every $i \in \mathbb{N}$.

Put

$$y_n = \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} x_n e_j, z_n = \sum_{s=1}^{\infty} \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} x_n e_j$$

and $y = \sum_{n=1}^{\infty} y_n$, $z = \sum_{n=1}^{\infty} z_n$. We will show that $\widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z) = \widetilde{M}_w(\lambda x)$, for any $\lambda > 0$. Let $\lambda > 0$ and put $a_n^s(\lambda) = M(\lambda x_n) \sum_{j=m_n^{(s)}}^{k_n^{(s)}} v_j$ for $n, s \in \mathbb{N}$. Let consider the infinite matrix

$$\begin{array}{cccccc} a_1^1(\lambda) & a_1^2(\lambda) & a_1^3(\lambda) & \ldots & a_1^s(\lambda) & \ldots \\ a_2^1(\lambda) & a_2^2(\lambda) & a_2^3(\lambda) & \ldots & a_2^s(\lambda) & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots & .. \\ a_n^1(\lambda) & a_n^2(\lambda) & a_n^3(\lambda) & \ldots & a_n^s(\lambda) & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots & .. \end{array}$$

For every $n \in \mathbb{N}$ the equality $\sum_{s=1}^{\infty} a_n^s(\lambda) = M(\lambda x_n) \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} v_j = w_n M(\lambda x_n)$ holds and thus $\sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s(\lambda) = \sum_{n=1}^{\infty} w_n M(\lambda x_n) < \infty$ for every $\lambda > 0$. By $a_n^s(\lambda) \geq 0$ for every $n, s \in \mathbb{N}$ and Theorem 3 it follows that for any two permutations $\pi, \sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n,s}^{\infty} a_{\pi(n)}^{\sigma(s)}(\lambda)$ is convergent and there hold the equalities

$$\sum_{n,s}^{\infty} a_{\pi(n)}^{\sigma(s)}(\lambda) = \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s(\lambda) = \widetilde{M}_w(\lambda x).$$

Consequently there exist two permutations $\pi, \sigma : \mathbb{N} \to \mathbb{N}$, such that we can write the chain of equalities

$$\begin{aligned} \widetilde{M}_w(\lambda y) &= \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} a_n^{p-n}(\lambda) \\ &= \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} v_j M(\lambda x_n) \\ &= \sum_{n,s}^{\infty} a_{\pi(n)}^{\sigma(s)}(\lambda) \\ &= \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s(\lambda) = \widetilde{M}_w(\lambda x). \end{aligned} \tag{2}$$

Similarly if we put $b_n^s(\lambda) = M(\lambda x_n)\sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} v_j$ for $n, s \in \mathbb{N}$ we get the chain of equalities

$$\begin{aligned} \widetilde{M}_w(\lambda z) &= \sum_{p=2}^{\infty}\sum_{n=1}^{p-1} b_n^{p-n}(\lambda) \\ &= \sum_{p=2}^{\infty}\sum_{n=1}^{p-1}\sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} u_j M(\lambda x_n) \\ &= \sum_{n,s}^{\infty} b_{\pi(n)}^{\sigma(s)}(\lambda) \\ &= \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} b_n^s(\lambda) = \widetilde{M}_w(\lambda x). \end{aligned} \tag{3}$$

2) If $\lambda = 1$ then by $\widetilde{M}_w(y) = \widetilde{M}_w(z) = \widetilde{M}_w(x) = 1$ we get that that if $x \in S_{(\ell_M(w), \|\cdot\|)}$ then $y, z \in S_{(\ell_M(w), \|\cdot\|)}$.

3) If $x \in S_{(\ell_M(w), |||\cdot|||)}$, then by 1) it follows that $\frac{1}{\lambda}(1 + \widetilde{M}_w(\lambda y)) = \frac{1}{\lambda}(1 + \widetilde{M}_w(\lambda z)) = \frac{1}{\lambda}(1 + \widetilde{M}_w(\lambda x))$ for any $\lambda > 0$. Therefore if $|||x||| = 1$, then $|||y||| = |||z||| = |||x||| = 1$. □

Lemma 3. *Let M be an Orlicz function with the Δ_2–condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then for every $x \in S_{(\ell_M(w), |||\cdot|||)}$ and $k > 1$, there exists a unique $d_{x,k} > 1$, such that $\widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$.*

Proof Let $x \in S_{(\ell_M(w), |||\cdot|||)}$ and $k > 1$ be arbitrary chosen and fixed, then we define the function $S(d) : [1, +\infty) \to \mathbb{R}$ by $S(d) = \widetilde{M}_w\left(\frac{kx}{d}\right) = \sum_{i=1}^{\infty} w_i M\left(\frac{kx_i}{d}\right)$. By the inequality $M\left(\frac{kx_i}{d}\right) \le M(kx_i)$ and the convergence of the series $\sum_{i=1}^{\infty} w_i M(kx_i)$ it follows that $\sum_{i=1}^{\infty} w_i M\left(\frac{kx_i}{d}\right)$ is uniformly convergent on $[1, +\infty)$. Thus S is a continuous function. It is easy to see that S is strictly decreasing function on $[1, +\infty)$. Therefore by the inequalities

$$f(1) = \widetilde{M}_w(kx) \ge |||kx|||_M - 1 = k - 1 > \frac{k-1}{2}$$

and

$$\lim_{d\to+\infty} f(d) = \lim_{d\to+\infty} \widetilde{M}_w\left(\frac{kx}{d}\right) = 0 < \frac{k-1}{2}$$

we get that there is a unique $d_{x,k} > 0$, such that $f(d_{x,k}) = \widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$. □

Lemma 4. *([22]) For a Köthe sequences space $(X, \|\cdot\|)$, the Riesz angle $a(X)$ can be expressed as*

$$a(X) = \sup\{\|(|x| \vee |y|)\| : x, y \in S_X, |x| \wedge |y| = 0\},$$

where $|x| \wedge |y| = \min\{|x|, |y|\}$.

5 Proof of main results

Proof of Theorem 1: Let $x = \{x_n\}_{n=1}^{\infty} \in S_{(\ell_M(w),\|\cdot\|)}$ be arbitrary chosen. By $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$ it follows that we can choose two subsequences $v = \{v_i\}_{i=1}^{\infty}$, $u = \{u_i\}_{i=1}^{\infty}$ of w, such that $v, u \in \Lambda$, $v \cap u = \emptyset$, $v \cup u = w$.

It follows from Lemma 1 that there exist sequences of naturals $\{m_i^{(s)}\}_{i=1}^{\infty}$, $\{k_i^{(s)}\}_{i=1}^{\infty}$, $\{\alpha_i^{(s)}\}_{i=1}^{\infty}$, $\{\beta_i^{(s)}\}_{i=1}^{\infty}$, $s \in \mathbb{N}$, such that

$$1 \le m_1^{(1)} \le k_1^{(1)}, \quad 1 \le \alpha_1^{(1)} \le \beta_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)}, \quad m_i^{(s)} \le k_i^{(s)}, \quad k_i^{(s)} < m_{i+1}^{(s-1)}$$

$$\text{for} \quad n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1,$$

$$\beta_{n-1}^{(1)} < \alpha_1^{(n)}, \quad \alpha_i^{(s)} \le \beta_i^{(s)}, \quad \beta_i^{(s)} < \alpha_{i+1}^{(s-1)}$$

$$\text{for} \quad n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1$$

and there holds the equalities

$$\sum_{s=1}^{\infty} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i = \sum_{s=1}^{\infty} \sum_{j=\alpha_i^{(s)}}^{\beta_i^{(s)}} u_j$$

for every $i \in \mathbb{N}$.

We can put

$$y_n = \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} x_n e_j, \quad z_n = \sum_{s=1}^{\infty} \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} x_n e_j$$

and $y = \sum_{n=1}^{\infty} y_n$, $z = \sum_{n=1}^{\infty} z_n$.

By using Lemma 2 we get that $y, z \in S_{(\ell_M(w),\|\cdot\|)}$ and $\widetilde{M}_w(\lambda x) = \widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z)$ for any $\lambda \in \mathbb{R}$.

By the choice of the subsequences $v, u \subset w$ we have that $|y| \wedge |z| = 0$ and $\|(|y| \vee |z|)\| = \|y + z\|$. Therefore we can write the chain of equalities:

$$\begin{aligned} 1 &= \widetilde{M}_w\left(\frac{y+z}{\|y+z\|}\right) \\ &= \widetilde{M}_w\left(\frac{y}{\|y+z\|}\right) + \widetilde{M}_w\left(\tfrac{z}{\|y+z\|}\right) \\ &= 2\widetilde{M}_w\left(\frac{x}{\|(|y| \vee |z|)\|}\right). \end{aligned}$$

Consequently it follows that for every $x \in S_{(\ell_M(w),\|\cdot\|)}$, there exists $k_x = \|(|y| \vee |z|)\|$, such that $\widetilde{M}_w\left(\frac{x}{k_x}\right) = \frac{1}{2}$. By Lemma 4 we get the inequality

$$a(X) \ge \sup\left\{k_x : \widetilde{M}_w\left(\frac{x}{k_x}\right) = \frac{1}{2}, x \in S_X\right\}$$

where $X = (\ell_M(w), \|\cdot\|)$.

On the other hand let put

$$D = \sup\left\{k_x : \widetilde{M}_w\left(\frac{x}{k_x}\right) = \frac{1}{2}, x \in S_{(\ell_M(w),\|\cdot\|)}\right\}.$$

It follows from Lemma 4 that for every $\varepsilon > 0$ there are $x, y \in S_{(\ell_M(w),\|\cdot\|)}$, $|x| \wedge |y| = 0$, such that $\|(|x| \vee |y|)\| > a(\ell_M(w)) - \varepsilon$.

Since

$$\begin{aligned} \widetilde{M}_w\left(\frac{(|x| \vee |y|)}{d}\right) &= \widetilde{M}_w\left(\frac{x}{d}\right) + \widetilde{M}_w\left(\frac{y}{d}\right) \\ &\le \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

we get the inequality $\|(|x| \vee |y|)\| \le d$, which implies $a(\ell_M(w)) \le d + \varepsilon$. By the arbitrariness of $\varepsilon > 0$ we obtain that $d \ge a(\ell_M(w))$. □

Proof of Theorem 2: Let us denote

$$d = \sup_{|||x|||=1} \inf_{k>1}\left\{d_{x,k} : \widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}\right\}. \tag{4}$$

For any $\varepsilon > 0$ there exist $x = \{x_n\}_{n=1}^\infty \in S_{(\ell_M(w),|||\cdot|||)}$ and $k > 1$, such that $d_{x,k} \ge d - \varepsilon$.

By $w = \{w_i\}_{i=1}^\infty \in \Lambda$ it follows that we can choose two subsequences $v = \{v_i\}_{i=1}^\infty$, $u = \{u_i\}_{i=1}^\infty$ of w, such that $v, u \in \Lambda$, $v \cap u = \emptyset$, $v \cup u = w$.

It follows from Lemma 1 that there exist sequences of naturals $\{m_i^{(s)}\}_{i=1}^\infty$, $\{k_i^{(s)}\}_{i=1}^\infty$, $\{\alpha_i^{(s)}\}_{i=1}^\infty$, $\{\beta_i^{(s)}\}_{i=1}^\infty$, $s \in \mathbb{N}$, such that

$$1 \le m_1^{(1)} \le k_1^{(1)}, \quad 1 \le \alpha_1^{(1)} \le \beta_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)}, \quad m_i^{(s)} \le k_i^{(s)}, \quad k_i^{(s)} < m_{i+1}^{(s-1)}$$

$$\text{for } \ n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1,$$

$$\beta_{n-1}^{(1)} < \alpha_1^{(n)}, \quad \alpha_i^{(s)} \le \beta_i^{(s)}, \quad \beta_i^{(s)} < \alpha_{i+1}^{(s-1)}$$

$$\text{for } \ n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1$$

and there hold the equalities

$$\sum_{s=1}^\infty \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i = \sum_{s=1}^\infty \sum_{j=\alpha_i^{(s)}}^{\beta_i^{(s)}} u_j$$

for every $i \in \mathbb{N}$.

We can put

$$y_n = \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} x_n e_j, z_n = \sum_{s=1}^{\infty} \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} x_n e_j$$

and $y = \sum_{n=1}^{\infty} y_n$, $z = \sum_{n=1}^{\infty} z_n$.

By using Lemma 2 we see that $y, z \in S_{(\ell_M(w),|||\cdot|||)}$, and $\widetilde{M}_w(\lambda x) = \widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z)$ for any $\lambda \in \mathbb{R}$. Let put

$$D_1 = \inf_{0<k\leq 1} \frac{1}{k} \left(1 + \widetilde{M}_w \left(\frac{k(y+z)}{d-\varepsilon}\right)\right)$$

and

$$D_2 = \inf_{k>1} \frac{1}{k} \left(1 + \widetilde{M}_w \left(\frac{k(y+z)}{d-\varepsilon}\right)\right)$$

Therefore by the chain of inequalities

$$\begin{aligned} D_2 &= \inf_{k>1} \frac{1}{k} \left(1 + \widetilde{M}_w \left(\frac{k(y+z)}{d-\varepsilon}\right)\right) \\ &= \inf_{k>1} \frac{1}{k} \left(1 + 2\widetilde{M}_w \left(\frac{kx}{d-\varepsilon}\right)\right) \\ &\geq \inf_{k>1} \frac{1}{k} \left(1 + 2\widetilde{M}_w \left(\frac{kx}{d_{x,k}}\right)\right) \\ &= \inf_{k>1} \frac{1}{k} \left(1 + 2\frac{k-1}{2}\right) = 1 \end{aligned}$$

and

$$\begin{aligned} \left|\left|\left|\tfrac{y+z}{d-\varepsilon}\right|\right|\right| &= \min\{D_1, D_2\} \\ &\geq \min\{1, D_2\} = 1 \end{aligned}$$

we get that $|||y+z||| \geq d - \varepsilon$. By the arbitrariness of $\varepsilon > 0$ and Lemma 4 we obtain the inequality $a((\ell_M(w), |||\cdot|||)) \geq d$.

On the other hand for any $\varepsilon > 0$ there are $x, y \in S_{(\ell_M(w),|||\cdot|||)}$, $|x| \wedge |y| = 0$, such that there holds the inequality $|||(|x| \vee |y|)||| > a((\ell_M(w), |||\cdot|||)) - \varepsilon$. It follows from (4) and Lemma 3, that for every $\varepsilon > 0$ there are $k, h > 1$, such that $d_{x,k} < d + \varepsilon$, $d_{y,h} < d + \varepsilon$, where $d_{x,k}$ and $d_{y,h}$ are the solutions of the equations $\widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$ and $\widetilde{M}_w\left(\frac{hy}{d_{y,h}}\right) = \frac{h-1}{2}$ respectively. WLOG, we may assume that $1 < h \leq k$.

By the chain of inequalities

$$\begin{aligned} \left|\left|\left|\tfrac{(|x|\vee|y|)}{d+\varepsilon}\right|\right|\right| &\leq \tfrac{1}{h}\left(1+\widetilde{M}_w\left(\tfrac{h(|x|\vee|y|)}{d+\varepsilon}\right)\right) \\ &= \tfrac{1}{h}\left(1+\widetilde{M}_w\left(\tfrac{hx}{d+\varepsilon}\right)+\widetilde{M}_w\left(\tfrac{hy}{d+\varepsilon}\right)\right) \\ &= \tfrac{1}{h}+\tfrac{1}{h}\widetilde{M}_w\left(\tfrac{hx}{d_{h,x}}\right)+\tfrac{1}{h}\widetilde{M}_w\left(\tfrac{hy}{d_{k,y}}\right) \\ &\leq \tfrac{1}{h}+\tfrac{1}{h}\widetilde{M}_w\left(\tfrac{hx}{d_{h,x}}\right)+\tfrac{1}{k}\widetilde{M}_w\left(\tfrac{ky}{d_{k,y}}\right) \\ &= \tfrac{1}{h}+\tfrac{1}{h}\tfrac{h-1}{2}+\tfrac{1}{k}\tfrac{k-1}{2} \\ &= 1+\tfrac{1}{2h}-\tfrac{1}{2k}<\tfrac{3}{2} \end{aligned}$$

we obtain the inequality $|||(|x|\vee|y|)|||\leq\dfrac{3}{2}\,d+\dfrac{3}{2}\varepsilon$ and hence $a(\ell_M(w))<\dfrac{3}{2}\,d+\dfrac{5\varepsilon}{2}$. Since $\varepsilon>0$ is arbitrary chosen it follows that $a(\ell_M(w))\leq\dfrac{3}{2}\,d$. □

6 Some estimates of the Riesz angle in weighted Orlicz sequence spaces

For the estimation of the Riesz angle in weighted Orlicz sequence spaces we will need some well known indices. For an Orlicz function M we consider the index function $G_M(u)=\frac{M^{-1}(u)}{M^{-1}(2u)}$, $u\in(0,+\infty)$ [16]. Following [20], [22] we define the indices:

$$\begin{aligned} \alpha_M^0 &= \liminf_{u\to 0} G_M(u), \\ \alpha_M^{+\infty} &= \liminf{}_{u\to\infty} G_M(u), \\ \alpha_M^{0,+\infty} &= \min\{\alpha_M^0,\alpha_M^{+\infty}\}, \\ \widetilde{\alpha}_M &= \inf\left\{\tfrac{M^{-1}(u)}{M^{-1}(2u)} : u\in(0,+\infty)\right\}. \end{aligned} \tag{5}$$

Let mention that for an Orlicz sequences spaces ℓ_M only the behavior of the Orlicz function M at zero is important and therefore the above indices are defined only at zero in [20], [22].

Theorem 4. *Let M be an Orlicz function with the Δ_2–condition and $w=\{w_i\}_{i=1}^{\infty}\in\Lambda$. Then*

$$\frac{1}{\alpha_M^{0,\infty}}\leq a((\ell_M(w),\|\cdot\|))=\frac{1}{\widetilde{\alpha}_M}.$$

Proof We will prove first that $a((\ell_M(w),\|\cdot\|))=\frac{1}{\widetilde{\alpha}_M}$.
Chose arbitrary $x=\{x_i\}_{i=1}^{\infty}\in S_{(\ell_M(w),\|\cdot\|)}$ and put $u_i=\frac{1}{2}M(x_i)$. It is easy to check the equality

$M\left(x_i.G_M\left(\frac{M(x_i)}{2}\right)\right) = \frac{1}{2}M(x_i)$. For any $u_i \in (0,+\infty)$ the inequality $\widetilde{\alpha}_M \leq G_M(u_i)$ holds. Then

$$\begin{aligned} \widetilde{M}_w(\widetilde{\alpha}_M x) &= \sum_{i=1}^{\infty} w_i M(\widetilde{\alpha}_M x_i) \\ &\leq \sum_{i=1}^{\infty} w_i M(x_i G_M(u_i)) \\ &= \sum_{i=1}^{\infty} w_i M\left(x_i G_M\left(\tfrac{M(x_i)}{2}\right)\right) \\ &= \sum_{i=1}^{\infty} \frac{w_i M(x_i)}{2} = \frac{1}{2}. \end{aligned} \tag{6}$$

For every $x = \{x_i\}_{i=1}^{\infty} \in S_{(\ell_M(w),\|\cdot\|)}$ there exists k_x, such that $\widetilde{M}_w\left(\frac{x}{k_x}\right) = \frac{1}{2}$ and thus by (6) it follows the inequality $k_x \leq \frac{1}{\widetilde{\alpha}_M}$. Therefore

$$\begin{aligned} a(X) &= \sup\left\{k_x : \widetilde{M}_w\left(\frac{x}{k_x}\right) = \frac{1}{2}, x \in S_X\right\} \\ &\leq \frac{1}{\widetilde{\alpha}_M}, \end{aligned}$$

where $X = (\ell_M(w), \|\cdot\|)$.

Now we will prove that $a((\ell_M(w), \|\cdot\|)) \geq \frac{1}{\widetilde{\alpha}_M}$.

For any $u \in (0,+\infty)$ there are sequences of naturals $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$, such that $p_n \leq q_n < p_{n+1}$, for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\sum_{i=p_n}^{q_n} w_i = \frac{1}{u}$.

Put $x = \sum_{n=1}^{\infty}\sum_{i=p_n}^{q_n} M^{-1}(u)e_i$. It is easy to see that $\widetilde{M}_w(x) = 1$ and thus $x \in S_{(\ell_M(w),\|\cdot\|)}$. By the equality

$$\widetilde{M}_w\left(\frac{x}{\frac{M^{-1}(u)}{M^{-1}(u/2)}}\right) = \frac{1}{2}$$

we get that for any $u \in (0,+\infty)$ there holds

$$\frac{M^{-1}(u)}{M^{-1}(u/2)} \leq a((\ell_M(w), \|\cdot\|))$$

the inequality and therefore

$$\begin{aligned} \tfrac{1}{\widetilde{\alpha}_M} &= \sup\left\{\tfrac{M^{-1}(u)}{M^{-1}(u/2)} : u \in (0,+\infty)\right\} \\ &\leq a(\ell_M(w)). \end{aligned}$$

Thus we have proven that

$$a((\ell_M(w), \|\cdot\|)) = \frac{1}{\widetilde{\alpha}_M}. \tag{7}$$

The proof of the inequality $\frac{1}{\alpha_M^{0,\infty}} \leq a((\ell_M(w), \|\cdot\|))$ follows directly by equality (7) and the inequality $\alpha_M^{0,\infty} \geq \widetilde{\alpha}_M$. □

Theorem 5. *Let M be an Orlicz function with the Δ_2–condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then*

$$\frac{1}{\alpha_M^{0,\infty}} \leq a((\ell_M(w), |||\cdot|||)) \leq \frac{3}{2\ \widetilde{\alpha}_M}.$$

Proof I) We will prove first that

$$a((\ell_M(w), |||\cdot|||)) \leq \frac{3}{2\ \widetilde{\alpha}_M}.$$

Let chose arbitrary $x = \{x_i\}_{i=1}^{\infty} \in S_{(\ell_M(w), |||\cdot|||)}$. By $M \in \Delta_2$ and the equality

$$1 = |||x||| = \inf\left\{\frac{1}{k}\left(1 + \widetilde{M}_w(kx)\right) : k > 0\right\}$$

it follows that there exists $k_0 > 0$, such that the equality $\widetilde{M}_w(k_0 x) = k_0 - 1$ holds [16].

Put $u_i = \dfrac{1}{2} M(k_0 x_i)$. Then similarly to (6) we can write the inequality:

$$\begin{aligned}
\widetilde{M}_w(k_0 \widetilde{\alpha}_M x) &= \sum_{i=1}^{\infty} w_i M(k_0 \widetilde{\alpha}_M x_i) \\
&\leq \sum_{i=1}^{\infty} w_i M(k_0 x_i G_M(u_i)) \\
&= \sum_{i=1}^{\infty} w_i M\left(k_0 x_i G_M\left(\frac{M(k_0 x_i)}{2}\right)\right) \\
&= \sum_{i=1}^{\infty} \frac{w_i M(k_0 x_i)}{2} = \frac{k_0 - 1}{2}
\end{aligned}$$

and consequently we get that $d_{x,k_0} < \frac{1}{\widetilde{\alpha}_M}$, where $d_{x,k}$ is the solution of the equation $\widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$, $k > 1$. Thus for any $k > 1$ there holds the inequality

$$\inf\{d_{x,k} : k > 1\} < d_{x,k_0} < \frac{1}{\widetilde{\alpha}}. \tag{8}$$

By the arbitrary choice of $x = \{x_i\}_{i=1}^{\infty} \in S_{(\ell_M(w), |||\cdot|||)}$ and (8) we obtain that

$$\begin{aligned}
a((\ell_M(w), |||\cdot|||)) &\leq \frac{3}{2} \sup_{|||x|||=1} \inf\{d_{x,k} : k > 0\} \\
&\leq \frac{3}{2\ \widetilde{\alpha}_M}.
\end{aligned}$$

From the inequalities $\|\cdot\| \leq |||\cdot||| \leq 2\|\cdot\|$ it follows that for any $x, y \in S_{(\ell_M(w), |||\cdot|||)}$, with $|x| \wedge |y| = 0$ there holds the inequality

$$\begin{aligned}
a(X) &= \sup\{|||(|x| \vee |y|)||| : |||x|||, |||y||| = 1\} \\
&\geq \sup\{\|(|x| \vee |y|)\| : \|x\|, \|y\| \leq 1\} \qquad (9) \\
&= a((\ell_M(w), \|\cdot\|)),
\end{aligned}$$

where $X = (\ell_M(w), |||\cdot|||)$.

Therefore we get that

$$a((\ell_M(w), |||\cdot|||)) \geq a((\ell_M(w), \|\cdot\|)) = \frac{1}{\widetilde{\alpha}_M}.$$

II) By (9) we obtain the inequality

$$a((\ell_M(w), |||\cdot|||)) \geq a((\ell_M(w), \|\cdot\|)) \geq \frac{1}{\alpha_M^{0,\infty}}.$$

□

Definition 2. *We say that the Orlicz function M satisfies the ∇_2 condition if there exists $l > 1$, such that $M(x) \leq \frac{1}{2l}M(lx)$, for every $x \in [0, +\infty)$ and we denote this by $M \in \nabla_2$.*

The function M^{-1} is a concave function and thus $\frac{M^{-1}(t)}{M^{-1}(2t)} \geq \frac{1}{2}$. According to ([16], p. 22) $M \in \nabla_2$ iff $\liminf_{t\to+\infty} \dfrac{M^{-1}(t)}{M^{-1}(2t)} > \dfrac{1}{2}$ and $\liminf_{t\to 0} \dfrac{M^{-1}(t)}{M^{-1}(2t)} > \dfrac{1}{2}$ i.e.

$$M \in \nabla_2 \Leftrightarrow \alpha_M^0 > 1/2 \quad \text{and} \quad \alpha_M^{+\infty} > 1/2. \tag{10}$$

Corollary 1. *Let M be an Orlicz function with $M \in \Delta_2$ and $w \in \Lambda$ is a weight sequence. Then*

a) $M \notin \nabla_2$ iff $a((\ell_M(w), \|\cdot\|)) = 2$;

b) $M \in \nabla_2$ iff $a((\ell_M(w), \|\cdot\|)) < 2$.

Proof a) Let $M \notin \nabla_2$. Then from (10) it follows that $\alpha_M^0 = 1/2$ or $\alpha_M^{+\infty} = 1/2$ and thus by Theorem 4 it follws that $a((\ell_M(w), \|\cdot\|)) \geq 2$. Therefore by the inequalities $1 \leq a((\ell_M(w), \|\cdot\|)) \leq 2$ it follows that $a(\ell_M(w), \|\cdot\|) = 2$.

Let $a((\ell_M(w), \|\cdot\|)) = 2$. There are three cases $\alpha_M^0 = 1/2$, $\alpha_M^{+\infty} = 1/2$ or there exists $t_0 \in (0, +\infty)$, such that $\frac{M^{-1}(t_0)}{M^{-1}(2t_0)} = 1/2$.

Let holds $\alpha_M^0 = 1/2$ or $\alpha_M^{+\infty} = 1/2$ then by (10) it follows that $M \notin \nabla_2$.

Let there exists $t_0 \in (0, +\infty)$, such that $\frac{M^{-1}(t_0)}{M^{-1}(2t_0)} = 1/2$. Then we can write the equality $\frac{M^{-1}(2t_0) - M^{-1}(t_0)}{2t_0 - t_0} = \frac{M^{-1}(t_0)}{t_0}$ and consequently by the concavity of the function M^{-1} it follows that the points $(0, 0)$, $(t_0, M^{-1}(t_0))$ and $(2t_0, M^{-1}(2t_0))$ lie on a line. Thus the function M^{-1} is linear on the segment $[0, 2t_0]$ and therefore $\alpha_M^0 = 1/2$. Therefore by (10) it follows that $M \notin \nabla_2$.

b) The proof follows directly from a). Indeed let holds $M \in \nabla_2$, but do not holds $a((\ell_M(w), \|\cdot\|)) < 2$. Then $a((\ell_M(w), \|\cdot\|)) = 2$ and by a) it follows that $M \notin \nabla_2$ which is a contradiction.

Let holds $a((\ell_M(w), \|\cdot\|)) < 2$, but do not holds $M \in \nabla_2$. Then $M \notin \nabla_2$ and by a) it follows that $a((\ell_M(w), \|\cdot\|)) = 2$ which is a contradiction. □

For the next Corollary we will need the indices form [19].

Put $F_M(t) = \frac{tp(t)}{M(t)}$, $t \in (0, +\infty)$ where p is the right derivative of M. Let us define

$$A_M^0 = \liminf_{t \to 0^+} F_M(t), \quad B_M^0 = \limsup_{t \to 0^+} F_M(t)$$

$$A_M^{+\infty} = \liminf_{t \to +\infty} F_M(t), \quad B_M^{+\infty} = \limsup_{t \to +\infty} F_M(t)$$

The above indices are connected by the formulas ([12], p.149), ([16], p.27).

$$\frac{1}{A_M^0} + \frac{1}{B_N^0} = \frac{1}{A_N^0} + \frac{1}{B_M^0} = 1$$

and

$$\frac{1}{A_M^{+\infty}} + \frac{1}{B_N^{+\infty}} = \frac{1}{A_N^{+\infty}} + \frac{1}{B_M^{+\infty}} = 1,$$

where N is the complementary function to M.

The inequalities

$$2^{-1/A_M^0} \leq \alpha_M^0 \leq 2^{-1/B_M^0} \tag{11}$$

$$2^{-1/A_M^{+\infty}} \leq \alpha_M^{+\infty} \leq 2^{-1/B_M^{+\infty}} \tag{12}$$

hold [17]. Let us mention that inequalities (11) are proven in [17]. The proof of inequalities (12) is similar. We are sure that inequalities (12) are proven somewhere. Just for completeness we will prove (12) by using the technique from [17].

If $B_M^{+\infty} = \infty$ then clearly $\alpha_M^{+\infty} = \liminf_{u \to \infty} \frac{M^{-1}(u)}{M^{-1}(2u)} \leq 1 = 2^{-1/B_M^{+\infty}}$. Assume that $B_M^{+\infty} < \infty$. For any $\varepsilon > 0$ there exists $t_0 > 0$ such that $\frac{tp(t)}{M(t)} = F_M(t) < B_M^{+\infty} + \varepsilon$ for ever $t \in [t_0, +\infty)$. Then for any $t_0 \leq t_1 < t_2 < +\infty$ we have

$$\begin{aligned} \log \tfrac{M(t_2)}{M(t_1)} &= \int_{t_1}^{t_2} \tfrac{p(t)}{M(t)} dt \leq \int_{t_1}^{t_2} \tfrac{B_M^{+\infty}+\varepsilon}{t} dt \\ &= \log \left(\tfrac{t_2}{t_1}\right)^{B_M^{+\infty}+\varepsilon}. \end{aligned}$$

Put $t_1 = M^{-1}(u)$ and $t_2 = M^{-1}(2u)$. Thus for any $u \in [M(t_0), +\infty)$ there holds the inequality

$$\frac{M^{-1}(u)}{M^{-1}(2u)} \leq 2^{-1/B_M^{+\infty}+\varepsilon}.$$

By the arbitrary choice of $\varepsilon > 0$ it follows the proof of the right side of the inequality (12). The proof of the left side is similar.

If $\lim_{t \to 0^+} F_M(t)$ exists we denote it by C_M^0 and if $\lim_{t \to +\infty} F_M(t)$ exists we denote it by $C_M^{+\infty}$. We put $C_M^{0,+\infty} = \min\{C_M^0, C_M^{+\infty}\}$.

Corollary 2. *Let M be an Orlicz function with the Δ_2–condition and $w \in \Lambda$ is a weight sequence. Then:*

a) If F_M is an increasing function on $(0,+\infty)$, then $a(\ell_M(w), \|\cdot\|) = 2^{1/C_M^0}$;

b) If F_M is a decreasing function on $(0,+\infty)$, then $a(\ell_M(w), \|\cdot\|) = 2^{1/C_M^{+\infty}}$

c) If there is $t_0 \in (0,+\infty)$, such that F_M is a increasing on $(0, M^{-1}(t_0))$ and decreasing on $(M^{-1}(t_0),+\infty)$, then $a(\ell_M(w), \|\cdot\|) = 2^{1/C_M^{0,+\infty}}$

Proof a) If F_M is an increasing function in $(0,+\infty)$, then $C_M^0 = \lim_{t\to 0+} F_M(t)$ exists and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is increasing in $(0,+\infty)$ [20]. Then from (11) we get

$$\alpha_M^0 = 2^{-\frac{1}{C_M^0}} = \lim_{t\to 0} G(u) = \widetilde{\alpha}_M$$

and therefore $a(\ell_M(w), \|\cdot\|) = 2^{1/C_M^0}$.

b) If F_M is a decreasing function in $(0,+\infty)$, then $C_M^{+\infty} = \lim_{t\to+\infty} F_M(t)$ exists and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is decreasing in $(0,+\infty)$ [20]. Then by (11) we obtain

$$\alpha_M^{+\infty} = 2^{-\frac{1}{C_M^{+\infty}}} = \lim_{t\to+\infty} G(u) = \widetilde{\alpha}_M$$

and therefore $a(\ell_M(w), \|\cdot\|) = 2^{1/C_M^{+\infty}}$.

c) If F_M is an increasing function in $(0, M^{-1}(t_0))$, then $C_M^0 = \lim_{t\to 0+} F_M(t)$ exists and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is increasing in $(0, t_0/2)$ and if F_M is a decreasing function in $(M^{-1}(t_0),+\infty)$, then $C_M^{+\infty} = \lim_{t\to+\infty} F_M(t)$ exists and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is decreasing in $(t_0/2,+\infty)$ [20].

From (11) it follows that $\alpha_M^0 = 2^{-\frac{1}{C_M^0}}$, $\alpha_M^{+\infty} = 2^{-\frac{1}{C_M^{+\infty}}}$ and hence $\widetilde{\alpha}_M = 2^{-\frac{1}{C_M^{0,+\infty}}}$. By Theorem 4 we get that $a(\ell_M(w), \|\cdot\|) = 2^{1/C_M^{0,+\infty}}$. □

7 Examples

Example 1 Let $M_1(t) = 2|t|^p + |t|^{2p}$, $p \in [1,+\infty)$. Then $F_{M_1}(t) = 2p\left(1 - \frac{1}{t^p+2}\right)$, for $t \in [0,+\infty)$. The function F_{M_1} is an increasing function and $\lim_{t\to 0} F_{M_1}(t) = p$. By Corollary 2 we get that $a(\ell_{M_1}(w), \|\cdot\|) = \sqrt[p]{2}$.

Example 2 Let $M_2(t) = \frac{|t|^p}{\log(1+|t|)}$, $p \in [1,+\infty)$. Then $F_{M_2}(t) = p - \frac{t}{(1+t)\log(1+t)}$, for $t \in [0,+\infty)$. The function F_{M_2} is an increasing function and $\lim_{t\to 0} F_{M_2}(t) = p - 1$. By Corollary 2 we get that $a(\ell_{M_2}(w), \|\cdot\|) = 2^{\frac{1}{p-1}}$.

Example 3 Let $M_3(t) = |t|^p \log^r(1+|t|)$, $p \in [1,+\infty)$, $r \in (0,+\infty)$. Then

$$F_{M_3}(t) = p + \frac{rt}{(1+t)\log(1+t)}, \text{ for } t \in [0,+\infty).$$

The function F_{M_3} is a decreasing function and $\lim_{t\to+\infty} F_{M_3}(t) = p$. By Corollary 2 we get that $a(\ell_{M_3}(w), \|\cdot\|) = 2^{1/p}$.

Example 4 Let $q \geq 2$ and $p \in [q-1, 2q-1]$. We define the function

$$M_4(t) = \begin{cases} |t|^p(1+\log|t|), & |t| \geq 1 \\ \frac{2q-p-1}{q}|t|^q + \frac{p-q+1}{q}|t|^{2q}, & |t| \leq 1. \end{cases}$$

The function M_4 is an Orlicz function. Then

$$F_{M_4}(t) = \begin{cases} p + \frac{1}{1+\log t}, & t \geq 1 \\ 2q\left(1 - \frac{a}{2(a+bt^q)}\right), & t \in [0,1], \end{cases}$$

where $a = \frac{2q-p-1}{q}$ and $b = \frac{p-q+1}{q}$. The index function F_{M_4} is increasing on $[0,1]$ and is decreasing on $[0,+\infty)$, $\lim_{t\to 0} F_{M_4}(t) = q$ and $\lim_{t\to+\infty} F_{M_4}(t) = p$. By Corollary 2 we get that $a(\ell_{M_4}(w), \|\cdot\|) = 2^{\frac{1}{\min\{p,q\}}}$.

Example 5 Let $M_5(t) = (1+|t|)\log(1+|t|) - |t|$. Then $F_{M_5}(t) = \frac{t\log(1+t)}{(1+t)\log(1+t)-t}$ is a decreasing function on $(0,+\infty)$ and $\lim_{t\to+\infty} F_{M_5}(t) = 1$. Thus $a(\ell_{M_5}(w), \|\cdot\|) = 2$. By Corollary 1 it follows that $M_6 \notin \nabla_2$.

Competing interests

The author declares that there are no competing interests.

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References

1. Benavides, TD, Rodriguez RJ: **Some geometric coefficients in Orlicz sequence spaces**. *Nonlinear Anal.* 1993, **20(4)**:349–358.
2. Betiuk-Pilarska, A. Prus, S: **Banach lattices which are N-order uniformly noncreasy** *Journal of Mathematical Analysis and Applications* 2013, **399(2)**:459–471.
3. Borwein, JM. Sims, B: **Non–expansive mappings on Banach lattices and related topics**. *Houston J. Math.* 1984, **10**:339–356.
4. Chen, ST: **Geometry of Orlicz spaces**. *PhD thesis*, Warszawa, 1996.
5. Chui, Y. Hudzik, H: **On the Garcia–Falset coefficient in Orlicz sequence spaces equipped with the Orlicz norm**. *J. Funct. Spaces Appl.* 2000, 60–68.
6. Fihtengolts, GM: *Course in Differential and Integral Calculus II.* State Publisher Physics–Mahtematical Literature: Moskow 1962.
7. Fuentes, F. Hernandez, F: **On weighted Orlicz sequence spaces and their subspaces**. *Rocky Mountain J. Math.* 1988, **18**:585–599.

8. García–Falset, J.: **Stability and fixed points for nonexpansive mappings**. *Houston J. Math.* 1994, **20(3)**:495–506.
9. Hernandez, F: **On the galp of weighted Orlicz sequence spaces II**. *Arch. Math.* 1985, **45**:158–168.
10. Jincai, W: **The Riesz angle of Orlicz function spaces**. *Comment. Math.* 2000, **40**:209–223.
11. Kaminska, A. Mastylo, M: **The Schur and (weak) Dunford–Pettis property in Banach lattices**. *J. Australian Math. Soc.* 2002, **73**:251–278.
12. Lindenstrauss, J. Tzafriri, L: *Classical Banach spaces I, Sequence spaces.* Springer–Verlag: Berlin 1977.
13. Maleev, R. Zlatanov B: **Cotype of Weighted Orlicz Sequence Spaces**. *C. R. Acad. Bulgare Sci.* 2000, **53**:9-12.
14. Maurey, B: *Points fixes des contractions de certains faiblement compacts de* L^1. Séminaire d'Analyse Fonctionnellecv Exposé **N**° VIII: Ecole Polytechnique, Palaiseau 1981.
15. Pierats, V Ruiz, C: **On ℓ_p copies in Musielak–Orlicz sequence spaces**. *Arch. Math.* 1992, **58**:164–173.
16. Rao, M. Ren, Z: *Theory of Orlicz Spaces.* Marcel Dekker: Ney York 1991.
17. Rao, M. Ren, Z: **Packing in Orlicz sequence spaces**. *Studia Math.* 1997, **126(3)**:235–251.
18. Ruiz, C.: **On the Structure of Rosenthal's Space X_φ in Orlicz Function Spaces**. *Rocky Mountain J. Math.* 1997, **27**:1223–1238.
19. Simonenko, I: **Interpolation and extrapolation of linear operators in Orlicz spaces**. *Mat. Sb.* 1964, **63**:536–553.
20. Yan, YQ: **Some Results on Packing in Orlicz Sequence Spaces**. *Studia Math.* 2001, **147**:73–88.
21. Yan, YQ: **Riesz angles of Orlicz sequence spaces**. *Comment.Math.Univ.Carolin.* 2002, **43(1)**:133–147.
22. Yan, YQ: **On Some Expressions for the Riesz Angle of Orlicz Sequence Spaces**. *Sib. Math. J.* 2003, **44(4)**:739–747.
23. Zlatanov, B: **On Weak Uniform Normal Structure in Weighted Orlicz Sequence Spaces**. *J. Math. Anal. Appl.* 2008, **341**:1042-1054.