SOME PROPERTIES OF REFLECTION OF QUADRANGLE ABOUT POINT*

Boyan Zlatanov

Plovdiv University "Paisii Hilendarski", Faculty of Mathematics and Informatics

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ABSTRACT: We show the power of the simultaneous usage of GeoGebra and Maple for generalizing and proving of geometry problems. We present a simple school problem, where with the help of the dynamics in GeoGebra new geometric properties are recognize and then we prove them with the help of Maple. We state an open problem for an investigation. We suggest a new construction for GeoGebra that can optimize the construction process in the extended Euclidian plane.
KEYWORDS: Dynamic Geometry Software, Computer Algebra Systems, reflection about point, conic section, loci, homogenous coordinates.

1. INTRODUCTION

Dynamic Geometry Software (DGS) and Computer Algebra Systems (CAS) have been widely used in teaching mathematics, solving problems and research. A classical usage of DGS is presented in [Pec12], where the problem is visualized and the dynamics is used to recognize geometry properties like invariant points, lines, circles etc., then using this knowledge a conjecture is stated and proved or disproved. Some applications of GeoGebra for the finding of loci are made in [Ant10, GE11, Pet10, Sor10]. We would like to mention that in many cases DGS helps students not only to visualize the problem or to suggest new geometric properties, but also helps to hint ideas for the proof [FT10, KTZ13]. A good example for generalizing and discovering of new types of objects is given in [Bar11], where a new class of central cyclides is found and a full classification of them is made. Another benefit of the DGS and especially of GeoGebra is that they give a linking of Geometry and Algebra as it is shown [HJ07]. We would like to mention that DGS could help the teacher to optimize the teaching process [Cho10, KTZ13]. The project Fibonacci [***] has made a large step towards introducing of GeoGebra to the Bulgarian teachers.
There are geometric problems that are possible to be solved without to much writing, once we have observed the idea of the proof [KTZ13], but there are problems in geometry, when the solution involves analytic geometry, that requires a lot of writing and calculations [GN08, GN11]. The CAS are of great help for these type of problems as it is shown in [Ger09], where Maple is used for calculating of non trivial geometric problems. A classical geometric problem that involves a minimization is investigated in [Ger09], where Maple is used for the actual calculations of the examples, and DGS is used for the visualization of the problem. Hilbert geometry in a triangle is investigated in [MRG10]. The illustrations of some of the concepts such as Hilbert distance, projective and affine coordinates are presented with Maple. The problem of determining the minimum surface area of solids obtained when the graph of a differentiable function is revolved about horizontal lines is investigated in [Tod08]. Solutions for this problem are given with the help of Maple and several potential difficulties are identified, when using CAS. A Maple procedures based on integration and transformation methods is presented and used to evaluate signed areas and volumes. The procedures are designed with formal parameters which can be easily used or modified by instructors and students [XYS12], which shows another benefit of the CAS – the possibility to generate procedures that can be used for large classes of problems. It is shown that CAS enables to solve many elementary and non-elementary problems of classical geometry, which in the past could not be solved for the complexity of involved equations or the degree of the problem [Kar98].
Following the above ideas we present a simple school problem and with GeoGebra, we recognize new geometric properties and we prove them with the help of Maple.
The Maple file and the sketches can be downloaded from http://fmi-plovdiv.org/GetResource?id=1440.

2. PRELIMINARY RESULTS

We will start with a classical school problem, which can be solved with basic facts from the Elementary Geometry.
Problem 1: *Let ABCD be a quadrangle and the points P and P', Q and Q', R and R' be the midpoints of the segments AB and CD, BC and AD, AC and BD, respectively. Prove that the quadrangles PQP'Q', PRP'R' and QRQ'R' are parallelograms (Fig. 1).*

The solution uses the well known properties of the mid segment of a triangle (*PQ // AC // P'Q' , PQ' // BD // P' Q*).

The generalization of the above problem as like as the generalization of the main problem require the quadrangle *ABCD* to be considered as a complete quadrangle in the terms of the projective geometry:

Definition 1:*([Cox49], p. 14) Four points A, B, C, D, of which no three are collinear, are the vertices of a complete quadrangle ABCD, of which the six sides are the lines AB, AC, AD, BC, BD, CD. The intersections of opposite sides, namely, U = AB ∩ CD, U = BC ∩ AD, W = AC ∩ BD are called diagonal points and are the vertices of the diagonal triangle .*

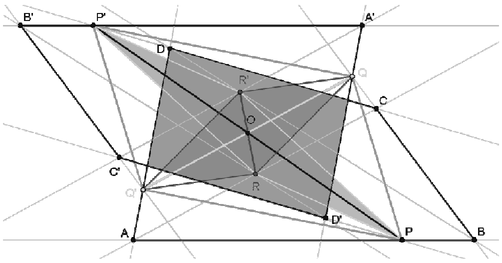

Figure 1.

The present investigation is made in the Euclidian plane extended with all its infinite points and its infinite line. Therefore we will use homogeneous coordinates. Let us remember: If *(x,y)* is a point in the Euclidean plane we will put as its homogenous coordinates in the extended Euclidian plane *(tx,ty,t)*, $t \in R \backslash \{0\}$. For any $t_1,t_2 \neq 0$ the points (t_1x, t_1y, t_1) and (t_2x, t_2y, t_2) are one and the same finite point. The points at infinity are denoted with *(x,y,0)*. For any $t_1,t_2 \neq 0$, $x^2+y^2 \neq 0$, the points $(t_1 x\ t_1 y, 0)$ and $(t_2 x, t_2 y, 0)$ are one and the same infinite point. Note that the triad *(0,0,0)* is omitted and does not represent any point. The origin is represented by *(0,0,1)*.

The equation of a line in Cartesian coordinates is $l: ax+by+c=0$ and in homogenous coordinates is $l: ax+by+ct=0$. The equation of a line *l*, passing through two points (a_1,b_1,t_1) and (a_2,b_2,t_2) is:

$$l : \begin{vmatrix} x & y & t \\ a_1 & b_1 & t_1 \\ a_2 & b_2 & t_2 \end{vmatrix} = 0 .$$

The equation of a curve of the second power in Cartesian coordinates is

$k: ax^2 + by^2 + 2cxy + 2dx + 2ey + f=0$

and its equation in homogenous coordinates is

$k: ax^2+by^2+2cxy+2dxt+2eyt+ft^2=0.$

For a quadratic form

$$k : ax^2 + by^2 + ft^2 + 2cxy + 2dxt + 2eyt = 0$$

the matrixes

$$A = \begin{pmatrix} a & c & d \\ c & b & e \\ d & e & f \end{pmatrix}, \quad A_{33} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

are used for determining the type of the curve *k*. It is degenerated if and only if *det(A)=0.* The curve *k* is: parabola if and only if $det(A_{33})=0$, hyperbola if and only if $det(A_{33})>0$, ellipse if and only if $det(A_{33})<0$.

3. MAIN PROBLEM

We will use the notation “a reflection about point”, which is used in GeoGebra. It is a synonym for “a reflection in a point” ([Cox49], p. 49]) and for “central symmetry” ([Jam92], p. 411).

It is not difficult to observe that the points *P* and *P'*, *Q* and *Q'*, *R* and *R'* introduced in Problem 1 are corresponding points for a reflection in the point *O=PP'∩QQ'∩RR'*, which is the centroid point of *ABCD.* Thus we get a natural generalization of Problem 1.

Problem 2: *Let ABCD be a complete quadrangle and Φ be a reflection in a point O. Let us denote A' B' C' D' =Φ(ABCD) and P=AB∩C'D', P' ABCD =A'B'∩CD, Q=BC∩A'D', Q'=B'C'∩AD, R=AC∩B'D', R'=A'C'∩BD . Prove that the quadrangles PP'QQ', PP'RR', QQ'RR' are parallelograms (Fig. 2.1 and Fig. 2.2).*

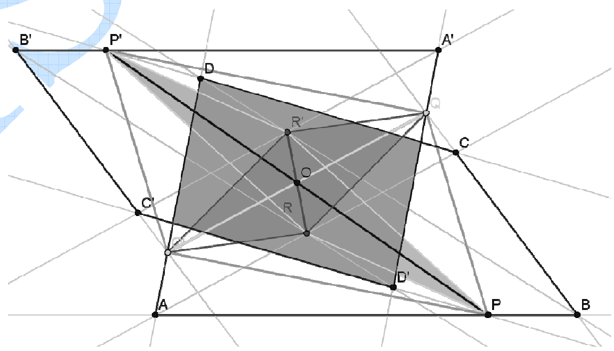

Figure 2.1.

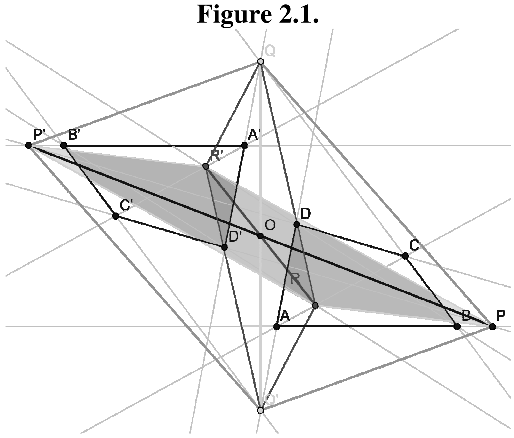

Figure 2.2.

Solution: Let *O* be an arbitrary finite point in the plane of the quadrangle *ABCD.* The reflection *Φ* has the property $\Phi = \Phi^{-1}$. That is way from the equalities

$$\Phi(P = AB \cap C'D') = \Phi(AB) \cap \Phi(C'D') \\ = A'B' \cap CD = P',$$

(1) $$\Phi(Q = BC \cap A'D') = \Phi(BC) \cap \Phi(A'D') \\ = B'C' \cap AD = Q',$$

$$\Phi(R = AC \cap B'D') = \Phi(AC) \cap \Phi(B'D') \\ = A'C' \cap BD = R'$$

it follows that the segments *PP'*, *QQ'*, *RR'* have a common midpoint *O*. Therefore the quadrangles *PQP'Q'*, *QRQ'R'*, *PRP'R'* are parallelograms (Fig 2). □

If we apply dynamics on the free point *O* in Problem 2, we observe that the parallelograms *PQP'Q'*, *QRQ'R'*, *PRP'R'* are changing their position, the length of their sides and the angles, but they remain either parallelograms or **their vertexes become collinear**. This observation leads to the following problem:

Main problem: *Let ABCD be a complete quadrangle and Φ be a reflection in a points a center O. Let us denote A' B' C' D' =Φ(ABCD) and P=AB∩C'D', P' ABCD =A'B'∩CD, Q=BC∩A'D', Q'=B'C'∩AD, R=AC∩B'D', R'=A'C'∩BD . Find the loci of the point O, when the parallelograms PQP'Q', QRQ'R', PRP'R' degenerate into segments that are laying at one line.*

An equivalent formulation of the **Main problem** is: *Let ABCD be a complete quadrangle and Φ be reflection in a point O. Let us denote A' B' C' D' =Φ(ABCD) and P=AB∩C'D', P' ABCD =A'B'∩CD, Q=BC∩A'D', Q'=B'C'∩AD, R=AC∩B'D', R'=A'C'∩BD . Find the loci of the point O, when the points P, Q, R, P', Q', R' are collinear (Fig. 3).*

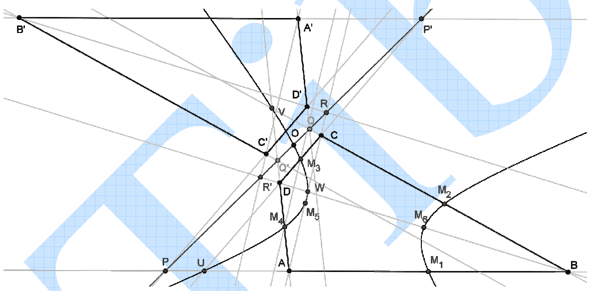

Figure 3.

Solution: We will use GeoGebra and Maple for our investigation of the problem. In the sequel we will use the sign “>” to indicate the Maple input. We will use Maple for the calculation to find the loci of the point *O*, when the points *P, Q, R, P', Q', R'* are collinear.

Without loss of generality we can assume that the homogenous coordinates of the points *A* and *B* are:

(2) $A(0,0,1), B(1,0,1)$.

If not we can translate the quadrangle *ABCD*, so that the point *A* to coincide with *(0,0,1)*, then we can rotate the figure around *A* until the point *B* coincides with *(b,0,1)* and at the end we can use a homothety with center *A(0,0,1)*, so that the point *B* to coincide with *(1,0,1)*. By all these transformations the construction will be preserved. Let us mention that with the help of Maple it is possible to make all the calculations without assuming that *A(0, 0, 1), B(1, 0, 1)*.

The easiest way to write the equations of lines and to check if three points are collinear is by using determinants. That is why we will use the package “LinearAlgebra” in Maple. Let us put

(3) $C(c_1, c_2, 1), D(d_1, d_2, 1), O(x, y, 1).$

$> with(LinearAlgebra) : a_1 := 0 : a_2 := 0 : b_1 := 1 : b_2 := 0 :$

With the choice of the third coordinate of the points *A, B, C, D, O* to be equal to *1* we reject the opportunity any one of these points to be infinite one. Indeed, when the centre *O* of reflection becomes infinite point, then the reflection *Φ* in a point converts into translation, which is not involutory collineation, which is very important for our investigation.

The finite points *A, B, C, D* are vertexes of a quadrangle, when any three of them are not collinear. Three points are collinear if and only if the determinant of the matrixes of their coordinates is equal to zero. We define the determinants of the matrixes with elements the homogeneous coordinates of the triads of points *(A, B, C), (A, B, D), (B, C, D), (A, C, D)*:

$> Determinant(Matrix(3,[[a_1,a_2,1],[b_1,b_2,1],[c_1,c_2,1]])) \neq 0;$

$> Determinant(Matrix(3,[[a_1,a_2,1],[b_1,b_2,1],[d_1,d_2,1]])) \neq 0;$

$> Determinant(Matrix(3,[[b_1,b_2,1],[c_1,c_2,1],[d_1,d_2,1]])) \neq 0;$

$> Determinant(Matrix(3,[[a_1,a_2,1],[c_1,c_2,1],[d_1,d_2,1]])) \neq 0;$

Consequently the quadrangle *ABCD* exists if and only if the following conditions are satisfied:

(4) $$c_2 d_2 \neq 0, c_1 d_2 - c_2 d_1 \neq 0, \\ c_1 d_2 - c_2 d_1 + c_2 - d_2 \neq 0.$$

We do not exclude the case, when the quadrangle is not a convex figure. The convexity of the quadrangle will be used later to determine the type of the loci of *O*.

Let us denote the coordinates of the vertexes of the image *A'B'C'D' = Φ(ABCD)* by

(5) $$A'(a_3, a_4, 1), B'(b_3, b_4, 1), \\ C'(c_3, c_4, 1), D'(d_3, d_4, 1).$$

Using (2) and (3) we define in Maple the coordinates of the vertexes of the image *A'B'C'D'*

$> a_3 := 2 \cdot x - a_1 : a_4 := 2 \cdot y - a_2 : b_3 := 2 \cdot x - b_1 : b_4 := 2 \cdot y - b_2 :$

$> c_3 := 2 \cdot x - c_1 : c_4 := 2 \cdot y - c_2 : d_3 := 2 \cdot x - d_1 : d_4 := 2 \cdot y - d_2 :$

and we obtain *A'(2x, 2y, 1), B'(2x-1, 2y, 1) , C'(2x-* c_1*, 2y –* c_2*, 1) , D'(2x-* d_1*, 2y –* d_2 *, 1).*

The points *P, Q* and *R* are intersection points of the pairs of lines *(AB, C'D'), (AD, B'C')* and *(AC, B'D'),* respectively. Therefore we need to find the equations of the lines *AB, C'D', AD, B'C', AC* and *B'D'*. We define the functions $G_i : R^3 \to R$*, i=1,2,…6,* which are the determinants, that define the equations of the lines *AB*, *C'D'*, *AD*, *B'C' AC* and *B'D'*:

$> G_1 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[a_1,a_2,1],[b_1,b_2,1]]))$:

$> G_2 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[c_3,c_4,1],[d_3,d_4,1]]))$:

$> G_3 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[b_1,b_2,1],[c_1,c_2,1]]))$:

$> G_4 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[a_3,a_4,1],[d_3,d_4,1]]))$:

$> G_5 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[a_1,a_2,1],[c_1,c_2,1]]))$:

$> G_6 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[b_3,b_4,1],[d_3,d_4,1]]))$:

We find the following equations of the lines *AB*, *C'D'*, *BC*, *A'D'*, *AC* and *B'D'* in homogenous coordinates.

$AB: \quad v=0,$

$C'D': \quad (d_2-c_2)u+(c_1-d_1)v$
$\quad +(c_1d_2-d_1c_2+x(2c_2-2d_2)+y(2d_1-2c_1))t=0,$

$BC: \quad -c_2u+(c_1-1)v+c_2t=0,$

$A'D': \quad d_2u-d_1v+(2yd_1-2xd_2)t=0,$

$AC: \quad -c_2u+c_1v=0,$

$B'D': \quad d_2u+(1-d_1)v$
$\quad +(d_2-2xd_2-2y+2d_1y)t=0$

By solving the systems of equations

$$(6)\quad \begin{cases} AB: v=0 \\ C'D': (d_2-c_2)u+(c_1-d_1)v \\ +(c_1d_2-d_1c_2-2xd_2+2xc_2+2yd_1-2yc_1)t=0, \end{cases}$$

$$(7)\quad \begin{cases} BC: -c_2u+(c_1-1)v+c_2t=0 \\ A'D': d_2u-d_1v+(2yd_1-2xd_2)t=0 \end{cases}$$

and

$$(8)\quad \begin{cases} AC: -c_2u+c_1v=0 \\ B'D': d_2u+(1-d_1)v \\ +(d_2-2xd_2-2y+2d_1y)t=0 \end{cases}$$

we will obtain the homogeneous coordinates of the points *P*, *Q*, and *R*, respectively.

Using Maple we solve the above systems of equations. We denote the solutions of the systems (6), (7) and (8) with *p, q* and *r*, respectively.

$> p := solve(\{G_1(u,v,t)=0, G_2(u,v,t)=0\},[u,v]);$

$> q := solve(\{G_3(u,v,t)=0, G_4(u,v,t)=0\},[u,v]);$

$> r := solve(\{G_5(u,v,t)=0, G_6(u,v,t)=0\},[u,v]);$

From the solutions *p, q* and *r* we obtain the following homogenous coordinates of the points *P*, *Q* and *R*:

$P=(Px,Py,Pt)$
$\quad =(c_1d_2-2d_2x+2c_2x-2c_1y+2d_1y,\, 0,\, c_2-d_2)$

$Q=(Qx,Qy,Qt)$
$\quad =(-(2xd_2-2d_1y-2c_1d_2x+2c_1d_1y+d_1c_2),$
$\quad -c_2(2d_1y-2d_2x+d_2),\, c_1d_2-d_1c_2-d_2)$

and

$R=(Rx,Ry,Rt)=(-c_1(d_2-2y-2d_2x+2d_1y)\,,$
$\quad -c_2(d_2-2y+2d_2x+2d_1y),\, c_1d_2-d_1c_2+c_2)$

We define these coordinates in Maple.

$> Px := c_1\cdot d_2-2\cdot x\cdot d_2+2\cdot x\cdot c_2-2\cdot y\cdot c_1+2\cdot y\cdot d_1-d_1c_2$:

$> Py := 0 : Pt := c_2-d_2$:

$> Qx := 2\cdot y\cdot d_1-2\cdot x\cdot d_2+2\cdot x\cdot c_1\cdot d_2-2\cdot y\cdot c_1\cdot d_1-d_1\cdot c_2$:

$> Qy := c_2(2\cdot x\cdot d_2-2\cdot y\cdot d_1-d_2) : Qt := c_1\cdot d_2-d_1\cdot c_2-d_2$:

$> Rx := c_1\cdot(2\cdot y-d_2+2\cdot x\cdot d_2-2\cdot y\cdot d_1)$:

$> Ry := c_2(2\cdot y-d_2+2\cdot x\cdot d_2-2\cdot y\cdot d_1) : Rt := c_1\cdot d_2-d_1\cdot c_2+c_2$:

According to the properties of any homology the triads of points *(O,P,P')*, *(O,Q,Q')* and *(O,R,R')* are always collinear. Hence in order to solve the problem it is enough to find the conditions when the triads of points *(O,P,Q)*, *(O,P,R)* and *(O,Q,R)* are collinear. That is why we define the functions $F_i:R^2\to R$, $i=1,2,3$, which are the determinants of the matrixes of the coordinates of the triads *(O,P,Q)*, *(O,P,R)*, *(O,Q,R)*.

$> F_1 := (x,y) \to Determinant(Matrix(3,[[x,y,1],[Px,Py,Pt],[Qx,Qy,Qt]]))$:

$> F_2 := (x,y) \to Determinant(Matrix(3,[[x,y,1],[Px,Py,Pt],[Rx,Ry,Rt]]))$:

$> F_3 := (x,y) \to Determinant(Matrix(3,[[x,y,1],[Qx,Qy,Qt],[Rx,Ry,Rt]]))$:

With the help of the function "*collect*" we simplify the quadratic forms $F_i(x,y)$ and we see that $F_i(x,y)=F_j(x,y)$ for any $i\neq j$, $i,j=1,2,3$.

$> collect(F_1(x,y),\{x,y\}); collect(F_2(x,y),\{x,y\}); collect(F_3(x,y),\{x,y\});$

$$\begin{aligned} F_i(x,y) &= (2c_2^2d_2-2c_2d_2^2)x^2 \\ &+(2c_1^2d_2-2c_1d_2+2d_1c_2-2c_2d_1^2)y^2 \\ &+(4d_1c_2d_2-4c_1c_2d_2)yx \\ &(c_2d_2^2-c_2^2d_2+2c_1c_2d_2^2-2d_1d_2c_2^2)x \\ &+(2c_1c_2d_2-2d_1c_2d_2)y \\ &+(d_1^2c_2^2-d_1c_2^2+c_1d_2^2-c_1^2d_2^2)y \\ &-c_1c_2d_2^2+d_1d_2c_2^2 \end{aligned}$$

Therefore we get that the collinearity of one of the triads of points *(O,P,Q)*, *(O,P,R)*, *(O,Q,R)* is equivalent to the collinearity of the other two triads of points and is equivalent to the collinearity of the seven points *(O,P,P',Q,Q',R,R')*.

Let us denote with *k* the curve defined by the equation $F_1(x,y)=0$.

We define the function $F(x,y,t):R^3\to R$, which will be the function that determines the quadratic form *k* in homogeneous coordinates.

$$\begin{aligned} k: F(x,y,t) \;=\; & 2d_2c_2(c_2-d_2)x^2 \\ &+2(d_1c_2-c_1d_2+c_1^2d_2-d_1^2c_2)y^2 \\ &+c_2d_2(d_1c_2-c_1d_2)t^2 \\ &+4c_2d_2(c_1-d_1)xy \\ &+c_2d_2(d_2-c_2+2c_1d_2-2d_1c_2)xt \\ &+2(d_1d_2c_2-c_1c_2d_2)yt \\ &+(d_1c_2^2-c_1d_2^2+c_1^2d_2^2-d_1^2c_2^2)yt \end{aligned}$$

Thus we obtain that the parallelograms *PQP'Q'* , *QRQ'R'* , *PRP'R'* degenerate into segments that are laying at one line when the center $O(x,y,1)$ of the reflection Φ lays on the curve of the second power $k{:}F(x,y,t)=0$.

$$
\begin{aligned}
> F := \ & (x,y,t) \to 2\cdot d_2\cdot c_2(c_2-d_2)\cdot x^2 \\
& +2\cdot(d_1\cdot c_2-c_1\cdot d_2+c_1^2\cdot d_2-d_1^2\cdot c_2)\cdot y^2 \\
& +c_2\cdot d_2\cdot(c_2\cdot d_1-d_2\cdot c_1)\cdot t^2+4\cdot c_2\cdot d_2\cdot(d_1-c_1)\cdot x\cdot y \\
& +c_2\cdot d_2\cdot(d_2-c_2+2\cdot c_1\cdot d_2-2\cdot d_1\cdot c_2)\cdot x\cdot t \\
& +2(c_1\cdot c_2\cdot d_2-d_1\cdot d_2\cdot c_2)\cdot y\cdot t \\
& +(c_1\cdot d_2^2-d_1\cdot c_2^2-c_1^2\cdot d_2^2+d_1^2\cdot c_2^2)\cdot y\cdot t:
\end{aligned}
$$

The easiest way find the curve k is by finding five points that are laying on it. By experiments with the sketch in GeoGebra we have observed that the midpoints of the sides and the diagonal points of the quadrangle $ABCD$ are supposed to lay on the curve k. Let denote by $M_i(m_i,n_i,t_i)$, $i=1,2,\ldots,6$ the midpoints of the segments AB, BC, CD, DA, AC, BD, respectively.

$$> m_1 := \frac{a_1+b_1}{2} : n_1 := \frac{a_2+b_2}{2} : t_1 := \frac{1+1}{2} : m_2 := \frac{b_1+c_1}{2} :$$
$$> n_2 := \frac{b_2+c_2}{2} : t_2 := \frac{1+1}{2} : m_3 := \frac{c_1+d_1}{2} : n_2 := \frac{c_2+d_2}{2} :$$
$$> t_3 := \frac{1+1}{2} : m_4 := \frac{a_1+d_1}{2} : n_4 := \frac{a_2+d_2}{2} : t_4 := \frac{1+1}{2} :$$
$$> m_5 := \frac{a_1+c_1}{2} : n_5 := \frac{a_2+c_2}{2} : t_5 := \frac{1+1}{2} :$$
$$> m_6 := \frac{b_1+d_1}{2} : n_6 := \frac{b_2+d_2}{2} : t_6 := \frac{1+1}{2} :$$

The calculation in Maple shows that $F(m_i,n_i,t_i)=0$ for every $i=1,2,\ldots,6$ and consequently $M_i \in k$ for every $i=1,2,\ldots,6$.

$> print(factor(F(m_1,n_1,t_1)), factor(F(m_2,n_2,t_2)), factor(F(m_3,n_3,t_3)));$
$> print(factor(F(m_4,n_4,t_4)), factor(F(m_5,n_5,t_5)), factor(F(m_6,n_6,t_6)));$

We will show that the three diagonal points $U=AB\cap CD$, $V=BC\cap AD$, $W=AC\cap BD$ lay on the curve k, too.

We have the equations of the lines $AB{:}G_1(x,y,t)=0$, $BC{:}\ G_3(x,y,t)=0$ and $AC{:}G_5(x,y,t)=0$. We will define the lines AC, AD and BD in a similar fashion.

$> G_7 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[c_1,c_2,1],[d_1,d_2,1]])):$
$> G_8 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[a_1,a_2,1],[d_1,d_2,1]])):$
$> G_9 := (u,v,t) \to Determinant(Matrix(3,[[u,v,t],[b_1,b_2,1],[d_1,d_2,1]])):$

The coordinates of the points U, V, W are the solutions of the following systems, respectively:

$$\left|\begin{matrix} G_1(x,y,t)=0 \\ G_7(x,y,t)=0 \end{matrix}\right., \quad \left|\begin{matrix} G_3(x,y,t)=0 \\ G_8(x,y,t)=0 \end{matrix}\right., \quad \left|\begin{matrix} G_5(x,y,t)=0 \\ G_9(x,y,t)=0. \end{matrix}\right.$$

We solve the above systems of linear equations in Maple.

$> u1 := solve(\{G_1(u,v,t)=0, G_7(u,v,t)=0\},[u,v]);$
$> v1 := solve(\{G_8(u,v,t)=0, G_3(u,v,t)=0\},[u,v]);$
$> w1 := solve(\{G_9(u,v,t)=0, G_5(u,v,t)=0\},[u,v]);$

We get the homogeneous coordinates of the points U (Ux, Uy, Ut), V (Vx, Vy, Vt) and W (Wx, Wy, Wt).

$> Ux := d_1\cdot c_2-c_1\cdot d_2 : Uy := 0 : Ut := c_2-d_2 :$
$> Vx := -d_1\cdot c_2 : Vy := -c_2\cdot d_2 : Vt := c_1\cdot d_2-d_1\cdot c_2-d_2 :$
$> Wx := c_1\cdot d_2 : Wy := d_2\cdot c_2 : Wt := c_1\cdot d_2-d_1\cdot c_2+c_2 :$

With the help of Maple we calculate $F(Ux, Uy, Ut)=0$, $F(Vx, Vy, Vt)=0$ and $F(Wx, Wy, Wt)=0$.

$> print(factor(F(Ux,Uy,Ut)), factor(F(Vx,Vy,Vt)), factor(F(Wx,Wy,Wt)));$

We will investigate the type of the curve k with the help of the matrixes A and A_{33}. We define in Maple matrixes A and A_{33} and we calculate their determinants.

$> a := 2\cdot d_2\cdot c_2\cdot(d_2-c_2):$
$> b := 2\cdot(c_1\cdot d_2-d_1\cdot c_2+d_1^2\cdot c_2-c_1^2\cdot d_2):$
$> c := 4\cdot c_2\cdot d_2\cdot(c_1-d_1):$
$> d := c_2\cdot d_2\cdot(c_2\cdot(1+2\cdot d_1)-d_2\cdot(1+2\cdot c_1)):$
$> e := c_2\cdot d_1\cdot(2\cdot d_2-c_2\cdot(d_1-1))+c_1\cdot d_2\cdot(d_2\cdot(c_1-1)-2\cdot c_2):$
$> f := c_2 d_2\cdot(c_1\cdot d_2-d_1\cdot c_2):$

$$> l := Determinant\left(Matrix\left(3,\left[\left[a,\frac{c}{2},\frac{d}{2}\right],\left[\frac{c}{2},b,\frac{e}{2}\right],\left[\frac{d}{2},\frac{e}{2},f\right]\right]\right)\right):$$

$$> l_{33} := Determinant\left(Matrix\left(2,\left[\left[a,\frac{c}{2}\right],\left[\frac{c}{2},b\right]\right]\right)\right)$$

The calculation in Maple gives us

$\det(A) = \dfrac{B_1B_2B_3B_4B_5B_6}{2}$, where $B_1 = c_2d_2$,

$B_2 = c_2-d_2$, $B_3 = c_1d_2+c_2-d_1c_2-d_2$, $B_4 = c_1d_2-d_1c_2$,
$B_5 = c_1d_2-d_1c_2-d_2$, $B_6 = c_1d_2-d_1c_2+c_2$.

The curve k is no degenerate if and only if $\det(A)\neq 0$. Taking into account (4) we obtain that $det(A)=0$ if and only if

$$(c_2-d_2)(c_1d_2-d_1c_2-d_2)(c_1d_2-d_1c_2+c_2)=0.$$

I) $c_2-d_2=0$ if and only if the pairs of opposite sides AB and CD are parallel.

II) $c_1d_2-d_1c_2-d_2=0$ if and only if the pairs of opposite sides AC and BD are parallel.

III) $c_1d_2-d_1c_2+c_2=0$ if and only if the pairs of opposite sides AD and BC are parallel.

Let there holds case I). We put $d_2=c_2$ and we find the function F.

$> d_2 := c_2 : factor(F(x,y,t));$

$$F(x,y,t) = c_2(c_1-d_1)(c_2t-2y).$$
$$(c_1y+c_2t-2c_2x+d_1y-y)$$

Because of (4) and $C\neq D$ it follows that $c_2(d_1-c_1)\neq 0$. Hence F consists of two intersecting lines g_1: $y(c_1+d_1-1)-2c_2x+c_2t=0$ and g_2: $2y-c_2t=0$. It is easy to confirm the fact M_1, $M_3 \in g_1$ and M_2, $M_4 \in g_2$. Indeed, we define the functions, that determine the lines g_1 and g_2

$> g_1 : (x,y,t) \to (c_1+d_1-1)\cdot y-2\cdot c_2\cdot x+c_2\cdot t :$
$> g_2 : (x,y,t) \to 2\cdot y-c_2\cdot t :$

to check that M_1, $M_3 \in g_1$ and M_2, $M_4 \in g_2$.

We get that $g_1(m_1,n_1,t_1))=0$, $g_1(m_3,n_3,t_3))=0$, $g_2(m_2,n_2,t_2)=0$ and $g_2(m_4,n_4,t_4)=0$. Therefore the point $g_1 \cap g_2$ is the centroid of $ABCD$.

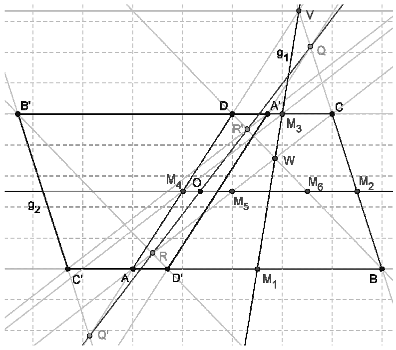

Figure 4.1.

I.1) It is easy to observe that the points of the line g_2: $y=c_2/2$ do not satisfy the conditions of Problem 3, because in this case the pairs of lines *(AB, C'D')* and *(A'B', CD)* are coinciding lines (Fig. 4.1) and the points *P* and *P'* are not clearly defined.

We can prove this observation not only with elementary geometry but with the help of Maple. The lines *AB* and *CD* are parallel if and only if $d_2=c_2$ and $O\in g_2$ if and only if $y=c_2/2$. We get that in this case the lines

AB: $G_1(u,v,t)=v=0$ and *C'D'*: $G_2(u,v,t)=(c_1-d_1)v=0$ coincide.

$$> d_2 := c_2 : y := \frac{c_2}{2} : G_1(u,v,t); G_2(u,v,t);$$

It can be observed in this case that the points *(O, Q, R)* are collinear, because $\begin{vmatrix} x & y & 1 \\ Qx & Qy & Qt \\ Rx & Ry & Rt \end{vmatrix} = 0 \cdot$

$$> Determinant\left(Matrix\left(3,\left[[x,y,1],[Qx,Qy,Qt],[Rx,Ry,Rt]\right]\right)\right);$$

I.2) Let *O* lays on the line g_1: $y(c_1+d_1-1)-2c_2x+c_2t=0$, except the point $g_1 \cap g_2$. Now we will show that the points *P*, *Q* and *R* are collinear (Figure 4.2).

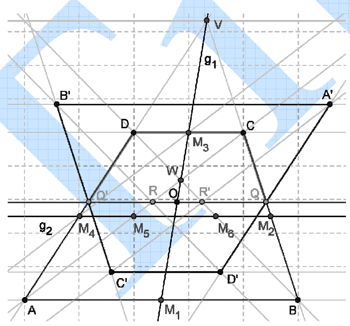

Figure 4.2.

Really the point *P* is an intersection point of two different parallel lines *AB* and *C'D'*. Thus the homogenous coordinates of the infinity point *P* are *(1,0,0)*. The solution of the equation

$$(9) \begin{vmatrix} Px & Py & Pt \\ Qx & Qy & Qt \\ Rx & Ry & Rt \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ Qx & Qy & Qt \\ Rx & Ry & Rt \end{vmatrix} = 0$$

gives the condition

$$2 \cdot c_2^2 \cdot (c_2 - 2 \cdot c_2 \cdot x + (c_1 - 1 + d_1) \cdot y) = 0$$

for the points *P*, *Q* and *R* to be collinear i.e. the vector *QR* to be collinear with the vector *p(1,0)*.

$$> unassign('x') : unassign('y') : unassign('t') :$$

$$> Determinant\left(Matrix\left(3,\left[[1,0,0],[Qx,Qy,Qt],[Rx,Ry,Rt]\right]\right)\right) = 0;$$

From (9) we get that the points *P*, *Q* and *R* are collinear if and only if $y(c_1+d_1-1)-2c_2x+c_2t=0$ and therefore *P*, *Q* and *R* are collinear if and only if the centre *O* of the reflection *Φ* in a point lays on the line g_1:$y(c_1+d_1-1)-2c_2x+c_2t=0$, except the point $g_1 \cap g_2$.

Let there holds case II), i.e. $c_1d_2-d_1c_2-d_2=0$. We will consider two subcases:

a) $c_1 \neq 1$; b) $c_1 = 1$.

Let there holds a). We put $d_2 = \dfrac{d_1c_2}{c_1-1}$ and we find

$$> d_2 := \frac{d_1 \cdot c_2}{c_1 - 1} c_2 : factor(F(x,y,t));$$

$$F(x,y,t) = \frac{d_1c_2g_3(x,y,t)g_4(x,y,t)}{(c_1-1)^2},$$

where g_3 : $(2c_1-2)y - 2xc_2 + c_2t=0$ and g_4 : $(c_12-c_1d_1-1+d_1)y+(d_1c_2-c_1c_2+c_2)x+d_1c_2t=0$.

Because of (4) and $D\neq A$ it follows that $d_1.c_2\neq 0$ and consequently *F* consists of the two intersecting lines g_3, g_4. It is easy to confirm that M_1, $M_3\in g_3$, and M_2, $M_4\in g_4$. Indeed, we define the functions, that determine the lines g_3 and g_4

$$> g_3 : (x,y,t) \to (2 \cdot c_1 - 2) \cdot y - 2 \cdot c_2 \cdot x + c_2 \cdot t :$$

$$> g_4 : (x,y,t) \to (c_1^2 - c_1 \cdot d_1 - 1 + d_1) \cdot y - c_2 \cdot (d_1 + 1 - c_1) \cdot x - c_2 \cdot d_1 \cdot t :$$

to check whether M_1, $M_3\in g_3$, and M_2, $M_4\in g_4$. We calculate

$$> factor(g_3(m_1,n_1,t_1)); factor(g_3(m_3,n_3,t_3));$$

$$> factor(g_4(m_2,n_2,t_2)); factor(g_4(m_4,n_4,t_4));$$

Therefore the point $g_3 \cap g_4$ is the centroid of *ABCD*.

II.a.1) It is easy to observe that the points of the line g_3 do not satisfy the conditions of Problem 3, because in this case the pairs of lines *(BC, A'D')* and *(AD, B'C')* are coinciding lines (Fig. 5.1) and the points *Q* and *Q'* are not clearly defined.

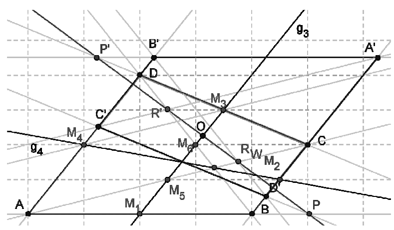

Figure 5.1.

We can prove this observation not only with elementary geometry but with the help of Maple. The lines *AD* and *BC* are parallel if and only if c_1d_2-

d_1c_2-d_2=0 and $O \in g_3$ if and only if $(2c_1-2)y - 2xc_2 + c_2 = 0$. We get that in this case the lines
BC: $G3(u, v, t)=(c_1-1)v-c_2u+c_2t=0$
and

$$A'D': G4(u,v,t) = \frac{-d_1}{c_1-1}\left((c_1-1)v - c_2u + c_2t\right) = 0$$

coincide.

$$> d_2 := \frac{d_1 \cdot c_2}{c_1-1} : y := \frac{2 \cdot c_2 \cdot x - c_2}{2 \cdot c_1 - 2} : G_3(u,v,t); G_4(u,v,t);$$

It can be observed in this case that the points *(O,R,P)* are collinear, because the determinant

$$> Determinant\left(Matrix\left(3, \left[[x, y, 1], [Px, Py, Pt], [Rx, Ry, Rt]\right]\right)\right);$$

is equal to zero.

II.a.2) Let O lays on the line g_4 : $y(c_1^2 - c_1d_1 - 1) + x(c_2 + c_2d_1 - c_1c_2) - d_1c_2t = 0$, except the point $g_3 \cap g_4$. Now we will show that the points P,Q and R are collinear (Fig. 5.2).

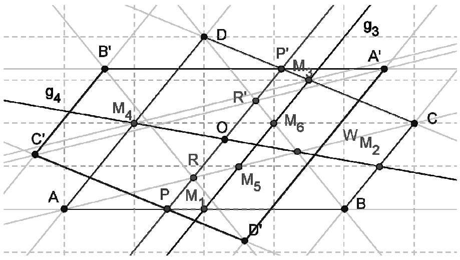

Figure 5.2.

Really the point Q is an intersection point of two different parallel line $A'D'$ and BC. Thus the homogenous coordinates of the infinite point Q are $(c_1-1, c_2, 0)$. The solution of the equation

$$(10) \quad \begin{vmatrix} c_1-1 & c_2 & 0 \\ Rx & Ry & Rt \\ Px & Py & Pt \end{vmatrix} = 0$$

gives the condition the points P, Q and R to be collinear i.e. the vector RP to be collinear with the vector $q(c_1-1, c_2)$.

From (10) we get that the points P, Q and R are collinear if and only if

$$y(c_1^2 - c_1d_1 + d_1 - 1) + x(c_2 + c_2d_1 - c_1c_2) - td_1c_2 = 0$$

and therefore P, Q and R are collinear if and only if the centre O of the reflection Φ in a point lays on the line g_4, except the point $g_3 \cap g_4$.

$$> unassign('x') : unassign('y') : unassign('t') :$$

$$> Determinant\left(Matrix\left(3, \left[[c_1-1, c_2, 0], [Rx, Ry, Rt], [Px, Py, Pt]\right]\right)\right);$$

Let there holds b). Then $d_1c_2=0$. Because of (4) it follows that $d_1=0$. We find the function F

$$> unassign('d2') : unassign('c2') :$$

$$> c_1 := 1 : d_1 := 0 : factor(F(x, y, t));$$

Because of (4) it follows that F consists of two intersecting lines g_5 : $2x - t=0$ and g_6 : $2y+(d_2-c_2)x - d_2t=0$. It is easy to confirm that M_1, $M_3 \in g_5$, and M_2, $M_4 \in g_6$. Indeed, we define the functions, that determine the lines g_5 and g_6

$$> g_5 : (x, y, t) \to 2 \cdot x - t :$$

$$> g_6 : (x, y, t) \to 2 \cdot y + (d_2 - c_2) \cdot x - d_2 \cdot t :$$

to check whether M_1, $M_3 \in g_5$, and M_2, $M_4 \in g_6$.

$$> factor(g_5(m_1, n_1, t_1)); factor(g_5(m_3, n_3, t_3));$$

$$> factor(g_6(m_2, n_2, t_2)); factor(g_6(m_4, n_4, t_4));$$

Therefore the point $g_5 \cap g_6$ is the centroid of *ABCD*.

II.b.1) It is easy to observe that the points of the line g_5 do not satisfy the conditions of Problem 3, because in this case the pairs of lines *(BC, A'D')* and *(AD, B'C')* are coinciding lines (Fig. 5.3) and the points Q and Q' are not clearly defined. We can prove this observation not only with elementary geometry but with the help of Maple.

The lines AD and BC are parallel and $O \in g_5$ if and only if $2xc_2 - c_2t=0$. We get that in this case the lines BC: $G3(u, v, t)=u-t=0$ and $A'D'$: $G4(u, v, t)=u-t=0$ coincide.

$$> c_1 := 1 : d_1 := 0 : x := \frac{1}{2} : factor(G_3(u,v,t)); factor(G_4(u,v,t));$$

It can be observed in this case that the points *(O, R, P)* are collinear because the determinant

$$> Determinant\left(Matrix\left(3, \left[[x, y, 1], [Px, Py, Pt], [Rx, Ry, Rt]\right]\right)\right);$$

is equal to zero.

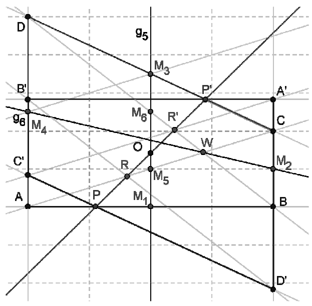

Figure 5.3.

II.b.2) Let O lays on the line g_6: $2y+(d_2-c_2)x - d_2t=0$, except the point $g_5 \cap g_6$. Now we will show that the points P, Q and R are collinear (Fig. 5.4).

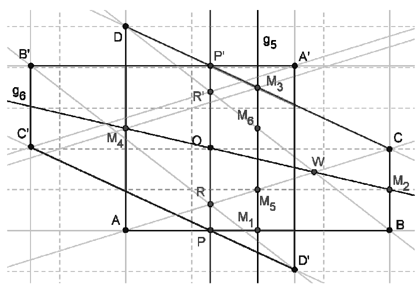

Figure 5.4.

Really the point Q is an intersection point of two different parallel lines $A'D'$ and BC. Thus the homogenous coordinates of the infinite point Q are $(0, 1, 0)$. The solution of the equation

$$(11)\quad \begin{vmatrix} 0 & 1 & 0 \\ Rx & Ry & Rt \\ Px & Py & Pt \end{vmatrix} = 0$$

gives the condition the points P,Q and R to be collinear i.e. the vector RP to be collinear with the vector $q_1(0,1)$.

From (11) we get that the points P, Q and R are collinear if and only if $2y+(d_2-c_2)x - d_2t=0$ and therefore P, Q and R are collinear if and only if the centre O of the reflection Φ in a point lays on the line g_6, except the point $g_5 \cap g_6$.

$> unassign('x') : unassign('y') : unassign('t') :$

$> factor\left(Determinant\left(Matrix\left(3,\left[\left[0,1,0\right],\left[Px,Py,Pt\right],\left[Rx,Ry,Rt\right]\right]\right)\right)\right) = 0;$

Let there holds III), i.e. $c_1d_2-d_1c_2+c_2=0$. We will consider two subcases: a) $c_1 \neq 0$; b) $c_1=0$.

Let there holds a). We put $d_2 = \dfrac{(d_1-1)c_2}{c_1}$ and we find the function F.

$> unassign('d1') : unassign('c1') :$

$> d_2 := \dfrac{(d_1-1)\cdot c_2}{c_1} : factor(F(x,y,t));$

Because of (4) and $B \neq D$ it follows that F consists of two intersecting lines g_7: $2c_1y-2c_2x+c_2t=0$ and g_8 : $(c_1^2-c_1-c_1d_1)y+(c_2d_1-c_1c_2-c_2)x+c_1c_2t=0$. It is easy to confirm the fact $M_1, M_3, M_2, M_4 \in g_7$ and $M_5, M_6 \in g_8$. Indeed, we define the function, that determine the lines g_7 and g_8

$> g_7 : (x,y,t) \to 2\cdot c_1 \cdot y - 2\cdot c_2 \cdot x - c_2 \cdot t :$

$> g_8 : (x,y,t) \to (c_1^2 - c_1 \cdot d_1 - c) \cdot y + (c_2 \cdot d_1 - c_1 \cdot c_2 - c_2) \cdot x - c_1 \cdot c_2 \cdot t :$

to check whether $M_1, M_3, M_2, M_4 \in g_5$ and $M_5, M_6 \in g_6$

$> factor(g_7(m_1,n_1,t_1)); factor(g_7(m_2,n_2,t_2));$

$> factor(g_7(m_3,n_3,t_3)); factor(g_7(m_4,n_4,t_4));$

$> factor(g_8(m_5,n_5,t_5)); factor(g_8(m_6,n_6,t_6));$

III.a.1) It is easy to observe that the points of the line g_7:$2c_1y-2c_2x+c_2t=0$ do not satisfy the conditions of Problem 3, because in this case the pairs of lines $(AC, B'D')$ and $(A'C', BD)$ are coinciding lines (Fig. 6.1) and the points R and R' are not clearly defined.

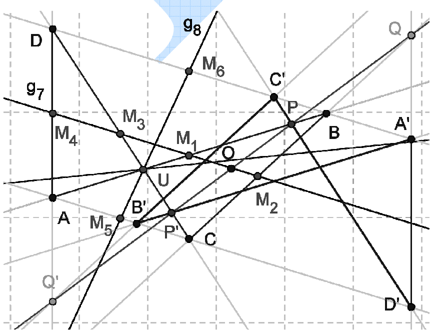

Figure 6.1.

We can prove these observations not only with elementary geometry but with the help of Maple. The lines AC and BD are parallel if and only if $c_1d_2-d_1c_2+c_2=0$ and $O \in g_7$ if and only if $2c_1y-2c_2x+c_2=0$. We get that in this case the lines AC:$G5(u, v, t)=-c_2u+c_1v=0$ and $B'D'$:$G6(u, v, t)=(d_1-1)(-c_2u+c_1v)/c_1$ coincide.

$> unassign('x') : unassign('y') : unassign('t') :$

$> d_2 := \dfrac{(d_1-1)\cdot c_2}{c_1} : y := \dfrac{2\cdot c_2 \cdot x - c_2}{2\cdot c_1} : G_5(u,v,t) : factor(G_6(u,v,t));$

It can be observed in this case that the points (O,Q,P) are collinear, because the determinant

$> Determinant\left(Matrix\left(3,\left[\left[x,y,1\right],\left[Qx,Qy,Qt\right],\left[Px,Py,Pt\right]\right]\right)\right);$

is equal to zero.

III.a.2) Let O lays on the line g_8: $(c_1^2-c_1-c_1d_1)y+(c_2d_1-c_1c_2-c_2)x+c_1c_2t=0$, except the point $g_7 \cap g_8$. Now we will show that the points P, Q and R are collinear (Fig. 6.2).

The point R is an intersection point of two different parallel lines AC and $B'D'$. Thus the homogenous coordinates of the point R are $(c_1, c_2, 0)$. The solution of the equation

$$(12)\quad \begin{vmatrix} c_1 & c_2 & 0 \\ Qx & Qy & Qt \\ Px & Py & Pt \end{vmatrix} = 0$$

gives the condition the points P, Q and R to be collinear i.e. the vector PQ to be collinear with the vector $r(c_1, c_2)$.

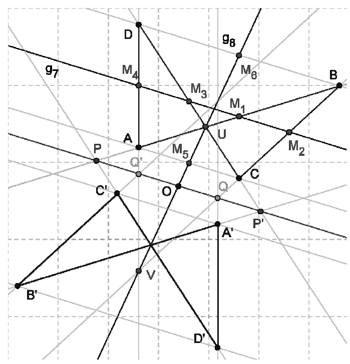

Figure 6.2.

$> unassign('x') : unassign('y') : unassign('t') :$

$> factor\left(Determinant\left(Matrix\left(3,\left[\left[c_1,c_2,0\right],\left[Px,Py,Pt\right],\left[Qx,Qy,Qt\right]\right]\right)\right)\right) = 0;$

From (12) we get that the points P, Q and R are collinear if and only if $(c_1^2-c_1-c_1d_1)y+(c_2d_1-c_1c_2-c_2)x+c_1c_2t=0$ and therefore P,Q and R are collinear if and only if the centre O of reflection Φ in a point lays on the line g_8, except the point $g_7 \cap g_8$.

Let there holds b), i.e. $c_1=0$. Then $c_2(d_1-1)=0$. Because of (4) it follows that $d_1=1$. We find the function $F(x,y,t) = c_2d_2g_9(x,y,t)g_{10}(x,y,t)$, where g_9: $2x-t$ and g_{10}: $2y+(c_2-d_2)x-c_2t$.

$> unassign('d2') : unassign('c2') :$

$> c_1 := 0 : d_1 := 1 : factor(F(x,y,t));$

Because of (4) it follows that F consists of the two intersecting lines g_9 and g_{10}. It is easy to confirm that

M_1, M_2, M_3, $M_4 \in g_9$ and M_5, $M_6 \in g_{10}$. Indeed, we define the functions, that determine the lines g_9 and g_{10}

$> g_9 : (x, y, t) \rightarrow 2 \cdot x - t :$

$> g_8 : (x, y, t) \rightarrow 2 \cdot y + (c_2 - d_2) \cdot x - c_2 \cdot t :$

to check that M_1, M_2, M_3, $M_4 \in g_9$ and M_5, $M_6 \in g_{10}$

$> factor(g_9(m_1, n_1, t_1)); factor(g_9(m_2, n_2, t_2));$

$> factor(g_9(m_3, n_3, t_3)); factor(g_9(m_4, n_4, t_4));$

$> factor(g_1(m_5, n_5, t_5)); factor(g_{10}(m_6, n_6, t_6));$

Therefore the point $g_9 \cap g_{10}$ is the centroid of *ABCD*.

III.b.1) It is easy to observe that the points of the line g_9 do not satisfy the conditions of Problem 3, because in this case the pairs of lines *(AC, B'D')* and *(BD, A'C')* are coinciding lines (Fig. 6.3) and the points *R* and *R'* are not clearly defined.

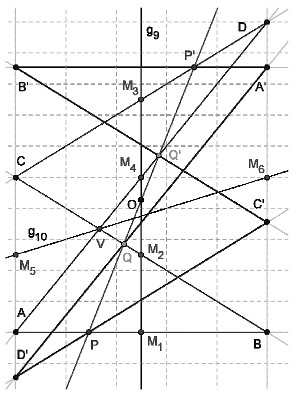

Figure 6.3.

We can prove these observation not only with elementary geometry but with the help of Maple. The lines *AC* and *BD* are parallel and $O \in g_9$ if and only if $2x\text{-}t=0$. We get that in this case the lines *AC*: $G5(u,v,t)=c_2u=0$ and *B'D'* : $G6(u,v,t)=d_2u=0$ coincide.

$> c_1 := 0 : d_1 := 1 : x := \frac{1}{2} : factor(G_5(u, v, t)) : factor(G_6(u, v, t));$

It can be observed in this case that the points *(O,P,Q)* are collinear, because the determinant

$> Determinant\left(Matrix\left(3, \left[[x, y, 1], [Px, Py, Pt], [Qx, Qy, Qt]\right]\right)\right);$

is equal to zero.

III.b.2) Let *O* lays on the line g_{10}: $2y+(c_2\text{-}d_2)x\text{-}c_2t=0$, except the point $g_9 \cap g_{10}$. Now we will show that the points *P*,*Q* and *R* are collinear (Fig. 6.4). Really the point *R* is an intersection point of two different parallel lines *B'D'* and *AC*.

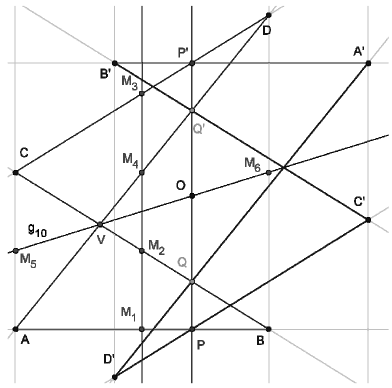

Figure 6.4.

Thus the homogenous coordinates of the infinite point *R* are *(0,1,0)*. The solution of the equation

$$(13) \quad \begin{vmatrix} 0 & 1 & 0 \\ Qx & Qy & Qt \\ Px & Py & Pt \end{vmatrix} = 0$$

gives the condition the points *P*,*Q* and *R* to be collinear i.e. the vector *RP* to be collinear with the vector *r(0,1)*.

From (13) we get that the points *P*, *Q* and *R* are collinear if and only if $2y+(c_2\text{-}d_2)x\text{-}c_2t=0$ and therefore *P*, *Q* and *R* are collinear if and only if the centre *O* of the reflection *Φ* in a point lays on the line g_{10}, except the point $g_9 \cap g_{10}$.

$> unassign('x') : unassign('y') : unassign('t') :$

$> factor\left(Determinant\left(Matrix\left(3, \left[[0,1,0], [Px, Py, Pt], [Qx, Qy, Qt]\right]\right)\right)\right) = 0;$

The determinant of A_{33} gives the type of the curve *k*.

$> factor\left(Determinant\left(A_{33}\right)\right);$

Thus we get

$det(A33)=4c_2d_2(d_1c_2\text{-}c_1d_2)(c_1d_2+c_2\text{-}d_1c_2\text{-}d_2)$.

From (4) it follows that $det(A_{33}) \neq 0$. Therefore the loci of the point *O* could not be parabola.

We will consider two cases:

a) Non of the points *A*,*B*,*C*, *D* lays in the convex hull of the other three;

b) One of the points *A*,*B*,*C*,*D* lays in the convex hull of the other three.

Let us holds a), then there holds one of the next pairs of conditions:

- the points *A* and *C* lay in a different half planes determined by the line *BD*;
- the points *B* and *D* lay in a different half planes determined by the line *AC* (Fig. 7.1);

or

- the points *A* and *C* lay in one of the half planes determined by the line *BD*;
- the points *B* and *D* lay in one of the half planes determined by the line *AC* (Fig. 7.2);

Combining the above conditions we get that non of the points *A*, *B*, *C*, *D* lays in the convex hull if and only if

$G_9(a_1,a_2,1).G_9(c_1,c_2,1)\ G_5(b_1,b_2,1).G_5(d_1,d_2,1)$
$=d_2.(c_1d_2+c_2\text{-}d_1c_2\text{-}d_2)\ c_2.(c_1d_2\ \text{-}d_1c_2)>0.$

Therefore *det*(A_{33})>0 and the curve *k* is a hyperbola.
$> G_9(0,0,1)\cdot G_9(c_1,c_2,1)\cdot G_5(1,0,1)\cdot G_5(d_1,d_2,1);$
Let us holds b), then there holds one of the next conditions:

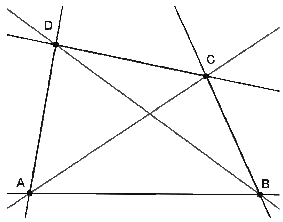

Figure 7.1.

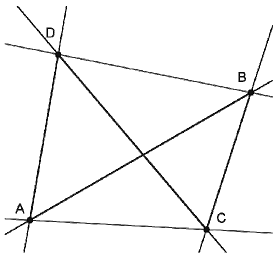

Figure 7.2.

•the points *A* and *C* lay in one of the half planes determined by the line *BD*;
•the points *B* and *D* lay in a different half planes determined by the line *AC* (Fig. 7.3);
or
•the points *A* and *C* lay in a different half planes determined by the line *BD*;
•the points *B* and *D* lay in one of the half planes determined by the line *AC* (Fig. 7.4);

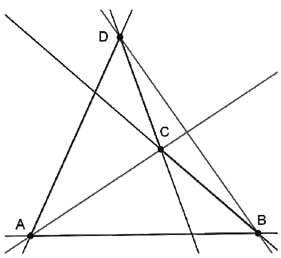

Figure 7.3.

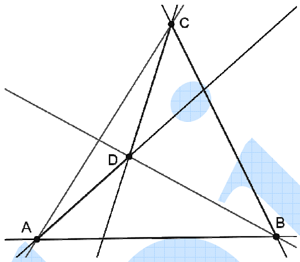

Figure 7.4.

Combining the above conditions we get that one of the points *A,B,C, D* lays in the convex hull of the other three if and only if $G_9(a_1,a_2,1).G_9(c_1,c_2,1)$ $G_5(b_1,b_2,1).G_5(d_1,d_2,1)=d_2.(c_1d_2+c_2-d_1c_2-d_2)\ c_2.(c_1d_2-d_1c_2)<0$.
Consequently we obtain that the type of the loci of the point *O* depends on the "convexity" of the vertexes *A, B ,C, D* of the quadrangle.

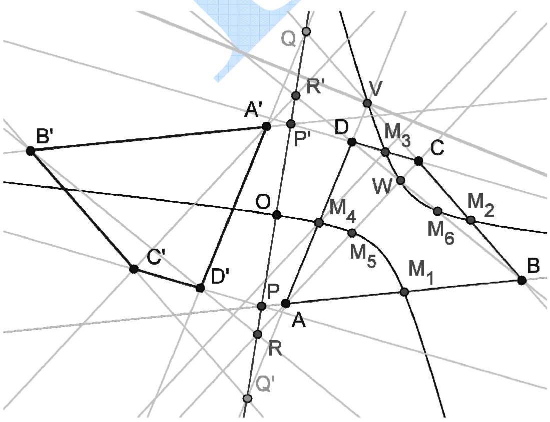

Figure 8.1.

i) loci of the point *O* is hyperbola if and only if non of the points *A,B,C, D* lays in the convex hull of the other three (Fig. 8.1 and 8.2);

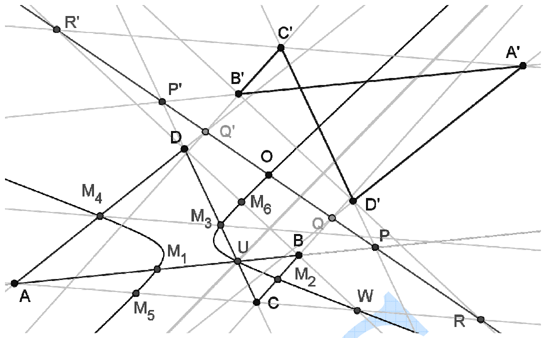

Figure 8.2.

ii) the loci of the point *O* is ellipse if and only if one of the points *A, B, C, D* lays in the convex hull of the other three (Fig. 8.3).

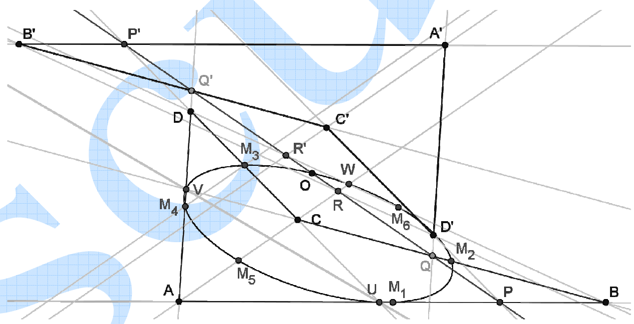

Figure 8.3.

□

4. THE SPECIAL FUNCTION OF DGS SAM "SWAP FINITE & INFINITE POINTS"

The benefits of the special function "Swap finite & infinite points" of DGS Sam are illustrated by many examples in [HJ07]. We suggest a construction, which simulates in GeoGebra the function "Swap" from [KTZ13]. This construction allows to optimize the construction process and to present the similarities and the differences for a whole class of homologies.
We will illustrate this with the next problem. Let us recall the definition of a harmonic homology first (see [Cox87], p. 55).
Definition 2: *A homology Φ with a center O and axis o is called harmonic, when any pair of corresponding points (A, A') are harmonic conjugated with the pair (O, A_1), where A_1 is the intersecting point of the axis o with the line AA'.*
There exist three possible harmonic homologies in the extended Euclidian plane:
a) *O* is a finite point, *o* is a finite line;
b) *O* is a finite point, *o* is the infinite line. Now the homology *Φ* is a reflection in the point *O*;

c) O is an infinite point, o is a finite line. If the lines passing through the infinite center O are orthogonal the axis o, then the homology Φ is a reflection in the line o.

Problem 3: *Let ABCD be a parallelogram.*

i) Let $\Phi(O_\infty, o, A\rightarrow A')$ be a homology, such that the pairs of points (A, A') and (O, A_1), where $A_1 = o \cap AA'$ are harmonic conjugates. Find the image $A'B'C'D'=\Phi(ABCD)$.

ii) Let $\Phi(O, o, A\rightarrow A')$ be a homology, such that the pairs of points (A, A') and (O, A_1), where $A_1 = o \cap AA'$ are harmonic conjugates. Find the image $A'B'C'D'=\Phi(ABCD)$.

iii) Let $\Phi(O, \omega, A\rightarrow A')$ be a homology, such that the pairs of points (A, A') and (O, A_1), where $A_1 = \omega \cap AA'$ are harmonic conjugates. Find the image $A'B'C'D'=\Phi(ABCD)$.

We would like to make some comments on the figures that give the construction.

1) The user can follow step-by-step the construction with the help of the Check Boxes 1-8 (Fig.9.1). The Check Boxes 1-8 control the visibility of the objects. If the user wants to make a more detailed presentation, then he can increase the number of the Check Boxes.

2) The slider $k=1,2,3$ is used to present the cases i) for $k=1$, ii) for $k=2$ or iii) for $k=3$. It is possible to change the slider k, i.e. to change the center and the axis of the homology, at any time of the presentation. This gives a possibility to compare the constrictions that are made in the different cases i), ii) and iii).

3) All this is obtained with the use of the operator “If”. We define the intersection points i_p, that are obtained throughout the constructions, with the help of the operator “If”. When the line o is a finite line then i_p is a finite point, which is the usual intersection of two finite lines. When the line o is the infinite line ω then i_p is the intersection of the infinite line ω with a finite line l , which is a vector collinear with l. The intersection point i_p is used to construct the next line in the construction. If i_p is a finite point then we construct a line through two finite points. If i_p is an intersection of ω with a finite line, i.e. a vector (an infinite point), then we construct a line through a point with direction the vector (infinite point) i_p.

The definitions of the seven lines, that depend on the finite or infinite choice of the point O are:

If[k $\stackrel{?}{=}$ 1, Line[i_3, u], If[k $\stackrel{?}{=}$ 2, Line[i_3, O], Line[O, w]]]

If[k $\stackrel{?}{=}$ 1, Line[A, u], Line[A, O]]

If[k $\stackrel{?}{=}$ 1, Line[Y_2, u], Line[Y_2, O]]

If[k $\stackrel{?}{=}$ 3, Line[i_4, z], Line[i_4, i_1]]

If[k $\stackrel{?}{=}$ 3, Line[i_5, v], Line[i_5, i_2]]

If[k $\stackrel{?}{=}$ 1, Line[B, u], Line[B, O]]

If[k $\stackrel{?}{=}$ 1, Line[D, u], Line[D, O]]

4) The lines through O are defined with the help of the operator “If” too. When O is a finite point then we construct a line through two finite points, when O_∞ is an infinite point we use the command a line defined by a point and a vector, which is the direction of O_∞.

Solution: Figure 9.1. presents the solution of case i), when the infinite center O_∞ is orthogonal to the axis o.

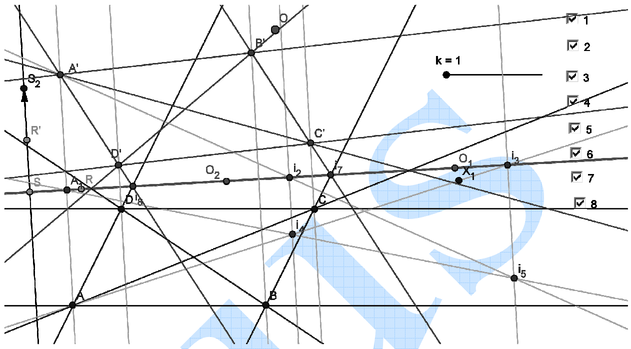

Figure 9.1.

Once we have done the construction in one of the cases then it is simple to generate all of the other cases. We can rotate the vector O_∞ to change the angle between o and O_∞ (Fig. 9.2)

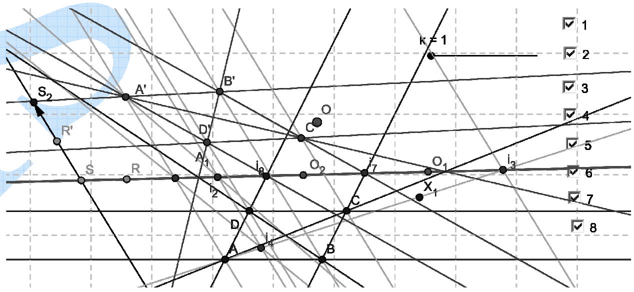

Figure 9.2.

The sketch of case ii) is obtained by swapping the points O_∞ and O for the construction. The swapping is obtained by changing $k=1$ to $k=2$. The solution is presented on Figure 9.3. It is possible to follow the construction with the help of the Check Boxes 1-8.

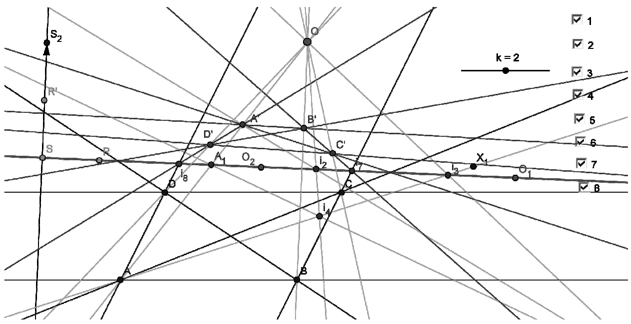

Figure 9.3.

The case iii) is obtained, when $k=3$, which “swaps” the finite line o with the infinite line ω and the infinite point O_∞ with the finite point O. In this case we get a reflection in a point (Fig. 9.4).

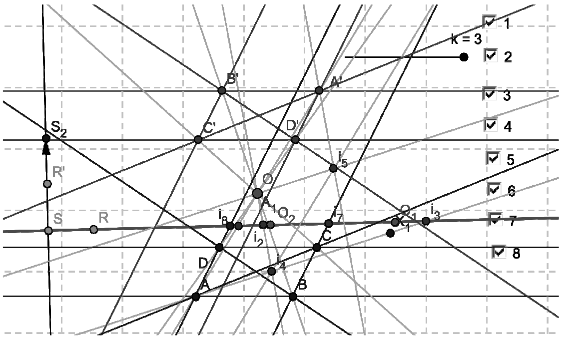

Figure 9.4.

The presented type of construction gives the possibility to see the similarities and the differences between the three cases.

We are working on writing of a code for GeoGebra to implement all the mentioned above operations as an easy to use tools (intersection of infinite line with a finite line, swapping of infinite point with a finite point, swapping infinite line with a finite line).

Problem 4: *The quadrangles PP'QQ', PP'RR', QQ'RR' defined in Problem 2 are parallelograms only in the cases b) and c) of Problem 3 (Fig. 9.1, 9.2, 9.4);*

Problem 5: *The points P, Q, R, P', Q', R', defined in Problem 2 lay on a conic section in all of the cases a), b) and c) of Problem 3.*

Proof: Indeed, it is enough to apply Pascal's theorem for the points *PQR P' Q' R'*. Because of the property of the harmonic homology ($\Phi=\Phi^{-1}$) it follows that the points $PQ \cap P'Q'=1$, $QR \cap Q'R'=2$ and $RP' \cap R'P=3$ lay on the axis of Φ. Hence *P, Q, R, P', Q', R'* are points of one conic section. □

5. OPEN PROBLEM

It is well known that any involutory collineation is a harmonic homology ([Cox49], p.55). The main problem in the present work deals with the case b) of Problem 3. It will be interesting to solve the main problem in the cases a) and c) of Problem 3, i.e.

Open Problem: *Let ABCD be a complete quadrangle and Φ be an arbitrary harmonic homology with center O and axis o. Let us denote A' B' C' D' =Φ(ABCD) and* $P=AB\cap C'D'$, $P'=A'B'\cap CD$, $Q=BC\cap A'D'$, $Q'=B'C'\cap AD$, $R=AC\cap B'D'$, $R'=A'C'\cap BD$.

Find the loci of the center O, when the points P, Q, R, P', Q', R' are collinear.

Find the loci of the axe o, when the points P, Q, R, P', Q', R' are collinear.

It is not difficult to see that the quadrangles *PP'QQ', PP'RR', QQ'RR'* from Problem 2 are parallelograms only in the cases b) and c) (Fig. 9.1, 9.2, 9.4). It should be easy to prove that the points *P, Q, R, P', Q', R'* lay on a conic section in all of the cases a), b) and c).

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