FIXED POINTS FOR MAPPINGS WITH A CONTRACTIVE ITERATE AT EACH POINT

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Abstract. We generalize the results of Sehgal and Guseman for mappings on a complete metric space with a contractive iterate condition at each point.

1. Introduction

Let $\big(X,\rho\big)$ be a metric space and $T : X \to X$ be a map. If there is $\alpha < 1$, such that $\rho(Tx, Ty) \leq \alpha\rho(x, y)$ holds for every $x, y \in X$, then T is called a contraction. A well known theorem of Banach [1] states that if X is a complete and T is a contraction, then T has a unique fixed point z and for any $x \in X$ the sequence of the successive approximations $\big\{T^n x\big\}_{n=1}^{\infty}$ converges to z, where $T^n x = T\big(T^{n-1}x\big)$. Sehgal has generalized this result in [8], by considering maps that a contractive at a point. The result of Sehgal has been generalized by Guseman [7]. In these two papers the authors have imposed weaker contraction conditions on the map, but again the fixed point can always be found by using the Picard iteration, beginning with some initial choice $x \in X$. Some other articles that consider mappings that are with a contrective iterate at a point are [2, 3, 4, 5, 6].

2. Main result

Theorem 2.1. *Let $\big(X,\rho\big)$ be a complete metric space and $T : X \to X$ be a map with the properties:*

(a) *there exist subsets $U_n \subseteq X$, $n \in \mathbb{N}$, such that $T : U_n \to U_n$, and $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$;*

(b) *there exists $\alpha \in (0,1)$, such that for every $x \in X$ there are $n_1, n_0 \in \mathbb{N}$, such that for every $u \in \bigcup_{n=n_0}^{\infty} U_n$ there holds the inequality*

$$\rho(T^{n_1}u, T^{n_1}x) \leq \alpha\rho(u, x).$$

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Then there exists $z \in X$ with the properties:

(i) *There exists $n_z \in \mathbb{N}$, such that $T^{n_z}z = z$;*
(ii) *For every $y \in X$ there is a sequence of naturals $\{q_i\}_{i=1}^{\infty}$, such that $\lim\limits_{i\to\infty} T^{q_i}y = z$;*
(iii) *For every $x \in \bigcap_{n=1}^{\infty} U_n$ there holds $\lim_{n\to\infty} T^n x = z$.*

Proof. Let $x \in \bigcap_{n=1}^{\infty} U_n$ be arbitrary chosen. By $T : U_n \to U_n$ it follows that for every $s \in \mathbb{N}$ we have $T^s x \in \bigcap_{n=1}^{\infty} U_n$. Following [7] and [8] let define the function $r(x) = \sup\{\rho(T^n x, x) : n \in \mathbb{N}\}$ for any $x \in X$. First we will prove that for any $x \in \bigcap_{n=1}^{\infty} U_n$ there holds the inequality $r(x) < +\infty$.

Let choose an arbitrary $x \in \bigcap_{n=1}^{\infty} U_n$, then there exist $n_1(x), n_0(x) \in \mathbb{N}$, such that the inequality $\rho(T^{n_1(x)}u, T^{n_1(x)}x) \le \alpha\rho(u, x)$ holds for any $u \in \bigcup_{n=n_0(x)}^{\infty} U_n$. Therefore for any $s \in \mathbb{N}$ there holds the inequality

$$\rho(T^{n_1(x)}(T^s x), T^{n_1(x)}x) \le \alpha\rho(T^s x, x).$$

Put $l(x) = \max\{\rho(T^s x, x), s = 1, 2, \dots, n_1\}$. For any $s \in \mathbb{N}$ there is $k \in \mathbb{N}$, such that $kn_1(x) < s \le (k+1)n_1(x)$, then we can write the chain of inequalities:

$$\begin{aligned}
\rho(T^s x, x) &\le \rho(T^{n_1(x)}(T^{s-n_1(x)}x), T^{n_1(x)}x) + \rho(T^{n_1(x)}x, x) \\
&\le \alpha\rho(T^{s-n_1(x)}x, x) + \rho(T^{n_1(x)}x, x) \le \alpha\rho(T^{s-n_1(x)}x, x) + l(x) \\
&\le \alpha\rho(T^{n_1(x)}(T^{s-2n_1(x)}x), T^{n_1(x)}x) + \alpha\rho(T^{n_1(x)}x, x) + l(x) \\
&\le \alpha^2\rho(T^{s-2n_1(x)}x, x) + (1+\alpha)l(x) \\
&\le \alpha^2\rho(T^{n_1(x)}(T^{s-3n_1(x)}x), T^{n_1(x)}x) + \alpha^2\rho(T^{n_1(x)}x, x) \\
&\quad +(1+\alpha)l(x) \\
&\le \alpha^3\rho(T^{s-3n_1(x)}x, x) + (1+\alpha+\alpha^2)l(x) \\
&\le \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\le \alpha^k\rho(T^{s-kn_1(x)}x, x) + (1+\alpha+\alpha^2+\dots+\alpha^{k-1})l(x) \\
&\le l(x)\sum_{j=0}^{k}\alpha^j \le \frac{l(x)}{1-\alpha}.
\end{aligned}$$

Thus for any $x \in \bigcap_{n=1}^{\infty} U_n$ there holds the inequality $r(x) \le \dfrac{l(x)}{1-\alpha} < +\infty$.

Now we will construct inductively a convergent sequence $\{x_i\}_{i=1}^{\infty}$. Let choose an arbitrary $x \in \bigcap_{n=1}^{\infty} U_n$. Put $m_0 = n_1(x)$, $x_1 = T^{m_0}x$, $m_1 = n_1(x_1)$, $x_2 = T^{m_1}x_1$. If we have defined $m_{i-1} \in \mathbb{N}$ and $x_i \in \bigcap_{n=1}^{\infty} U_n$, then put $m_i = n_1(x_i)$ and $x_{i+1} = T^{m_i}x_i$.

We will show that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Indeed by the inequalities

$$\begin{aligned}
\rho(x_{i+1}, x_i) &= \rho(T^{m_i}x_i, x_i) = \rho(T^{m_{i-1}}(T^{m_i}x_{i-1}), T^{m_{i-1}}x_{i-1}) \\
&\leq \rho(T^{m_i}x_{i-1}, x_{i-1}) = \alpha\rho(T^{m_{i-2}}(T^{m_i}x_{i-2}), T^{m_{i-2}}x_{i-2}) \\
&\leq \alpha^2\rho(T^{m_i}x_{i-2}, x_{i-2}) \leq \cdots \leq \alpha^i\rho(T^{m_i}x, x) \leq \alpha^i r(x)
\end{aligned}$$

we get

$$\begin{aligned}
\rho(x_{i+p}, x_i) &= \rho(x_{i+p}, x_{i+p-1}) + \rho(x_{i+p-1}, x_{i+p-2}) + \cdots \\
&\quad \cdots + \rho(x_{i+2}, x_{i+1}) + \rho(x_{i+1}, x_i) \\
&\leq r(x)\textstyle\sum_{s=i}^{i+p-1}\alpha^i \leq r(x)\frac{\alpha^i}{1-\alpha},
\end{aligned}$$

which ensures that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence and taking into account the completeness of X it follows that there is $z \in X$, such that $\lim_{i\to\infty} x_i = z$.

Let for the rest of the proof $\{x_i\}_{i=1}^{\infty}$ be the sequence, that was constructed above and z be the limit of the sequence $\{x_i\}_{i=1}^{\infty}$.

We will show that $\lim\limits_{s\to\infty} \rho(T^s x, z) = 0$. Choose $\varepsilon > 0$. There is $i_0 \in \mathbb{N}$, such that the inequalities $\rho(z, x_i) < \varepsilon/2$ and $\alpha^i r(x) < \varepsilon/2$ hold for every $i \geq i_0$. Put $s_0 = \sum_{i=0}^{i_0} m_i$. Then for every $s > s_0$ we can write the chain of inequalities

$$\begin{aligned}
\rho(z, T^s x) &\leq \rho(z, T^{m_{i_0}}x_{i_0}) + \rho(T^{m_{i_0}}x_{i_0}, T^s x) \\
&\leq \rho(z, x_{i_0+1}) + \alpha\rho(x_{i_0}, T^{s-m_{i_0}}x) \\
&= \frac{\varepsilon}{2} + \alpha\rho(T^{m_{i_0-1}}x_{i_0-1}, T^{s-m_{i_0}}x) \\
&\leq \frac{\varepsilon}{2} + \alpha^2\rho(x_{i_0-1}, T^{s-(m_{i_0}+m_{i_0-1})}x) \\
&\leq \cdots \leq \frac{\varepsilon}{2} + \alpha^{i_0+1}\rho(x, T^{s-s_0}x) \leq \frac{\varepsilon}{2} + \alpha^{i_0+1}r(x) < \varepsilon.
\end{aligned}$$

By the arbitrary choice of $\varepsilon > 0$ it follows that $\lim_{s\to\infty} \rho(T^s x, z) = 0$.

There are $n_1(z), n_0(z) \in \mathbb{N}$, such that $\rho(T^{n_1(z)}u, T^{n_1(z)}z) \leq \alpha\rho(u, z)$, holds for any $u \in \bigcup_{n=n_0(z)}^{\infty} U_n$. By the construction of the sequence $\{x_i\}_{i=1}^{\infty}$ it follows that for any $s, i \in \mathbb{N}$ there holds $T^s x_i \in \bigcap_{n=1}^{\infty} U_n$ and thus the inequality

$$\rho(T^{n_1(z)}x_i, T^{n_1(z)}z) < \alpha\rho(x_i, z) \tag{1}$$

holds for any $i \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary chosen, then there exists $j_0 \in \mathbb{N}$, such that for any $i \geq j_0$ there hold $\rho(x_i, z) < \varepsilon/3$ and $\alpha^i r(x) < \varepsilon/3$. Consequently by (1) we get $\rho(T^{n_1(z)}x_i, T^{n_1(z)}z) < \varepsilon/3$ and the chain of inequalities

$$\begin{aligned}
\rho(T^{n_1(z)}x_i, x_i) &= \rho(T^{m_{i-1}}(T^{n_1(z)}x_{i-1}), T^{m_{i-1}}x_{i-1}) \\
&\leq \alpha\rho(T^{n_1(z)}x_{i-1}, x_{i-1}) \\
&= \alpha\rho(T^{m_{i-2}}(T^{n_1(z)}x_{i-2}), T^{m_{i-2}}x_{i-2}) \\
&\leq \alpha^2\rho(T^{n_1(z)}x_{i-2}, x_{i-2}) \\
&\leq \cdots \leq \alpha^i\rho(T^{n_1(z)}x, x) \leq \alpha^i r(x) < \varepsilon/3.
\end{aligned}$$

We will proceed with proving of (i)–(iii).

(i) Indeed we have that for any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$, such that the inequality

$$\rho(T^{n_1(z)}z, z) \leq \rho(T^{n_1(z)}z, T^{n_1(z)}x_i) + \rho(T^{n_1(z)}x_i, x_i) + \rho(x_i, z) < \varepsilon$$

holds for every $i \geq j_0$ and thus by the arbitrary choice of $\varepsilon > 0$ it follows that $T^{n_1(z)}z = z$.

(ii) Let $y \in X$ be arbitrary. We will construct inductively a sequence $\{y_i\}_{i=1}^{\infty}$. Put $p_0 = n_1(y)$, $y_1 = T^{p_0}y$, $p_1 = n_1(y_1)$, $y_2 = T^{p_1}y_1$. If we have defined $p_{i-1} \in \mathbb{N}$ and y_i then put $p_i = n_1(y_i)$, $y_{i+1} = T^{p_i}y_i$ and $q_i = \sum_{j=0}^{i} p_j$. By the construction of the sequence $\{x_i\}_{i=1}^{\infty}$ it follows that for any $s, i \in \mathbb{N}$ there holds $T^s x_i \in \bigcap_{n=1}^{\infty} U_n$, thus

$$\begin{aligned}
\rho(T^{q_i}x, T^{p_i}y_i) &\leq \alpha\rho(T^{q_{i-1}}x, y_i) = \alpha\rho(T^{q_{i-1}}x, T^{p_{i-1}}y_{i-1}) \\
&\leq \alpha^2\rho(T^{q_{i-2}}x, y_{i-1}) \leq \cdots \leq \alpha^{i+1}\rho(x, y).
\end{aligned}$$

Therefore $\lim_{i\to\infty} \rho(T^{q_i}x, T^{p_i}y_i) = 0$. Since $\lim_{s\to\infty} \rho(T^s x, z) = 0$ and the triangle inequality

$$\rho(T^{p_i}y_i, z) \leq \rho(T^{p_i}y_i, T^{q_i}x) + \rho(T^{q_i}x, z)$$

we get that $\lim_{i\to\infty} \rho(T^{q_i}y, z) = \lim_{i\to\infty} \rho(T^{p_i}y_i, z) = 0$.

(iii) By (i) and the arbitrary choice of $x \in \bigcap_{n=1}^{\infty} U_n$ it follows that for every $y \in \bigcap_{n=1}^{\infty} U_n$ there is $z_y \in X$, such that $\lim_{s\to\infty} T^s y = z_y$. To finish the proof it enough to show that $z_y = z$. Let suppose the contrary i.e. there exists $y \in \bigcap_{n=1}^{\infty} U_n$, such that $\lim_{s\to\infty} T^s y = z_y \neq z$. We have just proved in (ii) that for any $y \in X$ there exists a sequences of natural numbers $\{q_i\}_{i=1}^{\infty}$, such that $\{T^{q_i}y\}_{i=1}^{\infty}$ is convergent to z and consequently $z_y = z$. □

Let for the proofs of the next Corollaries we use the notations from the proof of Theorem 2.1.

Corollary 2.1. *Let (X, ρ) be a complete metric space and $T : X \to X$ be a map with the properties:*

(a*) *there exist subsets $U_n \subseteq X$, $n \in \mathbb{N}$, such that $T : U_n \to U_n$, and $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$;*

(b*) *there exists $\alpha \in (0, 1)$, such that for every $x \in X$ there is $n_0 \in \mathbb{N}$, such that for every $u \in \bigcup_{n=n_0}^{\infty} U_n$ there holds the inequality*

$$\rho(Tu, Tx) \leq \alpha\rho(u, x).$$

Then there exists $z \in X$, such that:

(i*) *$Tz = z$;*

(ii*) *For every $y \in X$ holds $\lim\limits_{i\to\infty} T^i y = z$.*

Proof. The proof follows immediately by the proof of Theorem 2.1, because $n_1(z) = 1$, and $p_i = n_1(y_i) = 1$. □

Corollary 2.2. *Let (X, ρ) be a complete metric space and $T : X \to X$ be a map with the properties:*

(a) *there exist subsets $U_n \subseteq X$, $n \in \mathbb{N}$, such that $T : U_n \to U_n$, and $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$;*

(b) *there exists $\alpha \in (0, 1)$, such that for every $x \in X$ there are $n_1, n_0 \in \mathbb{N}$, such that for every $u \in \bigcup_{n=n_0}^{\infty} U_n$ there holds the inequality*

$$\rho(T^{n_1}u, T^{n_1}x) \leq \alpha\rho(u, x).$$

Let there holds one of the following conditions:

(c.1) *If there is $x \in \bigcap_{n=1}^{\infty} U_n$, such that T is continuous at $\lim_{i\to\infty} T^i x$;*

(c.2) *If there is $x \in \bigcap_{n=1}^{\infty} U_n$, such that $\lim_{i\to\infty} T^i x \in \bigcap_{n=1}^{\infty} U_n$;*

(c.3) *If the sets $\{U_n\}_{n=1}^{\infty}$ are closed,*

then there exists $z \in X$, such that $Tz = z$.

Proof. It follows by Theorem 2.1 that there exists $z \in X$, such that for any $u \in \bigcap_{n=1}^{\infty} U_n$ there holds $\lim_{i\to\infty} T^i u = z$ and there exists $n_1(z)$, such that $T^{n_1(z)} = z$.

Let holds (c.1), i.e. T be continuous at z. Choose arbitrary $u \in \bigcap_{n=1}^{\infty} U_n$ and put $u_i = T^i u$, then for every $\varepsilon > 0$ there exists $k_0 = k_0(u) \in \mathbb{N}$, such that for every $i \geq k_0$ there hold $\rho(z, Tu_i) = \rho(z, T^i u) < \varepsilon/2$ and $\rho(Tu_i, Tz) < \varepsilon/2$. Now by the inequality

$$\rho(z, Tz) \leq \rho(z, Tu_i) + \rho(Tu_i, Tz) < \varepsilon$$

we obtain that $Tz = z$.

Let holds (c.2). Suppose that $Tz = v \neq z$. Then $T^{n_1(z)}v = T^{n_1(z)}(Tz) = T(T^{n_1(z)}z) = Tz = v$. By the choice of $n_1(z) \in \mathbb{N}$ and Theorem 2.1 we get

$$\rho(v, z) = \rho(T^{n_1(z)}v, T^{n_1(z)}z) \leq \alpha\rho(v, z),$$

which is a contradiction. Consequently $Tz = z$.

Let holds (c.3). The proof follows form (c.2), because if $\{U_n\}_{n=1}^{\infty}$ are closed then for any $u \in \bigcap_{n=1}^{\infty} U_n$ will hold $\lim_{i\to\infty} T^i u \in \bigcap_{n=1}^{\infty} U_n$. □

Corollary 2.3. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a map with the properties:*

(a**) *there exist subsets $U_n \subseteq X$, $n \in \mathbb{N}$, such that $T : U_n \to U_n$, and $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$;*

(b**) *there exists $\alpha \in (0, 1)$, such that for every $x \in X$ there are $n_1, n_0 \in \mathbb{N}$, such that for every $u \in \bigcup_{n=n_0}^{\infty} U_n$ there holds the inequality*

$$\|T^{n_1}u - T^{n_1}x\| \leq \alpha\|u - x\|.$$

If there is $x \in \bigcap_{n=1}^{\infty} U_n$, such that the sequence $\{T^i x\}_{i=1}^{\infty}$ is weakly convergent to $T\left(\lim_{i\to\infty} T^i x\right)$, then there exists $z \in X$, such that $Tz = z$.

Proof. By Theorem 2.1 there is $z \in X$, such that for any $x \in \bigcap_{n=1}^{\infty} U_n$ we have $\lim_{i\to\infty} T^i x = z$. Let take $x \in \bigcap_{n=1}^{\infty} U_n$, such that $\{T^i x\}_{i=1}^{\infty}$ is weakly convergent to $T\left(\lim_{i\to\infty} T^i x\right) = Tz$. Then $\{T^i x - z\}_{i=1}^{\infty}$ is weakly convergent to $Tz - z$ and therefore we have $\|Tz - z\| \leq \underline{\lim}_{i\to\infty}\|T^i x - z\|$. Now using that $T^i x$ is convergent to z we get that $Tz = z$. □

3. Examples

Now we will illustrate the above results with some examples.

Example 1: Let $f : [0, 1] \to [0, 1]$ be a convex function, such that $\frac{2}{3}x < f(x) < \frac{4}{5}x$. We will construct inductively sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ and $\{\beta_{2n+1}\}_{n=1}^{\infty}$. Let $a_1 = 1$. Put a_{2n}, $n \in \mathbb{N}$ to be the solution of the equation

$$-a_{2n} + a_{2n-1} = f(a_{2n}), \tag{2}$$

β_{2n+1}, $n \in \mathbb{N}$ to be the solution of the equation

$$2a_{2n} + \beta_{2n+1} = f(a_{2n}) \tag{3}$$

and a_{2n+1}, $n \in \mathbb{N}$ to be the solution of the equation

$$2a_{2n+1} + f(a_{2n}) = 2a_{2n}. \tag{4}$$

Define $T : [0, 1] \to [0, 1]$ by

$$Tx = \begin{cases} -x + a_{2n-1}, & n \in \mathbb{N} \quad x \in [a_{2n}, a_{2n-1}] \\ \\ 2x + \beta_{2n+1}, & n \in \mathbb{N} \quad x \in [a_{2n+1}, a_{2n}] \end{cases}$$

and $T(0) = 0$. The graphic of the function T is shown in Fig. 1.

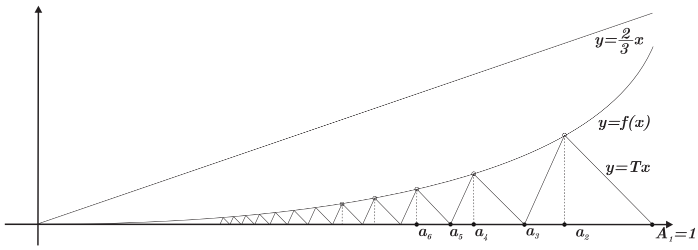

FIGURE 1. Graphic of the function in Example 1.

It is easy to see that $\lim_{n\to\infty} a_n = 0$. Indeed by (2) and (4) we have the inequalities $a_{2n} = a_{2n-1} - f(a_{2n}) \leq a_{2n-1} - \frac{2}{3}a_{2n}$ and $2a_{2n+1} = 2a_{2n} - f(a_{2n}) \leq 2a_{2n} - \frac{2}{3}a_{2n} = \frac{4}{3}a_{2n}$. After combining the last two inequalities we get that $a_{2n+1} \leq \frac{2}{5}a_{2n-1}$ holds for every $n \in \mathbb{N}$. By the inequalities $a_{2n+1} < a_{2n} < a_{2n-1}$ it follows that $\lim_{n\to\infty} a_n = 0$. Put $U_n = [0, a_{2n+1}]$ for $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} U_n = \{0\} \neq \emptyset$.

Let $x_0 \in [0,1]$ be arbitrary chosen and fixed. Then there is $k_0 \in \mathbb{N}$, such that $x \in [a_{2k_0+1}, a_{2k_0-1}]$. By $\lim_{n\to\infty} a_n = 0$ it follows that there is $n_0 \in \mathbb{N}$, such that the inequalities $T(u) < T(x_0)$ and $\frac{4}{5}u \leq \frac{4}{5}a_{2n+1} < \frac{4}{5}x_0 - T(x_0)$ hold for every $u \in \bigcup_{n=n_0}^{\infty} U_n$, $n \geq n_1$. Thus for any $u \in \bigcup_{n=n_0}^{\infty} U_n$ there hold the inequalities

$$|T(x_0) - T(u)| = T(x_0) - T(u) \leq \frac{4}{5}(x_0 - u) - Tu < \frac{4}{5}|x_0 - u|$$

and therefore by Corollary 2.2 it follows that T has a fixed point z and for every $x \in [0,1]$ the sequence $T^n x$ converges to z.

It is interesting in this example that there are open sets $V_n = (u_n, t_n)$, such that $\lim_{n\to\infty} u_n = 0$ and for any $v_n \in V_n$ there holds the inequality $|T^2 v_n - T v_n| > |T v_n - v_n|$. Indeed take $v_n = a_{2n-1} - a_{2n+1}$. By using the inequalities (2) and (4) we get

$$a_{2n} - v_n = 2a_{2n} - a_{2n-1} + a_{2n+1} = a_{2n} - \frac{3}{2}f(a_{2n}) < 0$$

and therefore $v_n \in (a_{2n}, a_{2n-1})$. Then $T(v_n) = -v_n + a_{2n-1} = a_{2n+1}$ and $T^2 v_n = T a_{2n+1} = 0$. Thus $|T^2 v_n - T v_n| = a_{2n+1}$ and $|T v_n - v_n| = a_{2n-1} - 2a_{2n+1}$, because $a_{2n-1} - 2a_{2n+1} > 0$. Consequently

$$\frac{|T^2 v_n - T v_n|}{|T v_n - v_n|} = \frac{a_{2n+1}}{a_{2n-1} - 2a_{2n+1}} = \frac{a_{2n} - \dfrac{f(a_{2n})}{2}}{2f(a_{2n}) - a_{2n}} > 1.$$

By the continuity of T it follows that there are open sets V_n, such that $v_n \in V_n$ and for any $v \in V_n$ there holds

$$\frac{|T^2v - Tv|}{|Tv - v|} > 1.$$

Example 2a: Let us consider the Cantor set. It is is obtained by repeatedly deleting the open middle thirds of a set of line segments (starting from $[0,1]$). More specifically, let K_1 be the unit interval $[0,1]$ with its middle third removed, that is $K_1 = [0,1/3] \cup [2/3,1]$. Let K_2 be K_1 with its middle thirds removed, that is $K_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$. Continuing in this manner, we generate a sequence of closed sets K_n. The Cantor set is defined by $C = \bigcap_{n=1}^{\infty} K_n \neq \emptyset$. There is and explicit formula for the open middle third sets that are removed. Put $V_0 = (1/3,2/3) \bigcup (1/9,2/9) \bigcup (7/9,8/9)$ and

$$V = V_0 \bigcup \left(\bigcup_{m=2}^{\infty} \bigcup_{k=-1}^{m-3} \left(\frac{3^{k+3}-8}{3^m}, \frac{3^{k+3}-7}{3^m} \right) \bigcup \left(\frac{3^{k+3}-2}{3^m}, \frac{3^{k+3}-1}{3^m} \right) \right),$$

then $C = [0,1] \backslash V$. Put $A = V_0 \backslash \mathbb{Q}$ and $B = V \backslash \big(V_0 \cup \mathbb{Q}\big)$ and let define $T_1 : A \to \mathbb{R}$ and $T_2 : B \to \mathbb{R}$ by

$$T_1(x) = \begin{cases} \frac{1}{2}x - \frac{1}{18}, & x \in \left(\frac{1}{3}, \frac{4}{9}\right) \backslash \mathbb{Q} \\ \frac{1}{4}x + \frac{1}{18}, & x \in \left(\frac{4}{9}, \frac{2}{3}\right) \backslash \mathbb{Q} \\ \frac{1}{2}x - \frac{1}{54}, & x \in \left(\frac{1}{9}, \frac{4}{27}\right) \backslash \mathbb{Q} \\ \frac{1}{4}x + \frac{1}{54}, & x \in \left(\frac{4}{27}, \frac{2}{9}\right) \backslash \mathbb{Q} \\ \frac{1}{2}x - \frac{7}{54}, & x \in \left(\frac{7}{9}, \frac{22}{27}\right) \backslash \mathbb{Q} \\ \frac{1}{4}x + \frac{2}{27}, & x \in \left(\frac{22}{27}, \frac{8}{9}\right) \backslash \mathbb{Q} \end{cases}$$

$$T_2(x) = \begin{cases} \frac{1}{2}x - \frac{3^{k+3}-8}{2.3^{m+1}}, & x \in \left(\frac{3^{k+3}-8}{3^m}, \frac{3^{k+4}-23}{3^{m+1}}\right) \backslash \mathbb{Q} \\ \frac{1}{4}x + \frac{3^{k+3}-7}{4.3^{m+1}}, & x \in \left(\frac{3^{k+4}-23}{3^{m+1}}, \frac{3^{k+3}-7}{3^m}\right) \backslash \mathbb{Q} \\ \frac{1}{2}x - \frac{3^{k+3}-2}{2.3^{m+1}}, & x \in \left(\frac{3^{k+3}-2}{3^m}, \frac{3^{k+4}-5}{3^{m+1}}\right) \backslash \mathbb{Q} \\ \frac{1}{4}x + \frac{3^{k+3}-1}{4.3^{m+1}}, & x \in \left(\frac{3^{k+4}-5}{3^{m+1}}, \frac{3^{k+3}-1}{3^m}\right) \backslash \mathbb{Q}. \end{cases}$$

Let define a map $T[0,1] \to [0,1]$ by

$$Tx = \begin{cases} \frac{1}{3}x, & x \in (\mathbb{Q} \cap V) \cup C \\ T_1(x), & x \in A \\ T_2(x), & x \in B. \end{cases}$$

An approximation of the graphic of T is plotted in Fig. 2.

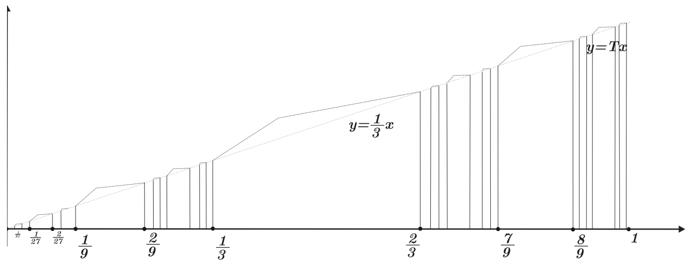

FIGURE 2. An approximation of the graphic of the function in Example 2a.

We will show that the map T with the sequence of sets $\{U_n\}_{n=1}^{\infty}$, where $U_n = K_n$, satisfies the conditions in Theorem 2.1.

Let $x_0 \in [0,1]$ be arbitrary chosen. There are two cases $x_0 \in V$ or $x_0 \in C$.
Case I) Let $x_0 \in V$.

If $x_0 \in V \backslash V_0$, then there are $m_0 \geq 2$ and $k_0 \in \{-1, 0, 1, \ldots m_0 - 3\}$, such that $x_0 \in \left(\frac{3^{k+3}-8}{3^m}, \frac{3^{k+3}-7}{3^m}\right) \bigcup \left(\frac{3^{k+3}-2}{3^m}, \frac{3^{k+3}-1}{3^m}\right)$. Put

$$r = \sup\left\{\lambda > 0 : (x_0 - \lambda, x_0 + \lambda) \subset \left(\frac{3^{k+3}-8}{3^m}, \frac{3^{k+3}-7}{3^m}\right) \cup \left(\frac{3^{k+3}-2}{3^m}, \frac{3^{k+3}-1}{3^m}\right)\right\}$$

Choose $n_0 = m_0 + 1$ and $n_1 \in \mathbb{N}$ be such that $(1/2)^{n_1 - 1} < (r/3)$. Then for every $u \in \bigcup_{n=n_0}^{\infty} U_n$ there hold the inequalities

$$|T^{n_1}x_0 - T^{n_1}u| \leq \left(\frac{1}{2}\right)^{n_1} x_0 + \left(\frac{1}{2}\right)^{n_1} u \leq \left(\frac{1}{2}\right)^{n_1 - 1} < \frac{r}{3} < \frac{1}{3}|x_0 - u|,$$

because $Tx \leq \dfrac{x}{2}$, for every $x \in [0,1]$.

If $x_0 \in V_0$, then choose $n_0 = 2$ and put $r = \sup\{\lambda > 0 : (x_0 - \lambda, x_0 + \lambda) \subset V_0\}$. Choose n_1 be such that $(1/2)^{n_1-1} < (r/3)$. Then for every $u \in \bigcup_{n=2}^{\infty} U_n$ there hold the inequalities

$$|T^{n_1}x_0 - T^{n_1}u| \leq \left(\frac{1}{2}\right)^{n_1} x_0 + \left(\frac{1}{2}\right)^{n_1} u \leq \left(\frac{1}{2}\right)^{n_1 - 1} < \frac{r}{3} < \frac{1}{3}|x_0 - u|.$$

Case II) Let $x_0 \in C$. Then for any $u \in \bigcup_{n=2}^{\infty} U_n$ there holds

$$|Tx_0 - Tu| \leq \frac{1}{3}|x_0 - u|,$$

because $Tx \geq \dfrac{x}{3}$ for every $x \in [0,1]$ and $Tx = \dfrac{x}{3}$ for every $x \in C$. Now we can apply Theorem 2.1. Let us mention the result

Theorem 3.1. [8] *Let X be a Banach space, and $T : X \to X$ a continuous mapping satisfying the condition: there exists a constant $\alpha \in (0,1)$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$*

$$\rho(T^{n(x)}y, T^{n(x)}x) \leq \alpha\rho(y,x).$$

Then T has a unique fixed point z and $\lim\limits_{s\to\infty} T^s x = z$ for each $x \in X$.

It is easy to see that for any $x \in V \cap \mathbb{Q}$ and for any $n_1 \in \mathbb{N}$ we can choose an irrational u, such that $|T^{n_1}x - T^{n_1}u| > |x - u|$ and thus the conditions in Theorem 3.1 are not satisfied.

Example 2b: Let $T : [0,1] \to [0,1]$ be defined as in Example 2a, for every $x \in (0,1)\backslash\{1/2\}$. Let $T(0) = 1$, $T(1) = 1/2$, $T(1/2) = 1$. Then all he conditions in Theorem 2.1 are satisfied with with the sets $U_n = [0, 1/n]$, $n \in \mathbb{N}$. It easy to see that $n_1(0) = 3$. Therefore $T^3(0) = 0$ and obviously there is no $x \in [0,1]$, such that $Tx = x$.

Example 3: Let $(X, \|\cdot\|)$ be a Banach space with a basis $\{e_i\}_{i=1}^{\infty}$, such that

$$\begin{array}{l} \text{if} \quad x = \sum_{i=1}^{\infty} x_i e_i, y = \sum_{i=1}^{\infty} y_i e_i \quad \text{with} \quad |x_i| \leq |y_i| \quad \text{for every} \quad i \in \mathbb{N}, \\ \text{then} \quad \|x\| \leq \|y\|. \end{array} \tag{5}$$

Let $\{\alpha_i\}_{i=1}^{\infty}$ be an increasing sequence of positive reals, convergent to 1. Let $T : X \to X$, be linear map defined by $Te_k = \alpha_k e_k$. Consider the sets $U_n = \left\{\sum_{i=1}^{n} \lambda_i e_i : \|\sum_{i=1}^{n} \lambda_i e_i\| \leq \frac{1}{n}\right\}$. Obviously $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

Let $x = \sum_{i=1}^{\infty} x_i e_i \in X$ be arbitrary chosen. There is $m_1 \in \mathbb{N}$, such that

$$\left\|\sum_{i=m_1}^{\infty} x_i e_i\right\| < \frac{1}{8}\|x\|,$$

which implies the inequality $\frac{7}{8}\|x\| \leq \|\sum_{i=1}^{m_1} x_i e_i\|$. Choose $m_2 \in \mathbb{N}$, such that $\frac{1}{m_2} \leq \frac{1}{8}\|x\|$ and put $m_0 = \max\{m_1, m_2\}$. Then for every $m \geq m_0$ and every $y \in U_m$ there hold the inequalities

$$\begin{aligned} \left\|\sum_{i=m+1}^{\infty} x_i e_i\right\| &\leq \frac{1}{8}\|x\| \leq \frac{1}{4}\left(\frac{3}{4}\|x\| - \frac{1}{m}\right) \leq \frac{1}{4}\left|\left\|\sum_{i=1}^{m} x_i e_i\right\| - \|y\|\right| \\ &\leq \frac{1}{4}\left|\|x\| - \|y\|\right| \leq \frac{1}{4}\|x - y\| \end{aligned}$$

and

$$\|y\| \leq \frac{1}{m} \leq \frac{1}{8}\|x\| \leq \frac{1}{4}\|x\| - \frac{1}{4}\|y\| \leq \frac{1}{4}\|x-y\|.$$

Choose $s_0 = s(x)$, so that $\max\{\alpha_i^{s_0} \leq 2^{-2} : i = 1,2,\dots,m_0\}$. Let $y \in \bigcup_{n=m_0}^{\infty} U_n$, then there is $n \geq m_0$, such that $y \in U_n$ and $y = \sum_{i=1}^{\infty} \lambda_i e_1$, where $\lambda_i = 0$ for every $i > n$. We can write the chain of inequalities:

$$\begin{aligned}
\|T^{s_0}x - T^{s_0}y\| &= \left\| \sum_{i=1}^{m_0} \alpha_i^{s_0}(x_i - \lambda_i)e_i + \sum_{i=m_0+1}^{\infty} \alpha_i^{s_0} x_i e_i \right\| \\
&\leq \left\| \sum_{i=1}^{m_0} \alpha_i^{s_0}(x_i - \lambda_i)e_i + \sum_{i=m_0+1}^{\infty} \alpha_{m_0}^{s_0}(x_i - \lambda_i)e_i \right\| \\
&\quad + \left\| \sum_{i=m_0+1}^{\infty} (\alpha_i^{s_0} - \alpha_{m_0}^{s_0})x_i e_i \right\| + \left\| \sum_{i=m_0+1}^{\infty} (\alpha_i^{s_0} - \alpha_{m_0}^{s_0})\lambda_i e_i \right\| \\
&\leq \frac{1}{4}\|x-y\| + \left\| \sum_{i=m_0+1}^{\infty} x_i e_i \right\| + \|y\| \\
&\leq \frac{1}{4}\|x-y\| + \frac{1}{4}\|x-y\| + \frac{1}{4}\|x-y\| = \frac{3}{4}\|x-y\|.
\end{aligned}$$

The sets U_n are closed for every $n \in \mathbb{N}$ and by Corollary 2.2 there exists $z \in X$, such that $Tz = z$. It is easy to see that $\lim_{s\to\infty} T^s e_1 = \sigma$ and σ is the fixed point of the map T, where σ is the zero vector in X. By Theorem 2.1 we have that for every $y \in X$ there is a sequence of naturals $\{q_i\}_{i=1}^{\infty}$, such that $\lim_{i\to\infty} T^{q_i} y = \sigma$. Therefore for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$, such that $\|T^{q_{k_\varepsilon}} y\| < \varepsilon$. By the fact that $0 < \alpha_i < 1$ for every $i \in \mathbb{N}$ it follows that for every $s \geq q_{k_\varepsilon}$ holds $|\alpha_i|^s \leq |\alpha_i|^{q_{k_\varepsilon}}$ for every $i \in \mathbb{N}$. Now by (5) it follows that for every $s \geq q_{k_\varepsilon}$ there holds the inequality $\|T^s y\| < \varepsilon$. By the arbitrary choice of $\varepsilon > 0$ we get that $\lim\limits_{s\to\infty} T^s y = \sigma$.

A Result that generalizes Theorem 3.1 is the following:

Theorem 3.2. [7] *Let X be a Banach space, and let $T : X \to X$ a mapping. Suppose there exists $B \subset X$ such that*

(G1) *$T(B) \subset B$;*

(G2) *for some $\alpha \in (0,1)$ and each $y \in B$ there is an integer $n(y) \geq 1$ with*

$$\rho(T^{n(y)}x, T^{n(y)}y) \leq \alpha\rho(x,y)$$

for all $x \in B$;

(G3) *for some $x \in B$, $\mathrm{cl}\{T^s x : s \in \mathbb{N}\} \subset B$.*

Then there is a unique $z \in B$ such that $T(z) = z$ and $\lim\limits_{s\to\infty} T^s x = z$ for each $x \in B$. Furthermore, if

$$\rho(T^{n(z)}x, T^{n(z)}z) \leq \alpha\rho(x,z) \tag{6}$$

for all $x \in X$, then z is unique in X and $\lim_{s\to\infty} T^s x = z$ for each $x \in X$.

It is easy to see in Example 3, that for every $0 < \alpha < 1$ and every $n \in \mathbb{N}$ there is $s_n \in \mathbb{N}$ such that

$$\|T^n e_{s_n} - T^n \sigma\| = \|T^n e_{s_n}\| = (\alpha_{s_n})^n \|e_{s_n}\| \geq \alpha \|e_{s_n}\| = \alpha \|e_{s_n} - \sigma\|,$$

which shows that the condition (6) in Theorem 3.2 is not satisfied.

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