Schur property and ℓ_p isomorphic copies in Musielak–Orlicz Sequence Spaces[1]

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ABSTRACT: We show that if the dual of a Musielak–Orlicz sequence space ℓ_Φ is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis then ℓ_Φ is saturated with complemented copies of ℓ_1 and has the Schur property. A sufficient condition is found for the isomorphic embedding of ℓ_p spaces into Musielak–Orlicz sequence spaces.

2000 Mathematics Subject Classification: 46B20, 46B45, 46E30, 46A45

1 Introduction

The notion of asymptotic ℓ_p spaces first appeared in [14], where the collection of spaces that are now known as stabilized asymptotic ℓ_p spaces were introduced. Later in [13] more general collection of spaces, known as asymptotic ℓ_p spaces, were introduced. Characterization of the stabilized asymptotic ℓ_∞ Musielak–Orlicz sequence space was given in [4].

A Banach space X is said to have the Schur property if every weakly null sequence is norm null. It is well known that ℓ_1 has the Schur property and it's dual ℓ_∞ is obviously stabilized asymptotic ℓ_∞ space with respect to the unit vector basis. A characterization of the Musielak–Orlicz sequence spaces ℓ_Φ possessing the Schur property, as well as sufficient conditions for ℓ_Φ and weighted Orlicz sequence spaces $\ell_M(w)$ to have the Schur property were found in [8]. Using an idea from [1] we find that if the dual of a Musielak–Orlicz sequence space is stabilized asymptotic ℓ_∞ space then it is saturated with complemented copies of ℓ_1 and has the Schur property. While simple necessary conditions for embedding of ℓ_p spaces into Musielak–Orlicz spaces ℓ_Φ were found in [16], the problem of finding analogous sufficient conditions, as it is done in [11] for Orlicz ℓ_M, appeared more difficult. We find a sufficient condition for the existence of an ℓ_p copy in ℓ_Φ in Paragraph 4.

2 Preliminaries

We use the standard Banach space terminology from [11]. Let us recall that an Orlicz function M is even, continuous, non-decreasing convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. A sequence $\Phi = \{\Phi_i\}_{i=1}^\infty$ of Orlicz functions is called a Musielak–Orlicz function or MO function in short.

The MO sequence space ℓ_Φ, generated by a MO function Φ is the set of all real sequences $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$. The Luxemburg's norm is defined by

$$\|x\|_\Phi = \inf\left\{r > 0 : \sum_{i=1}^\infty \Phi_i(x_i/r) \le 1\right\}.$$

We denote by h_Φ the closed linear subspace of ℓ_Φ, generated by all $x \in \ell_\Phi$, such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i) < \infty$ for every $\lambda > 0$.

If the MO function Φ consists of one and the same function M one obtains the Orlicz sequence spaces ℓ_M and h_M.

Let $1 \leq p_i$, $i \in \mathbb{N}$ be a sequence of reals. The MO sequence space ℓ_Φ, where $\Phi = \{t^{p_i}\}_{i=1}^\infty$ is called Nakano sequence space and is denoted by $\ell_{\{p_i\}}$. In [3] it was proved that two Nakano sequence spaces $\ell_{\{p_i\}}$, $\ell_{\{q_i\}}$ are isomorphic iff there exists $0 < C < 1$ such that

$$\sum_{i=1}^\infty C^{1/|p_i - q_i|} < \infty.$$

An extensive study of Orlicz and MO spaces can be found in [11], [15], [6] and [9].

Definition 2.1 *We say that the MO function Φ satisfies the δ_2 condition at zero if there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^\infty \in \ell_1$ such that for every $n \in \mathbb{N}$*

$$\Phi_n(2t) \leq K\Phi_n(t) + c_n$$

provided $t \in [0, \Phi_n^{-1}(\beta)]$.

The spaces ℓ_Φ and h_Φ coincide iff Φ has δ_2 condition at zero.

Recall that given MO functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms iff Φ is equivalent to Ψ, that is there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^\infty \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \leq \Psi_n(t) + c_n \quad \text{and} \quad \Psi_n(Kt) \leq \Phi_n(t) + c_n$$

hold for every $t \in [0, \min(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta))]$.

Throughout this paper M will always denote Orlicz function while Φ - an MO function. As the properties we are dealing with are preserved by isomorphisms without loss of generality we may assume that Φ consists entirely of non–degenerate Orlicz functions, such that for every $i \in \mathbb{N}$ the Orlicz function Φ_i is differentiable, $\Phi_i'(0) = 0$ and $\Phi_i(1) = 1$. Indeed, we can always choose a sequence $\{\alpha_i\}$, such that $\alpha_i \leq 1/2$, $i \in \mathbb{N}$, $\sum_{i=1}^\infty \Phi_i(\alpha_i) < \infty$ and consider the sequence of functions $\varphi_i(t) = \int_0^t \frac{\psi_i(s)}{s} ds$, where

$$\psi_i(t) = \begin{cases} \frac{\Phi_i(\alpha_i)}{\alpha_i^2} t^2, & 0 \leq t \leq \alpha_i \\ \Phi_i(t), & t \geq \alpha_i \end{cases} .$$

Obviously the MO function $\varphi = \{\varphi_i\}_{i=1}^\infty$ consists of differentiable functions and $\varphi_i'(0) = 0$ for every $i \in \mathbb{N}$.

For every $t \in [0, \alpha_i]$ we have $\varphi_i(\alpha_i) = \frac{\Phi_i(\alpha_i)}{2}$ and

$$\varphi_i(t) = \int_0^t \frac{\psi_i(s)}{s} ds = \int_0^t \frac{\Phi_i(\alpha_i)}{\alpha_i^2} s ds = \frac{\Phi_i(\alpha_i)}{2\alpha_i^2} t^2 .$$

For every $t \geq \alpha_i$ we have

$$\varphi_i(t) = \int_0^{\alpha_i} \frac{\Phi_i(\alpha_i)}{\alpha_i^2} s ds + \int_{\alpha_i}^t \frac{\Phi_i(s)}{s} ds = \frac{\Phi_i(\alpha_i)}{2} + \int_{\alpha_i}^t \frac{\Phi_i(s)}{s} ds .$$

By the convexity of Φ_i follows that

$$(1) \qquad \varphi_i(t) \leq \frac{\Phi_i(\alpha_i)}{2} + \Phi_i(t)$$

for every $t \geq 0$.

In order to get the opposite inequality we consider separately three cases:

I) Let $\alpha_i \leq \frac{t}{2}$ then

$$
\begin{aligned}
\varphi_i(t) &= \int_0^{\alpha_i} \frac{\psi_i(s)}{s} ds + \int_{\alpha_i}^{t/2} \frac{\psi_i(s)}{s} ds + \int_{t/2}^t \frac{\psi_i(s)}{s} ds \\
&\geq \frac{\Phi_i(\alpha_i)}{2} + \int_{t/2}^t \frac{\Phi_i(s)}{s} ds \geq \frac{\Phi_i(\alpha_i)}{2} + \Phi_i(t/2).
\end{aligned}
$$

II) Let $\frac{t}{2} \leq \alpha_i \leq t$ then

$$
\begin{aligned}
\varphi_i(t) &= \frac{\Phi_i(\alpha_i)}{2} + \int_{t/2}^t \frac{\Phi_i(s)}{s} ds - \int_{t/2}^{\alpha_i} \frac{\Phi_i(s)}{s} ds \\
&\geq \frac{\Phi_i(\alpha_i)}{2} + \Phi_i(t/2) - \Phi_i(\alpha_i) = \Phi_i(t/2) - \frac{\Phi_i(\alpha_i)}{2}.
\end{aligned}
$$

III) Let $t \leq \alpha_i$ then

$$\frac{\Phi_i(t)}{2} \leq \varphi_i(t) + \frac{\Phi_i(\alpha_i)}{2}.$$

Thus

$$(2) \qquad \frac{\Phi_i(t/2)}{2} \leq \varphi_i(t) + \frac{\Phi_i(\alpha_i)}{2}$$

for every $t \geq 0$. By (1) and (2) it follows that $\varphi \sim \Phi$ and thus $\ell_\varphi \cong \ell_\Phi$. To complete it is enough to normalize the functions φ_i by considering $\widetilde{\varphi} = \{\varphi_i/\varphi_i(1)\}_{i=0}^\infty$.

Definition 2.2 *For an Orlicz function M, such that $\lim_{t \to 0} M(t)/t = 0$ the function*

$$N(x) = \sup\{t|x| - M(t) : t \geq 0\},$$

is called function complementary to M.

Definition 2.3 *The MO function $\Psi = \{\Psi_j\}_{j=1}^\infty$, defined by*

$$\Psi_j(x) = \sup\{t|x| - \Phi_j(t) : t \geq 0\}, j = 1, 2, ..., n, ...$$

is called complementary to Φ.

Let us note that the condition $\lim_{t \to 0} M(t)/t = 0$ secures that the complementary function N is always non-degenerate. Observe that if N is function complementary to M, then M is complementary to N and if the MO function Ψ is complementary to the MO function Φ, then Φ is function complementary to Ψ. Throughout this paper the function complementary to the MO function Φ is denoted by Ψ.

It is well known that $h_M^* \cong \ell_N$ and $h_\Phi^* \cong \ell_\Psi$. Well known equivalent norm in ℓ_Φ is the Orlicz norm $\|x\|_\Phi^O = \sup\left\{\sum_{j=1}^\infty x_j y_j : \sum_{j=1}^\infty \Psi_j(y_j) \leq 1\right\}$, which satisfies the inequalities (see e.g.[7])

$$\|\cdot\|_\Phi \leq \|\cdot\|_\Phi^O \leq 2\|\cdot\|_\Phi.$$

We will use the Hölder's inequality: $\sum_{j=1}^\infty |x_j y_j| \leq \|x\|_\Phi^O \|y\|_\Psi$, which holds for every $x = \{x_j\}_{j=1}^\infty \in \ell_\Phi$ and $y = \{y_j\}_{j=1}^\infty \in \ell_\Psi$, where Φ and Ψ are complementary MO functions.

By $\{e_j\}_{j=1}^\infty$ and $\{e_j^*\}_{j=1}^\infty$ we denote the unit vector basis in h_Φ and h_Ψ respectively. For a Banach space X with a basis $\{v_i\}_{i=1}^\infty$ and element $x \in X$, $x = \sum_{i=1}^\infty x_i v_i$ we define $\operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$. We write $n \leq x$ if $n \leq \min\{\operatorname{supp} x\}$ and $x < y$ if $\max\{\operatorname{supp} x\} < \min\{\operatorname{supp} y\}$. We say that x is a block vector with respect to the basis $\{v_i\}_{i=1}^\infty$ if $x = \sum_{i=p}^q x_i v_i$ for some finite p and q and we say that x is a normalized block vector if it is a block vector and $\|x\| = 1$.

Definition 2.4 *A Banach space X is said to be stabilized asymptotic ℓ_∞ with respect to a basis $\{v_i\}_{i=1}^\infty$, if there exists a constant $C \geq 1$, such that for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$, so that whenever $N \leq x_1 < \ldots < x_n$ are successive normalized block vectors, then $\{x_i\}_{i=1}^n$ are C–equivalent to the unit vector basis of ℓ_∞^n, i.e.*

$$\frac{1}{C} \max_{1 \leq i \leq n} |a_i| \leq \left\|\sum_{i=1}^n a_i x_i\right\| \leq C \max_{1 \leq i \leq n} |a_i|.$$

The following characterization of the stabilized asymptotic ℓ_∞ MO sequence spaces is due to Dew:

Proposition 2.1 *(Proposition 4.5.1 [4]) Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a MO function. Then the following are equivalent:*
(i) h_Φ is stabilized asymptotic ℓ_∞ (with respect to its natural basis $\{e_j\}_{j=1}^\infty$);
(ii) there exists $\lambda > 1$ such that for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $N \leq p \leq q$ and $\sum_{j=p}^q \Phi_j(a_j) \leq 1$, then

$$\sum_{j=p}^q \Phi_j(a_j/\lambda) \leq \frac{1}{n}.$$

Let X be a Banach space. By $Y \hookrightarrow X$ we denote that Y is isomorphic to a subspace of X.

3 MO spaces with stabilized asymptotic ℓ_∞ dual with respect to the unit vector basis

We start with the following

Lemma 3.1 *Let Φ has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then every normalized block basis $\{x^{(n)}\}_{n=1}^\infty$ of the unit vector basis in ℓ_Φ contains a subsequence $\{x^{(n_i)}\}_{i=1}^\infty$ such that:*

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a) $\{x^{(n_i)}\}_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_1;

b) The closed subspace $[x^{(n_i)}]_{i=1}^{\infty}$ generated by $\{x^{(n_i)}\}_{i=1}^{\infty}$ is complemented in ℓ_Φ by means of a projection of norm less then or equal to 4λ, where λ is the constant from Proposition 2.1.

Proof: a) Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a normalized block basis of ℓ_Φ, where $x^{(n)} = \sum_{j=m_n+1}^{m_{n+1}} x_j^{(n)} e_j$, $\{m_n\}$ strictly increasing sequence of naturals. For every $n \in \mathbb{N}$ there exists $y^{(n)} = \sum_{j=1}^{\infty} y_j^{(n)} e_j^* \in h_\Psi$ such that

$$\sum_{j=1}^{\infty} \Psi_j(y_j^{(n)}) \leq 1 \quad \text{and} \quad \sum_{j=1}^{\infty} y_j^{(n)} x_j^{(n)} \geq 1/2.$$

WLOG we may assume that supp $y^{(n)} \equiv$ supp $x^{(n)}$. We claim that

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) = \lim_{n\to\infty} \sum_{j=m_n+1}^{m_{n+1}} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) = 0,$$

where $\lambda > 1$ is the constant from Proposition 2.1.

Indeed, by assumption h_Ψ is stabilized asymptotic ℓ_∞ space and according to Proposition 2.1 there exists $\lambda > 1$ such that for every $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ so, that whenever $m_n \geq N$ the inequality holds $\sum_{j=m_n+1}^{m_{n+1}} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) \leq 1/m$. Thus

$$\lim_{n\to\infty} \sum_{j=m_n+1}^{m_{n+1}} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) = 0.$$

Now passing to a subsequence we get a sequence $\{y^{(n_k)}\}_{k\in\mathbb{N}}$, $y^{(n_k)} = \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} e_j^*$ such that

$$\sum_{k=1}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) \leq 1.$$

Denote $y = \sum_{k=1}^{\infty} y^{(n_k)} = \sum_{k=1}^{\infty} \left(\sum_{j=p_k}^{q_k} y_j^{(n_k)} e_j^*\right)$. Obviously $y \in \ell_\Psi$ and $\|y\|_\Psi \leq \lambda$. Thus $\|y\|_\Psi^O \leq 2\|y\|_\Psi \leq 2\lambda$.

Let now $a = \{a_k\}_{k=1}^{\infty} \in \ell_1$. Then

$$\left\|\sum_{k=1}^{\infty} a_k x^{(n_k)}\right\|_\Phi \geq \frac{1}{\|y\|_\Psi^O} \sum_{k=1}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \left|a_k y_j^{(n_k)} x_j^{(n_k)}\right| \geq \frac{1}{2\lambda} \sum_{k=1}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} \left|a_k y_j^{(n_k)} x_j^{(n_k)}\right|$$

$$\geq \frac{1}{2\lambda} \sum_{k=1}^{\infty} |a_k| \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j^{(n_k)} \geq \frac{1}{4\lambda} \sum_{k=1}^{\infty} |a_k| = \frac{1}{4\lambda} \|a\|_1.$$

Obviously $\|\sum_{k=1}^{\infty} a_k x^{(n_k)}\|_\Phi \leq \|a\|_1$ and thus $\{x^{(n_k)}\}_{k=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_1.

b) Define now for each $k \in \mathbb{N}$ the functional $F_k : \ell_\Phi \to \mathbb{R}$ by

$$F_k(x) = \frac{1}{\sum\limits_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j^{(n_k)}} \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j$$

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and the map $P : \ell_\Phi \to \ell_\Phi$ by $P(x) = \sum_{k=1}^\infty F_k(x) x^{(n_k)}$. Then for every $k \in \mathbb{N}$ holds

$$\|F_k\| \leq 2\|y^{(n_k)}\|_\Psi \leq 2 \left(1 + \sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j \left(y_j^{(n_k)} \right) \right) \leq 4.$$ Furthermore P is a projection of ℓ_Φ onto $[x_{n_k}]_{k=1}^\infty$ with

$$
\begin{aligned}
\|P\| &= \sup_{\|x\|_\Phi \leq 1} \left\| \sum_{k=1}^\infty \frac{\sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j}{\sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} x_j^{(n_k)}} x^{(n_k)} \right\| \leq 2 \sup_{\|x\|_\Phi \leq 1} \sum_{k=1}^\infty \sum_{j=p_{n_k}}^{q_{n_k}} \left| y_j^{(n_k)} x_j \right| \\
&\leq 2 \sup_{\|x\|_\Phi \leq 1} \sum_{j=1}^\infty |y_j x_j| \leq 2 \sup_{\|x\|_\Phi \leq 1} \|y_j\|_\Psi^O \|x\|_\Phi \leq 4\lambda .
\end{aligned}
$$

$\square$

The following two theorems are simple corollaries of Lemma 3.1.

Theorem 1 *Let Φ has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then ℓ_Φ has the Schur property.*

Proof: The proof is an easy consequence of Kaminska, Mastylo characterization of MO spaces possessing Schur property ([8], Theorem 4.4). Consider a $\boldsymbol{\Phi}$–convex block of Φ, i.e. a sequence of convex functions $\left\{ M_i(t) = \sum_{j=n_i+1}^{n_{i+1}} \Phi_j(t\alpha_j) \right\}_{i=1}^\infty$, where n_i is a strongly increasing sequence in $\mathbb{N}$ and $\{\alpha_j\}_{j=1}^\infty$ is a sequence of positive numbers such that $\sum_{j=n_i+1}^{n_{i+1}} \Phi_j(\alpha_j) = 1$ for each $i \in \mathbb{N}$. It is easy to observe that the sequence $\left\{ u_i = \sum_{j=n_i+1}^{n_{i+1}} \alpha_j e_j \right\}_{i=1}^\infty$ is normalized block-basis of the unit vector basis of ℓ_Φ. Lemma 3.1 now implies that the closed linear span $[u_{i_k}]_{k=1}^\infty$ for appropriate subsequence $\{u_{i_k}\}_{k=1}^\infty$ is isomorphic to ℓ_1. On the other hand $[u_{i_k}]_{k=1}^\infty$ is obviously isometrically isomorphic to the MO space $\ell_{\{M_{i_k}\}}$, generated by the subsequence $\{M_{i_k}\}$ of the given $\boldsymbol{\Phi}$–convex block. Thus every $\boldsymbol{\Phi}$–convex block contains a subsequence equivalent to a linear function and therefore ℓ_Φ has the Schur property. $\square$

Theorem 2 *Let Φ has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then every subspace Y of ℓ_Φ contains an isomorphic copy of ℓ_1 which is complemented in ℓ_Φ.*

Proof: According to a well known result of Bessaga and Pelczinski [2] every infinite dimensional closed subspace Y of ℓ_Φ has a subspace Z isomorphic to a subspace of ℓ_Φ, generated by a normalized block basis of the unit vector basis of ℓ_Φ. Now to finish the proof it is enough to observe that by Lemma 3.1 the space Z contains a complemented subspace of ℓ_Φ, which is isomorphic to ℓ_1. $\square$

Remark: It is well known ([18]) that every subspace of MO sequence space ℓ_Φ with Φ satisfying the δ_2 condition, contains ℓ_p for some $p \in [1, \infty]$. If ℓ_Φ has in addition the Schur property, as no $\ell_p, p \neq 1$ has the Schur property, it follows that ℓ_Φ is ℓ_1 saturated.

4 ℓ_p copies in MO sequence spaces

Let Φ be a MO function consisting of differentiable Orlicz functions. Denote:

$$a(\Phi_n) = \sup\left\{p > 0 : p \leq \frac{x\Phi_n'(x)}{\Phi_n(x)}, x \in (0, 1]\right\} ;$$

$$b(\Phi_n) = \inf\left\{q > 0 : q \geq \frac{x\Phi_n'(x)}{\Phi_n(x)}, x \in (0, 1]\right\} .$$

The following indexes, introduced by Yamamuro ([17])

$$a(\Phi) = \liminf_{n\to\infty} a(\Phi_n) \ , \ \ b(\Phi) = \limsup_{n\to\infty} b(\Phi_n)$$

appear to be useful in the study of MO sequence spaces (see for example [11], [16], [8] and [12]). Obviously $1 \leq a(\Phi) \leq b(\Phi) \leq \infty$. By the results of Woo ([18]) and Katirtzoglou ([9]) it follows that an MO function Φ satisfies the δ_2 condition at zero iff $b(\Phi) < \infty$. Analogously to the case of the classical Orlicz sequence spaces if ℓ_p, , $p \geq 1$ or c_0 for $p = \infty$ is isomorphic to a subspace of h_Φ, then $p \in [a(\Phi), b(\Phi)]$ (see [16], [18]). However, the converse fails to be true in general (see [16]) for MO sequence spaces, which confirms their more complex structure. Sufficient conditions for the isomorphical embedding of ℓ_p, , $p \geq 1$ in h_Φ are given by the following:

Theorem 3 *Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a MO function and $p \in [a(\Phi), b(\Phi)]$. If there exist sequences $\{\tau_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$, $\{\varepsilon_j\}_{j=1}^\infty$ and constants $0 < k < 1 < K$ such that:*
1) $\varepsilon_j \geq 0$, $0 < y_j \leq 1$ $0 < \tau_j < 1$ for every $j \in \mathbb{N}$;
2) $\lim_{j\longrightarrow\infty} \tau_j = 0$;
3) $\sum_{j=1}^\infty \Phi_j(y_j) = \infty$;
4) $kt^{\varepsilon_j} \leq \dfrac{\Phi_j(ty_j)}{t^p \Phi_j(y_j)} \leq K(1/t)^{\varepsilon_j}$ for every $t \in [\tau_j, 1]$;
5) $\sum_{j=1}^\infty C^{1/\varepsilon_j} < \infty$ for some $0 < C < 1$,
then $\ell_p \hookrightarrow h_\Phi$.

Proof: The condition 5) obviously implies $\lim_{j\longrightarrow\infty} \varepsilon_j = 0$.

We may assume that $\tau_j < 1/2$ for every j. Indeed, by 2) we easily get $\tau_j < 1/2$, $j < j_0$ for some j_0 and can consider the MO sequence space $h_{\{\Phi_j\}_{j=j_0}^\infty} \cong h_\Phi$.

Consider first the case: $\sharp\{j \in \mathbb{N} : \Phi(y_j) \geq 1/2\} < \infty$. For the same reason as above we may assume that $\Phi(y_j) \leq 1/2$ for every $j \in \mathbb{N}$.

Find sequence of naturals $\{k_n\}_{n=1}^\infty$, $k_1 = 0$, such that for every $n \in \mathbb{N}$:

$$\frac{1}{2} \leq \sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j) < 1 \ , \ \ \Phi_{k_{n+1}}(y_{k_{n+1}}) \geq 1 - \sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j) .$$

Put

$$\varphi_n(t) = \sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j t) + \Phi_{k_{n+1}}(\overline{y}_{k_{n+1}} t) ,$$

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where

(3)
$$\sum_{j=k_n+1}^{k_{n+1}-1} \Phi_j(y_j) + \Phi_{k_{n+1}}(\overline{y}_{k_{n+1}}) = 1\,.$$

Obviously

(4)
$$\sum_{j=k_n+1}^{k_{n+1}} \Phi(y_j) < \frac{3}{2}$$

and $0 < \overline{y}_{k_{n+1}} \le y_{k_{n+1}}$. Let us note that $u_n = \sum_{j=k_n+1}^{k_{n+1}-1} y_j e_j + \overline{y}_{k_{n+1}} e_{k_{n+1}}$, $n = 1, 2, \dots$ represents a normalized block basis of the unit vector basis of h_Φ. Obviously the MO sequence space $h_{\overline{\Phi}}$, generated by the sequence $\{\varphi_n\}$ is isometrically isomorphic to $[u_n]_{n=1}^\infty$, which in turn is isomorphic to a subspace of h_Φ. Further we find a sequence of $\{n_m\}_{m=1}^\infty$, such that $\tau_j \le \frac{1}{m^2}$ for $j > k_{n_m}$. Following [11] and [10] we easily check that the functions φ_{n_m}, $m = 1, 2, \dots$ are equi–continuous in $[0, 1/2]$. Indeed, from

$$\Phi_j(t) = \int_0^t \Phi_j'(t)dt \ge \int_{t/2}^t \Phi_j'(t)dt \ge \frac{1}{2}t\Phi_j(t/2)$$

it follows immediately

$$\left| \frac{\Phi_j(\mu t_1)}{\Phi_j(\mu)} - \frac{\Phi_j(\mu t_2)}{\Phi_j(\mu)} \right| \le |t_1 - t_2| \frac{\mu \Phi_j'(\mu/2)}{\Phi_j(\mu)} \le 2|t_1 - t_2|$$

for every $0 \le t_1, t_2 \le 1/2$ and any $\mu > 0$. Now it is enough to apply the last inequality to the functions φ_{n_m}, taking into account (3). The functions φ_{n_m}, $m = 1, 2, \dots$ are also uniformly bounded in $[0, 1/2]$. Using the Arzela-Ascoli theorem by passing to a subsequence if necessary, which in order to simplify the notations we denote $\{\varphi_{n_m}\}_{m=1}^\infty$ too, we have that $\{\varphi_{n_m}\}_{m=1}^\infty$ converges uniformly to a function φ on $[0, 1/2]$, satisfying the inequalities $\|\varphi_{n_m} - \varphi\|_\infty \le \frac{1}{2^m}$ for every $m \in \mathbb{N}$. Obviously φ is an Orlicz function on $[0,1/2]$ as uniform limit of Orlicz functions and the MO sequence space $h_{\{\varphi_{n_m}\}}$ is isomorphic to the Orlicz space h_φ, when φ is non-degenerated. If we take into account that $h_{\{\varphi_{n_m}\}}$ is isometrically isomorphic to $[u_{n_m}]_{m=1}^\infty$ to finish the proof it is enough to show that h_φ and ℓ_p consist of the same sequences. Before starting the last part of the proof we mention that according to the result from [3], mentioned in the preliminaries, the condition 5) implies that the Nakano spaces $\ell_{\{p+\nu_j\varepsilon_j\}_{j=1}^\infty}$ are isomorphic to ℓ_p for every choice of the sequence of signs $\{\nu_j = \pm 1\}_{j=1}^\infty$.

Define the sets:

$$A_m = \left\{ j \in \mathbb{N} : k_{n_m} + 1 \le j \le k_{n_{m+1}}, \tau_j \ge \alpha_m \right\}$$

and

$$B_m = \left\{ j \in \mathbb{N} : k_{n_m} + 1 \le j \le k_{n_{m+1}}, \tau_j < \alpha_m \right\}.$$

It is obvious that $A_m \cap B_m = \emptyset$ and $A_m \cup B_m = \{k_{n_m} + 1, \dots, k_{n_{m+1}}\}$. Let $\delta_m = \max\{\varepsilon_j : k_{n_m} + 1 \le j \le k_{n_{m+1}}\}$. Then $\{\delta_m\}_{m=1}^\infty$ is a subsequence of $\{\varepsilon_j\}_{j=1}^\infty$ and thus by 5) we obtain $\sum_{m=1}^\infty C^{1/\delta_m} < \infty$. So the Nakano spaces $\ell_{\{p+\nu_m\delta_m\}}$ consist of the same sequences as ℓ_p for every choice of the signs $\{\nu_m = \pm 1\}$.

Let now $\{\alpha_j\}_{j=1}^\infty \in \ell_p$ i.e. $\sum_{j=1}^\infty \alpha_j^p < \infty$. We may assume that $\alpha_j \le 1/2$ for every $j \in \mathbb{N}$. Now we can write the chain of inequalities:

$$
\begin{aligned}
\sum_{m=1}^\infty \varphi_{n_m}(\alpha) &= \sum_{m=1}^\infty \left(\sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \Phi_j(\alpha_m y_j) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \right) \\
&\le \sum_{m=1}^\infty \sum_{j=k_{n_m}+1}^{k_{n_{m+1}}} \Phi_j(\alpha_m y_j) \le \sum_{m=1}^\infty \sum_{j \in A_m} \Phi_j(\alpha_m y_j) + \sum_{m=1}^\infty \sum_{j \in B_m} \Phi_j(\alpha_m y_j) \\
&\le \sum_{m=1}^\infty \sum_{j \in A_m} \Phi_j(\tau_m y_j) + \sum_{m=1}^\infty \sum_{j \in B_m} K \alpha_m^{p-\delta_m} \Phi_j(y_j) \\
&\le \sum_{m=1}^\infty \sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \tau_j \Phi_j(y_j) + K \sum_{m=1}^\infty \alpha_m^{p-\delta_m} \sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \Phi_j(y_j) \\
&\le \frac{3K}{2} \left\{ \sum_{m=1}^\infty \frac{1}{m^2} + \sum_{m=1}^\infty \alpha_m^{p-\delta_m} \right\} < \infty ,
\end{aligned}
$$

where we used that $0 < \overline{y}_{k_{n+1}} \le y_{k_{n+1}}$ for the second and (4) for the last inequality.
Let now $\alpha = \{\alpha_m\}_{m=1}^\infty \in \ell_{\{\varphi_{n_m}\}}$, i.e.

$$
\sum_{m=1}^\infty \varphi_{n_m}(\alpha_m) = \sum_{m=1}^\infty \left(\sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \Phi_j(\alpha_m y_j) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \right) < \infty .
$$

It is not difficult to check that for every $m \in \mathbb{N}$ the estimate holds:

$$
(5) \qquad |\alpha_m|^{p+\delta_m} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) \le \frac{1}{m^2} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) .
$$

Denote $A'_m = A_m \setminus \{n_{m+1}\}$ and $B'_m = B_m \setminus \{n_{m+1}\}$ Now taking into account (3), (4) and (5) we can write the chain of inequalities:

$$
\begin{aligned}
\sum_{m=1}^\infty |\alpha_m|^{p+\delta_m} &= \sum_{m=1}^\infty |\alpha_m|^{p+\delta_m} \left(\sum_{j=k_{n_m}+1}^{k_{n_{m+1}}-1} \Phi_j(y_j) + \Phi_{n_{m+1}}(\overline{y}_{k_{n_{m+1}}}) \right) \\
&\le \sum_{m=1}^\infty \left(|\alpha_m|^{p+\delta_m} \left(\sum_{j \in A'_m} \Phi_j(y_j) + \sum_{j \in B'_m} \Phi_j(y_j) \right) \right. \\
&\qquad \left. + \frac{1}{m^2} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) + \Phi_{n_{m+1}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \right) \\
&\le \sum_{m=1}^\infty \left(\sum_{j \in A'_m} (\tau_j)^{p+\delta_m} \Phi_j(y_j) + \sum_{j \in B'_m} |\alpha_m|^{p+\delta_m} \Phi_j(y_j) \right. \\
&\qquad \left. + \frac{1}{m^2} \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \right) \\
&\le \sum_{m=1}^\infty \frac{1}{m^2} \left(\sum_{j \in A'_m} \Phi_j(y_j) + \Phi_{k_{n_{m+1}}}(\overline{y}_{k_{n_{m+1}}}) \right) \\
&\qquad + \frac{1}{k} \sum_{m=1}^\infty \left(\sum_{j \in B'_m} \Phi_j(\alpha_m y_j) + \Phi_{k_{n_{m+1}}}(\alpha_m \overline{y}_{k_{n_{m+1}}}) \right) \\
&\le \frac{1}{k} \left(\sum_{m=1}^\infty \frac{1}{m^2} + \sum_{m=1}^\infty \varphi_{n_m}(\alpha_m) \right) < \infty ,
\end{aligned}
$$

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which concludes the proof.

Let now $1/2 \leq \Phi(y_{j_k}) \leq 1$ for some increasing sequence of naturals $\{j_k\}_{k=1}^{\infty}$. Passing to a subsequence if necessary we may assume that $\sum_{k=1}^{\infty} \tau_{j_k} < \infty$. Then

$$\Phi_{j_k}(t) \geq \Phi_{j_k}(ty_{j_k}) \geq kt^{p+\varepsilon_{j_k}} \Phi_{j_k}(y_{j_k}) \geq \frac{k}{2} t^{p+\varepsilon_{j_k}}$$

for every $t \in [\tau_{j_k}, 1]$. Consequently

$$(6) \qquad\qquad u^{p+\varepsilon_{j_k}} \leq \frac{2}{k} \Phi_{j_k}(u) + \tau_{j_k}.$$

holds for every $u \in [0, 1]$. Similarly

$$\Phi_{j_k}(t/2) \leq \Phi_{j_k}(ty_{j_k}) \leq 2^{p-\varepsilon_{j_k}} K \left(\frac{t}{2}\right)^{p-\varepsilon_{j_k}} \Phi_{j_k}(y_{j_k})$$

for every $t \in [\tau_{j_k}, 1]$. Thus

$$\Phi_{j_k}(u) \leq K_1 u^{p-\varepsilon_{j_k}}$$

holds for every $u \in [\tau_{j_k}/2, 1/2]$, where $K_1 = 2^p K$. So

$$(7) \qquad\qquad \Phi_{j_k}(u) \leq K_1 u^{p-\varepsilon_{j_k}} + \tau_{j_k}$$

holds for every $u \in [0, 1/2]$. Consequently by (6) and (7) follows that $\ell_p \cong \ell_{\{\Phi_{j_k}\}} \hookrightarrow \ell_\Phi$.
$\square$

Remark: If the conditions in Theorem 3 hold for a subsequence $\{\Phi_{n_k}\}_{k=1}^{\infty}$ then $\ell_p \hookrightarrow \ell_{\{\Phi_{n_k}\}} \hookrightarrow \ell_\Phi$.

Corollary 4.1 *Let* $\Phi = \{\Phi_j\}_{j=1}^{\infty}$ *be a MO function and* $\frac{\Phi_j(ty_j)}{\Phi_j(y_j)}$ *converges uniformly to* t^p *on* $[0,1]$ *for some sequence* $\{y_j\}_{j=1}^{\infty}$ *such that,* $0 < y_j \leq 1$, $\sum_{j=1}^{\infty} \Phi_j(y_j) = \infty$ *and* $p \in [a(\Phi), b(\Phi)]$. *Then* $\ell_p \hookrightarrow h_\Phi$.

Proof: Pick a decreasing sequence $\{\delta_k\}_{k=1}^{\infty}$, such that $\lim_{k\to\infty} \delta_k = 0$. There exists $j(k)$ such that for every $j \geq j(k)$ the inequalities hold:

$$(8) \qquad\qquad t^p - \delta_k < \frac{\Phi_j(ty_j)}{\Phi_j(y_j)} < t^p + \delta_k$$

for every $t \in [0,1]$. Thus (8) implies

$$(1/2)t^0 \leq 1 - \delta_k/t^p < \frac{\Phi_j(ty_j)}{t^p \Phi_j(y_j)} < 1 + \delta_k/t^p \leq 2(1/t)^0$$

for every $t \in [(2\delta_k)^{1/p}, 1]$ and for every $j \geq j(k)$. We define inductively sequences $\{r(k)\}$ and $\{s(k)\}$ in the following way. We put $r(1) = j(1)$ and choose $s(1)$ with $\sum_{j=r(1)}^{r(1)+s(1)} \Phi_j(y_j) > 1/2$. If $r(k)$, $s(k)$ are already chosen we put $r(k+1) = \max(r(k) + s(k), j(k+1))$ and choose $s(k+1)$ such that $\sum_{j=r(k+1)}^{r(k+1)+s(k+1)} \Phi_j(y_j) > 1/2$. Now we can

apply Theorem 3 for the subsequence $\{\Phi_{j_m}\}_{m=1}^{\infty}$ and the sequences $\{\varepsilon_m = 0\}$, $\{\tau_m = (2\delta_m)^{1/p}\}$, $m \in \mathbb{N}$, where for every m the index j_m is of the form $j_m = \sum_{i=1}^{k-1} s(i) + p$ for some $k \in \mathbb{N}$ and p with $1 \leq p \leq s(k)$, while $\varepsilon_m = 0$, $\delta_m = \delta_k$. $\square$

Remark: In particular if the sequence of Orlicz functions $\Phi = \{\Phi_j\}_{j=1}^{\infty}$ converges uniformly on $[0, 1]$ to t^p for some $p \in [a(\Phi), b(\Phi)]$ then $\ell_p \hookrightarrow h_\Phi$.

An easy to apply form of Theorem 3 is given by the following

Corollary 4.2 *Let $\Phi = \{\Phi_j\}_{j=1}^{\infty}$ be a MO function and $p \in [a(\Phi), b(\Phi)]$. If there exist sequences $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$, $\{\varepsilon_j\}_{j=1}^{\infty}$ such that:*
1) $\varepsilon_j \geq 0$, $0 < x_j \leq y_j \leq 1$ for every $j \in \mathbb{N}$;
2) $\lim_{j \longrightarrow \infty} \dfrac{x_j}{y_j} = 0$;
3) $\sum_{j=1}^{\infty} \Phi_j(y_j) = \infty$;
4) $p - \varepsilon_j \leq \dfrac{u\Phi_j'(u)}{\Phi_j(u)} \leq p + \varepsilon_j$ for every $u \in [x_j, y_j]$;
5) $\sum_{j=1}^{\infty} C^{1/\varepsilon_j} < \infty$ for some $0 < C < 1$,
then $\ell_p \hookrightarrow h_\Phi$.

For the prove it is enough to rewrite the inequalities from 4) in the form:

$$(9) \qquad p - \varepsilon_j \leq \frac{ty_j \Phi_j'(ty_j)}{\Phi_j(ty_j)} \leq p + \varepsilon_j \quad \text{for every} \quad t \in [x_j/y_j, 1].$$

After integration in (9) we easily get for every $n \in \mathbb{N}$:

$$(10) \qquad t^{p+\varepsilon_j} \Phi_j(y_j) \leq \Phi_j(ty_j) \leq t^{p-\varepsilon_j} \Phi_j(y_j)$$

for every $t \in [x_j/y_j, 1]$. Now we can apply Theorem 3 with $\tau_j = x_j/y_j$. $\square$

We will illustrate some applications of Theorem 3 and the necessity of some of the conditions in it by the following four examples. By examples 1) and 2) we show that conditions 2) and 3) in Theorem 3 could not be omitted.

The next example represents a convexfied analog to an example from [16]

Example 1: Let

$$f_n(x) = \begin{cases} x & if \quad x \geq 1/n^2 \\ n^2 x^2 & if \quad x \in [0, 1/n^2]. \end{cases}$$

Obviously

$$\frac{f_n(x)}{x} = \begin{cases} 1 & if \quad x \geq 1/n^2 \\ n^2 x & if \quad x \in [0, 1/n^2] \end{cases}$$

is an increasing function and therefore

$$\Phi_n(x) = \int_0^x \frac{f_n(t)}{t} dt = \begin{cases} x - \dfrac{1}{2n^2} & if \quad x \geq 1/n^2 \\ \dfrac{n^2}{2} x^2 & if \quad x \in [0, 1/n^2]. \end{cases}$$

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is an Orlicz function.

It is easy to check that

$$\frac{\Phi_n\left(\frac{t}{n^2}\right)}{t^2\Phi_n\left(\frac{1}{n^2}\right)} = 1$$

for every $n \in \mathbb{N}$ and every $t \in [0,1]$. Therefore for the sequences $\{y_n = 1/n^2\}_{n=1}^{\infty}$, $\{\varepsilon_n = 0\}_{n=1}^{\infty}$ and any arbitrary sequence $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \searrow 0$ all the conditions of Theorem 3 hold except for the condition 3) $(\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} 1/n^2 < \infty)$. Nonetheless $\ell_2 \not\hookrightarrow \ell_{\Phi_n}$ because the inequalities

$$\Phi_n(x) \le x \quad \text{and} \quad x \le \Phi_n(x) + \frac{1}{2n^2}, \quad \text{for every} \quad x \in [0,+\infty).$$

imply $\ell_1 \cong \ell_\Phi$.

Let for the next two examples $k_n = 2n\left(1 - \sqrt{1 - \frac{1}{n}}\right)$, $b_n = 1 - k_n$, $\alpha_n = 1 - \sqrt{1 - \frac{1}{n}}$, $n \in \mathbb{N}$. It is easy to see that $1/2n \le \alpha_n \le 1/n$.

Example 2: Consider the functions

$$\Phi_n(x) = \begin{cases} k_n x + b_n & if \quad x \ge \alpha_n \\[2mm] nx^2 & if \quad x \in \left[\frac{\alpha_n}{2}, \alpha_n\right] \\[2mm] \frac{n\alpha_n}{2}x & if \quad x \in \left[0, \frac{\alpha_n}{2}\right] \end{cases}$$

Obviously by the choice of the sequences k_n, b_n and α_n follows that Φ_n are Orlicz functions.

It is easily to check that

$$\frac{\Phi_n(t\alpha_n)}{t^2\Phi_n(\alpha_n)} = 1$$

for every $n \in \mathbb{N}$ and for every $t \in [1/2,1]$. Obviously $\sum_{n=1}^{\infty} \Phi_n(\alpha_n) = \sum_{n=1}^{\infty} n.\alpha_n^2 = \infty$. Therefore for the sequences $\{y_n = \alpha_n\}_{n=1}^{\infty}$, $\{\varepsilon_n = 0\}_{n=1}^{\infty}$ and $\{\tau_n = 1/2\}_{n=1}^{\infty}$ all the conditions of Theorem 3 hold except for the condition 2) $(\lim_{n\to\infty} \tau_n = 0)$. Nonetheless $\ell_2 \not\hookrightarrow \ell_{\Phi_n}$ because $\ell_1 \cong \ell_\Phi$.

Indeed consider now the Nakano sequence space $\ell_{\{p_n\}}$, where $p_n = 1 + \frac{1}{\ln n^2}$. According to [3] $\ell_1 \cong \ell_{\{p_n\}}$. It is easy to check that $x^{p_n} \le \Phi_n(x) \le x$, for every $x \in [0,1]$, because the solutions of the equation: $nx^2 = x^{p_n}$ are $x_1 = 0$ and $x_2 = \left(\frac{1}{n}\right)^{\frac{1}{2 - p_n}}$ and $x_2 < \frac{1}{4n} < \frac{\alpha_n}{2}$. Thus $\ell_1 \cong \ell_\Phi$ which in turn implies $\ell_2 \not\hookrightarrow \ell_{\Phi_n}$.

Similar calculations can be done in Examples 1) and 2) to show that conditions 2) and 3) in Corollary 4.2 do not hold.

The next example shows that the indexes

$$\alpha_\Phi = \liminf_{n\to\infty} \alpha_{\Phi_n}, \quad \beta_\Phi = \limsup_{n\to\infty} \beta_{\Phi_n},$$

where α_{Φ_n} and β_{Φ_n} are the Boyd indexes of Φ_n (see e.g. [11] p.143) are irrelevant when embedding of ℓ_p – spaces into ℓ_Φ are investigated. This fact is not surprising taking into account that among the MO functions Ψ equivalent to a given MO function Φ there exist such with $\alpha_\Psi = \beta_\Psi = 1$ ([18]).

Example 3: Let $\{t_n\}_{n=1}^\infty$ be a sequence such that $\lim_{n\to\infty} t_n = 0$ and $t_n < 1/2$ for every $n \in \mathbb{N}$. Let define the functions

$$\Phi_n(x) = \begin{cases} k_n x + b_n & if \quad x \geq \alpha_n \\[2mm] nx^2 & if \quad x \in \left[\frac{t_n}{n}, \alpha_n\right] \\[2mm] t_n x & if \quad x \in \left[0, \frac{t_n}{n}\right] \end{cases}$$

Obviously by the choice of the sequences k_n, b_n and α_n follows that Φ_n are Orlicz functions which are differentiable for every $x \in [0,1]$ except for $x = t_n/n$ and $x = \alpha_n$.

It easy to see that $\ell_1 \cong \ell_{\{\Phi_{2^n}\}} \hookrightarrow \ell_\Phi$ because $\Phi_{2^n}(x) \leq x \leq \Phi_{2^n}(x) + \alpha_{2^n}$ and $\sum_{n=1}^\infty \alpha_{2^n} < \infty$.

The conditions $\frac{u\Phi_n'(u)}{\Phi_n(u)} = 2$ for every $u \in [\frac{t_n}{n}, \alpha_n]$, $\sum_{n=1}^\infty \Phi_n(\alpha_n) = \infty$ and $\lim_{n\to\infty} \frac{t_n}{n\alpha_n} = 0$ ensure that by Corollary 4.2 $\ell_2 \hookrightarrow \ell_\Phi$.

To calculate the Boyd indexes we have to observe that the functions Φ_n are linear for $t \in [0, t_n/n]$ and thus $1 = \alpha_\Phi = \beta_\Phi$.

We have that $\frac{u\Phi_n'(u)}{\Phi_n(u)} = 1$ for every $u \in [0, t_n/n]$. So we obtain that $1 = a(\Phi) < b(\Phi) = 2$. Thus there exists a MO sequence space ℓ_Φ such that $\ell_2 \hookrightarrow \ell_\Phi$ and $2 \notin [\alpha_\Phi, \beta_\Phi]$.

Following [5] we will construct an example of a weighted Orlicz sequence space which contains an isomorphic copy of ℓ_1.

Example 4: Let the sequences $\{d_n\}_{n=1}^\infty$ and $\{a_n\}_{n=1}^\infty$ be such that $d_n \leq d_{n+1}$, $a_n \leq a_{n+1}$, $\lim_{n\to\infty} \frac{d_n}{d_{n+1}} = 0$, $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} a_n \frac{d_n}{d_{n+1}} = 0$ and $\sum_{n=1}^\infty C^{a_n} < \infty$ for some $0 < C < 1$. Let define the Orlicz function

$$M(x) = \begin{cases} x^2 & if \quad 0 \leq x \leq 1 \\[2mm] A_n x + B_n & if \quad d_n \leq x \leq d_{n+1}, \end{cases}$$

where $A_n = d_{n+1} + d_n$, $B_n = -d_{n+1}d_n$.

Let the sequence $w = \{w_n\}_{n=1}^\infty$ be defined by $w_n = \dfrac{1}{\Phi(d_{n+1})} = \dfrac{1}{d_{n+1}^2}$. Then $\ell_\Phi(w) \cong \ell_{\{\Phi_n\}}$, where $\Phi_n(x) = \dfrac{\Phi(d_{n+1}x)}{\Phi(d_{n+1})}$.

Thus

$$\frac{x\Phi_n'(x)}{\Phi_n(x)} = \frac{xd_{n+1}\frac{\Phi_n'(d_{n+1}x)}{\Phi_n(d_{n+1})}}{\frac{\Phi_n(d_{n+1}x)}{\Phi_n(d_{n+1})}} = \frac{xd_{n+1}A_n}{xd_{n+1}A_n + B_n}$$

for $\dfrac{d_n}{d_{n+1}} \leq s \leq 1$.

After easy calculations we obtain the inequalities:

$$1 - \frac{1}{a_n - 1} < 1 + \frac{d_n}{d_{n+1}} \leq \frac{x\Phi_n^{'}(x)}{\Phi_n(x)} \leq 1 + \frac{1}{a_n - 1}$$

for every $a_n \frac{d_n}{d_{n+1}} \leq x \leq 1$.

Thus $\sum_{n=1}^{\infty} C^{a_n - 1} = 1/C \sum_{n=1}^{\infty} C^{a_n} < \infty$ and we can apply Corollary 4.2 with $y_n = 1$, $x_n = a_n \frac{d_n}{d_{n+1}}$, $\varepsilon_n = \frac{1}{a_n - 1}$ to show that $\ell_1 \hookrightarrow \ell_\Phi(w) \cong \ell_{\{\Phi_n\}}$.

Remark: If

(11)
$$\sum_{n=1}^{\infty} \frac{d_n}{d_{n+1}} < 1/2$$

it is proved in [5] that $\ell_1 \cong \ell_M(w)$.

Remark: By choosing the sequences $\{d_n = n!\}_{n=1}^{\infty}$ and $\{a_n = \log n^2\}_{n=1}^{\infty}$ in Example 4 we get a weighted Orlicz sequence space $\ell_M(w)$ generated by an Orlicz function M which does not verify Δ_2–condition at infinity and a weight sequence $w = \left\{ w_n = \frac{1}{((n+1)!)^2} \right\}_{n=1}^{\infty}$, but containing an isomorphic copy of ℓ_1. Indeed $\frac{M(2n!)}{M(n!)} = 3 + n$ and thus M does not satisfy the Δ_2–condition at ∞. The sequences $\{d_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ verify the conditions imposed on them in Example 4 and thus $\ell_1 \hookrightarrow \ell_M(w)$.

Following [4] we define a sequence of real numbers $\{\psi_\lambda(j)\}_{j=1}^{\infty}$ by

$$\psi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : t > 0\}.$$

Proposition 4.1 *(Proposition 4.5.3 [4]) Let $\Phi = \{\Phi_j\}_{j=1}^{\infty}$ be a MO function. Suppose that for some $\lambda > 1$, $\lim_{j \to \infty} \psi_\lambda(j) = \infty$, then h_Φ is stabilized asymptotic ℓ_∞.*

Let us mention that in the proof of Proposition 4.1 it is choosen a_j such that $\sum_{j=p}^{q} \Phi(a_j) \leq 1$. Thus the function $\psi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : t > 0\}$ can be replaced by $\psi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : 0 < t \leq 1\}$.

Corollary 4.3 *Let Φ has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ If there exist sequences: $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ and $\{\varepsilon_j\}_{j=1}^{\infty}$ fulfilling:*
1') $\varepsilon_j > 0$, $0 < x_j \leq y_j \leq 1$ for every $j \in \mathbb{N}$;
2') $\lim_{j \to \infty} \frac{x_j}{y_j} = 0$;
3') $\sum_{j=1}^{\infty} \Phi_j(y_j) = \infty$;
4') $b(\Phi) - \varepsilon_j \leq \frac{u\Phi_j^{'}(u)}{\Phi_j(u)} \leq b(\Phi) + \varepsilon_j$ for any $u \in [x_j, y_j]$;
5') $\sum_{j=1}^{\infty} C^{1/\varepsilon_j} < \infty$ for some $0 < C < 1$.
and ℓ_Φ is ℓ_1 saturated, then holds:
a) $a(\Phi) = b(\Phi) = 1$;
b) h_Ψ is stabilized asymptotic ℓ_∞ respect to the basis $\{e_j^*\}_{j=1}^{\infty}$.

Proof: a) By [16] follows that if $\ell_1 \hookrightarrow \ell_\Phi$ then $1 \in [a(\Phi), b(\Phi)]$ and thus $a(\Phi) = 1$. Let $a(\Phi) \neq b(\Phi)$. By Corollary 4.2 follows that $\ell_{b(\Phi)} \hookrightarrow \ell_\Phi$, which is a contradiction. Thus $1 = a(\Phi) = b(\Phi)$.

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b) By a) we have $a(\Phi) = b(\Phi) = 1$. So we have $\lim_{j\to\infty} a(\Phi_j) = \lim_{j\to\infty} b(\Phi_j) = 1$. Then using the well known connections $1/a(\Phi_j) + 1/b(\Psi_j) = 1$ and $1/a(\Psi_j) + 1/b(\Phi_j) = 1$ (see [8]) follows that $\lim_{j\to\infty} a(\Psi_j) = \lim_{j\to\infty} b(\Psi_j) = \infty$. Then by the definition of the indexes $a(\Psi_j)$ and $b(\Psi_j)$ there is $\varepsilon > 0$, such that for every p_j, q_j: $0 < a(\Psi_j) - \varepsilon < p_j < a(\Psi_j)$ and $b(\Psi_j) < q_j < b(\Psi_j) + \varepsilon$ holds

$$2^{a(\Psi_j)-\varepsilon} < 2^{p_j} < \frac{\Psi_j(2t)}{\Psi_j(t)} < 2^{q_j} < 2^{b(\Psi_j)+\varepsilon}.$$

Thus

$$\lim_{j\to\infty}\left(\inf\left\{\frac{\Psi_j(2t)}{\Psi_j(t)} : t > 0\right\}\right) \geq \lim_{j\to\infty} 2^{p_j} = \infty$$

and by Propositon 4.1 follows that h_Ψ is stabilized asymptotic ℓ_∞ respect to the basis $\{e_j^*\}_{j=1}^\infty$. $\qquad\square$

Remark: Kaminska and Mastylo has given a some sufficient and some necessary conditions for Schur property in terms of the generating MO function Φ [8]. Sometimes we know only the complementary function Ψ. For example let the MO function $\Psi = \{\Psi_j\}_{j=1}^\infty$ be defined by $\Psi_j = e^{\alpha_j} e^{-\frac{\alpha_j}{|x|^{c_j}}}$, where $\lim_{j\to\infty} \alpha_j = \infty$ and $0 < c_j$. Then ℓ_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ because

$$\lim_{j\to\infty} \inf\left\{\frac{\Psi_j(2x)}{\Psi_j(x)} : 0 \leq x \leq 1\right\} = \lim_{j\to\infty} \inf\left\{e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}|x|^{c_j}}} : 0 \leq x \leq 1\right\} = \lim_{j\to\infty} e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}}} = \infty.$$

Thus we conclude that ℓ_Φ has Schur property without considering the functions Φ_n, $n \in \mathbb{N}$.

References

[1] J. ALEXOPOULOS, On subspaces of non–reflexive Orlicz spaces, *Quaestiones Mathematicae* **21** (3 and 4) (1998), 161-175.

[2] C. BESSAGA, A. PELCZYNSKI, On bases and unconditional convergence of series in Banach Spaces, *Studia Math.*, **17** (1958), 165–174.

[3] O. BLASCO, P. GREGORI, Type and Cotype in Nakano Sequence Spaces $\ell_{(\{p_n\})}$, preprint

[4] NEIL DEW, Asymptotic sructure of Banach spaces, PhD Thesis (St. John's College University of Oxford) 2002.

[5] F. FUENTES, F. HERNANDEZ, On weighted Orlicz sequence spaces and their subspaces, *Rocky Mountain Journal of Mathematics* **18** (1988), 585–599.

[6] H. HUDZIK, A. KAMINSKA, On uniformly convex and B–convex Musielak–Orlicz spaces, *Comment. Math. Prace Mat.* **25** (1985), 59–75.

[7] H. Hudzik, L. Maligranda, Amemiya norm equals Orlicz norm in general, *Indagationes Mathematicae* **11** (2000), 573–585.

[8] A. Kaminska, M. Mastylo, The Schur and (weak) Dunford–Pettis property in Banach lattices, *J. Australian Math. Soc.* **73** (2002), 251–278.

[9] E. Katirtzoglou, Type and cotype in Musielak–Orlicz, *J. Math. Anal. Appl.* **226** (1998), 431-455.

[10] K. Kircev, S. Troyanski, On Orlicz spaces associated to Orlicz functions not satisfying the Δ_2–condition, *Serdica* **1** (1975), 88-95.

[11] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces I, Sequence spaces, Springer–Verlag, Berlin, 1977.

[12] L. Maligranda, Indices and inerpolation, *Dissertationes* **234** (1985).

[13] B. Maurey, V.D. Milman, N. Tomczak–Jaegermann, Asymptotic infinite–dimensional theory of Banach spaces, *Geometric aspects of functional analysis (Israel 1992–1994)*, Operator Theory Advances and Applications **77** Birkhauser, (1995), 149-175.

[14] V.D. Milman, N. Tomczak–Jaegermann, Asymptotic ℓ_p spaces and bounded distortions. Banach spaces (Merida) *Contemporary Mathematics* **144** American Mathematical Society (1992), 173-195.

[15] J. Musielak, Lecture Notes in Mathematics **1034** Springer–Verlag, Berlin, 1983

[16] V. Peirats, C. Ruiz, On ℓ^p–copies in Musielak–Orlicz sequence spaces, *Arch. Math.* **58** (1992), 164-173.

[17] S. Yamamuro, Modulared Sequence Spaces, *J. Fasc. Sci. Hokkaido Univ. Ser. I* **13** (1954), 1-12.

[18] J. Woo, On modular sequence spaces, *Studia Mathematica* **XLVIII** (1973), 271-289.

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