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SOME NOTES ON THE UNIT GROUPS OF COMMUTATIVE GROUP ALGEBRAS

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Abstract. Let G be an abelian p-group and let K be a field of the first kind with respect to p of characteristic not equal to p such that the spectrum $s_p(K)$ of the field K with respect to p contains $\mathbb{N}$. Denote by KG the group algebra of G over K and by $S(KG)$ the p-component of the group of the normalized units in KG. We compute $\exp(S(KG)/G)$ and prove that if G is a separable group, then $S(KG)/G$ is separable, i.e. G is a nice subgroup of $S(KG)$.

Suppose G is a finite abelian group, G_p is the p-component of G and R is a finite commutative ring with identity of prime characteristic p without nilpotent elements. We compute the Ulm-Kaplansky invariants of the group $S(RG)/G_p$.

The indicated results correct some essential inaccuracies and incompleteness in the formulations and in the proofs of results in this direction of P. V. Danchev (2005 (Zbl. 1107.16030), 2003 (Zbl. 1035.16025) and 2004 (Zbl.1067.16054)). We note also that because of gaps some results of the same author (2003 (Zbl. 1035.16025), 2004 (Zbl. 1080.16022), 2005 (Zbl. 1107.16030 and Zbl. 1097.16007) and 2008 (Zbl. pre 05375552)) remain unproved.

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1. Introduction

Let RG be the group algebra of an abelian group G over a commutative ring R with identity. Denote by tG the torsion subgroup of G, by G_p be the p-component of G, by $U(RG)$ the multiplicative group of RG and by $S(RG)$ the Sylow p-subgroup of the group $V(RG)$ of normalized units of RG, i.e. the p-component of $V(RG)$. The investigations of this group begin with the fundamental papers of Berman, S. D. [B and B1]) in which a complete description of $S(RG)$ (up to isomorphism) is given, when G is a countable abelian p-group and R is a countable field of positive characteristic p such that if G is not a restricted direct product of cyclic groups, then the field R is perfect. Further Mollov T. Zh. [M1 and M2] calculates the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(RG)$ when G is an arbitrary abelian group and R is a field of positive characteristic p. When R is a commutative ring with identity of prime characteristic p without nilpotent elements A. Bovdi and Z. Pataj [BP] calculate the Ulm-Kaplansky invariants of $S(RG)$ under the following restriction: if the maximal divisible subgroup of G_p is not identity, then R is a p-divisible ring, i.e. $R^p = R$. Nachev, N. A and Mollov, T. Zh. [NM] calculate the invariants $f_\alpha(S)$ with the only restriction on G to be an abelian p-group. Nachev, N. A. [N3] calculates the invariants $f_\alpha(S)$ without restrictions on G and R. Moreover, in all indicated cases the authors give a full description, up to isomorphism, of the maximal divisible subgroup of $S(RG)$.

Let G be an abelian p-group and let K be a field of characteristic p. W. May [M] proves that $S(KG)$ is simply presented if and only if G is a simply presented abelian p-group provided the field R is perfect. Therefore, if K is a perfect field of characteristic p and G is a simply presented abelian p-group, then the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(KG)$ together with the description of the maximal divisible subgroup of $S(RG)$ give a full description, up to isomorphism, of the group $S(KG)$.

Let G be an abelian p-group and let K be a field whose characteristic is different from p. S. D. Berman and A. R. Rossa [BR, BR1] have given a description of the torsion subgroup $tV(KG)$ of $V(KG)$ when G is a countable abelian p-group and K is a field. Let R be a commutative ring with identity, such that the characteristic of R does not divide the orders of the elements of G. N. Nachev [N1, N2] has given a description of the torsion subgroup $tV(RG)$ of $V(RG)$ when G is an abelian p-group and R contains the p^nth roots of unity, $n \in \mathbb{N}$. G. Karpilovsky [K, 5.2.5 Theorem, p.126] has determined the isomorphism class of $U(\mathbb{Q}G)$ when G is a finitely generated abelian group.

We define

$$G^{p^n} = \{g^{p^n} | g \in G\},\ n \in \mathbb{N},\ G^1 = \bigcap_{n=1}^{\infty} G^{p^n}.$$

Denote by $|M|$ the cardinality of a set M. Let $\coprod_{\alpha} G$ denote the coproduct of α copies of G, where α is a cardinal number. T. Mollov [M5] gives a full description of the group $S(KG)$ when G is an arbitrary abelian p-group and the field K is of the second kind with respect to p. When the field K is of the first kind with respect to p then in the same paper the following decomposition

$$S(KG) \cong S^1(KG) \times S(K(G/G^1)),$$

is obtained where

$$S^1(KG) \cong \coprod_{|G|} Z(p^{\infty}),$$

when $G^1 \neq 1$ and $S^1(KG) = 1$ when $G^1 = 1$. (Here we note that in this paper there is a technical mistake since the first isomorphism is written as an equality. Obviously, we can write

$$S(KG) = S^1(KG) \times T, \quad T \cong S(K(G/G^1)).)$$

Therefore, the description of $S(KG)$ is reduced to a description when G is a separable group. In this case in [M7] the Ulm-Kaplansky invariant of $S(KG)$ are calculated. In this way T. Mollov [M1-M3, M5-M7] has described the torsion subgroup $tV(RG)$ of $V(RG)$ when G is an abelian group and R is a field. T. Mollov [M4] has also described $V(RG)$, up to isomorphism, when either (a) G is an infinite direct sum of cyclic p-groups and $R = \mathbb{Q}$ or (b) G is an abelian p-group and $R = \mathbb{R}$. Z. Chatzidakis and P. Pappas [ChP] have determined the isomorphic class of $U(RG)$ when the torsion abelian group G is a direct sum of countable groups and R is a field. N. Nachev and T. Mollov [NM1 and NM2] describe $U(RG)$, up to isomorphism, when G is an abelian p-group and at least one of the following conditions (a) or (b) is fulfilled:

(a) the first Ulm factor G/G^1 of G is a direct sum of cyclic groups and R is a field of the first kind with respect to p;

(b) R is a field of the second kind with respect to p.

If R is a direct product of m indecomposable rings $R_i,\ m \in \mathbb{N}$, T. Mollov and N. Nachev [MN] give a description of the unit group $U(RG)$ of RG in the following cases:

(a) when R_i is a ring of prime characteristic p_i, tG/G_{p_i} is finite and the exponent of tG/G_{p_i} belongs to R_i^*;

(b) when R_i is of characteristic zero, R_i has no nilpotents, tG is finite of exponent n and $n \in R_i^*$.

When G is an abelian p-group and K is a field of the first kind with respect to p such that $s_p(K) \supseteq \mathbb{N}$, then P. Danchev makes an attempt:

(i) to calculate $\exp(S(KG)/G)$ [D4, Proposition 2];

(ii) to calculate the Ulm-Kaplansky invariants of $S(KG)/G$ [D4, Theorem 1];

(iii) to prove that $S(KG)/G$ is a direct sum of cyclic groups, provided G is a direct sum of cyclic groups [D1, Theorem 7 (Direct Factor) and D2, Proposition (Structure) ($\circ\circ$)];

(iv) to prove that if G is a separable group, then $S(KG)/G$ is separable, i.e. G is a nice subgroup of $S(KG)$ ([D1], Proposition 16 (a)] or [D4], Lemma 3);

(v) to prove that $S(KG)$ is torsion complete if and only if G is bounded [D2, Theorem 1].

Besides, P. Danchev wants

(vi) to calculate the α-th Ulm-Kaplansky invariant of the group $S(RG)/G_p$ (α is an arbitrary ordinal), when G is an abelian group and R is a commutative perfect ring with identity of characteristic p without nilpotent elements such that G^{p^α} and R are finite [D3, Theorem 6 (i)];

(vii) to give a criterion, i.e. a necessary and sufficient conditions for $V(FG) = G$ when the group G is finite and F is a field [D5] and

(viii) to give a criterion for $V(RG) = G$ when either $\mathrm{supp}\,(G) \cap \mathrm{inv}\,(R) \neq \emptyset$ or RG is a modular group algebra where R is a commutative ring with identity, $\mathrm{supp}\,(G) = \{p | G_p \neq 1\}$ (p is a prime number) and $\mathrm{inv}\,(R) = \{p | p.1_R \in R^*\}$ (1_R is the identity of R and R^* is the multiplicative group of R) [D6].

We note that in the proofs and in the formulations of many assertions of [D1]-[D6] which are connected with (i)-(viii) there are essential gaps and mistakes. In Sections 2 we correct the proofs and the formulations of some assertions and we prove correctly the results (i) and (iv). Besides, in Section 3 we correct case (vi), since, in this case, the indicated invariant is given in [D3] incompletely and ambiguously. However, the results (ii), (iii), (v), (vii) and (viii) of Danchev remain unproved.

We note additionally that in the paper of W. May, T. Mollov and N. Nachev [MMN] are commented the gaps and the mistakes made in 6 papers [Da-Df] of Danchev but now we will not consider these articles.

The abelian group terminology is in agreement with [F].

2. On the unit groups of semisimple group algebras

Our basic aim in this section is to prove correctly the results (i) and (iv) which were noted in Section 1, to correct some preliminary assertions in the papers [D1]-[D4] and to motivate that the above results (ii), (iii), (v), (vii) and (viii) remain unproved.

We will multiplicatively write the abelian groups. We recall some well known definitions. We say that an abelian group G has *a finite exponent* n and we write $\exp(G) = n$ if $G^n = 1$ and n is the least natural with this property. Otherwise we say that the exponent of G is infinity and we write $\exp(G) = \infty$.

For the presentation of the main results we use the following notation. If A is a subgroup of the group G, then we will write $A \leq G$. We denote the order of $a \in G$ by $O(a)$ and by b/c the number bc^{-1}, when b and c are real numbers, $c \neq 0$.

Let μ_p be the group of the p^n-th roots of unity with n ranging over $\mathbb{N}$. The field K of characteristic not equal to p *is called of the first kind with respect to* p if $(K(\mu_p) : K)$ is infinite; otherwise it is of*the second kind with respect to* p. All *direct products* of groups are assumed to be *restricted direct products*, i.e. they are *direct sums* and the concept direct product will mean a restricted direct product. Let K_p be the p-component of the multiplitive group of a field K.

Let K be a field of the first kind with respect to p and let $\varepsilon_i,\ i \in \mathbb{N}$, be a primitive p^i-th root of unity. Then the group $K(\varepsilon_i)_p$, where $i = 1$ if $p \neq 2$ and $i = 2$ if $p = 2$, is cyclic. Hence for this i there exists a positive integer θ such that $|K(\varepsilon_i)_p| = p^\theta$. We call the number θ *the constant of the field* K *with respect to* p. The set

$$s_p(K) = \{i \in \mathbb{N}_0 | K(\varepsilon_i) \neq K(\varepsilon_{i+1})\}$$

is said to be *a spectrum of the field* K *with respect to* p [M6].

We will use the following lemma, which is proved in the book of G. Karpilovsky [K1, Lemma 12.34, p. 247]. We note that in this section the prime p is fixed.

Lemma 2.1. *Let* K *be a field of characteristic different from the prime* p. *Suppose* $q = 1$, *if* $p \neq 2$ *and* $q = 2$, *if* $p = 2$. *Then*

(i) if K *is a field of the second kind with respect to* p, *then* $K(\varepsilon_m) = K(\varepsilon_q)$ *for all* $m \geq q$ *and*

(ii) if K *is a field of the first kind with respect to* p *and* f *is the constant of* K *with respect to* p, *then*

$$K(\varepsilon_q) = K(\varepsilon_{q+1}) = ... = K(\varepsilon_f) \subset K(\varepsilon_{f+1}) \subset ...$$

The following lemma is a result of S. D. Berman [B1, Corollary of Lemma 2.5].

Lemma 2.2. *If K is a field of the first kind with respect to p, then $(K(\varepsilon_i) : K(\varepsilon_f)) = p^{i-f}$ for every $i \geq f$, where f is a constant of K with respect to p.*

Proposition 2 of (P.Danchev, [D4]) has the following formulation.

"For an abelian p-group G and $s_p(K) \supseteq \mathbb{N}$, we have

$$\exp(S(KG)/G) = i \in \mathbb{N} \Leftrightarrow \exp(G) = i \in \mathbb{N}."$$

We can give immediately the following *counterexample to Proposition* 2. Namely, let $G = \langle a \rangle$ be a cyclic of order 3, i.e. $p = 3$ and let $K = \mathbb{Z}_2$ be the field of two elements. It is not hard to see that $s_3(Z_2) = \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup 0$. Really, let ε_i be a primitive 3^i-th root of unity over Z_2, $i \in \mathbb{N}_0$. Since $\varepsilon_1 \in (\mathbb{Z}_2(\varepsilon_1))^*$ and $|\mathbb{Z}_2(\varepsilon_1)^*)| = 2^k - 1$, where k is the minimal integer with the property $3/(2^k - 1)$, then $k = 2$, i.e. $(\mathbb{Z}(\varepsilon_2):\mathbb{Z}_2) = 2$. Analogously, for ε_2 we obtain $9/(2^k - 1)$ hence $k = 6$, i.e. $(\mathbb{Z}(\varepsilon_2):\mathbb{Z}) = 6$. In this way $(\mathbb{Z}_2(\varepsilon_2) : \mathbb{Z}_2(\varepsilon_1)) = 3$. Then Lemma 2.1 implies $\mathbb{Z}_2(\varepsilon_i) \neq \mathbb{Z}_2(\varepsilon_{i+1})$, $i = 1, 2, \ldots$. Therefore, $s_3(\mathbb{Z}_2) = \mathbb{N}_0 \supseteq \mathbb{N}$. Besides the group algebra $\mathbb{Z}_2 G$ contains 8 elements. Since $1 + a + a^2$ is an idempotent of $\mathbb{Z}_2 G$ and $0, 1 + a, 1 + a^2, a + a^2$ does not belong to $V(KG)$, then $V(KG) = S(KG) = G$. Consequently, $\exp(S(KG)/G) = \exp\{1\} = 0 \neq 1 = \exp\{(G)\}$. Therefore, Proposition 2 of [D4] is not true.

In order to calculate $\exp(S(KG)/G)$ we will give some definitions and will prove some preliminary assertions.

Let G be a finite abelian p-group of exponent n, $s_p(K) \supseteq \mathbb{N}$ and K^* is the unit group of the field K. *Character* χ of the group G is a homomorphism $\chi : G \to (K(\varepsilon_n))^*$. Let e be an idempotent of KG which corresponds to χ and $\operatorname{Ker} e = \{g \in G \mid ge = e\}$. It is obviously that $\operatorname{Ker} e = \operatorname{Ker} \chi$. Besides $G/\operatorname{Ker} e$ *is a cyclic group*. Really, $\chi : G \to (K(\varepsilon_i))^* = \langle \varepsilon_i \rangle$ is a homomorphism of G onto $(K(\varepsilon_1))^*$. Therefore, $G/\operatorname{Ker} e \cong \langle \varepsilon_i \rangle$, i.e. $G/\operatorname{Ker} e$ is a cyclic group. If $\chi(g) = 1$ for every $g \in G$, then we will call χ *an identity character.*

Further, if the contrary is not assumed, we will denote by G an abelian p-group, by K a field of the first kind with respect to p and by e_0 the idempotent, which corresponds to the identity character.

Lemma 2.3. *Let G be a finite abelian p-group, let K be a field of the first kind with respect to p and let $s_p(K) \supseteq \mathbb{N}$. Suppose e is a minimal idempotent of KG different from e_0. Then the p-component $(KGe)_p$ of the ideal KGe, which is regarded as a field of identity e, consists of the elements ge, where g runs G.*

Proof. It is well known that $KGe \cong K(\varepsilon_i)$. Since $s_p(K) \supseteq \mathbb{N}$ and $e \neq e_0$, then $i = 0$ is impossible because $i = 0$ holds if and only if $G = \mathrm{Ker}\, e$ which is equivalent to $e = e_0$. Therefore, $i \neq 0$, $i \in s_p(K)$ and $(K(\varepsilon_i))_p$ is a cyclic group of order p^i. Let a be an element of G such that $G/\mathrm{Ker}\, e = \langle a\mathrm{Ker}\, e\rangle$. Then the image of ae by the isomorphism $\varphi : KGe \to K(\varepsilon_i)$ is ε_i. Consequently ae is a generating element of $(KGe)_p$, i.e. every element of $(KGe)_p$ has a form ge, $g \in G$. $\square$

The element $x = \alpha_1 g_1 + ... + \alpha_n g_n$, $\alpha_i \in K$, $g_i \in G$, is called normalized element if $\alpha_1 + ... + \alpha_n = 1$.

Lemma 2.4. *If G is a finite abelian p-group, $\mathbb{N} \subseteq s_p(K)$ and $x \in KG$, then $x \in S(KG)$ if and only if*

$$x = e_0 + g_1 e_1 + ... + g_n e_n, g_i \in G, \tag{2.1.}$$

where $e_0, e_1, ..., e_n$ form a full system of minimal orthogonal idempotents of KG. Therefore, $\exp(S(KG)) \leq \exp(G)$.

Proof. Let $x \in S(KG)$. It is well known that

$$KG = KGe_0 \oplus KGe_1 \oplus ... \oplus KGe_n.$$

Therefore, $x = \lambda_0 e_0 + \lambda_1 e_1 + ... + \lambda_n e_n$, $\lambda_i \in KG$, $i = 0, 1, ..., n$. Since x is a normalized element and $\mathrm{Ker}\, e_0 = G$, then $\lambda_0 = 1$. Besides $\lambda_i e_i \in (KGe_i)_p$, $(i = 1, 2, ..., n)$, and by Lemma 2.3, $\lambda_i e_i = g_i e_i$, $g_i \in G$, $i = 1, 2, ..., n$.

Conversely, if (2.1) holds, then obviously $x \in S(KG)$. $\square$

Lemma 2.5. *Let G be a finite abelian p-group and $\mathbb{N} \subseteq s_p(K)$. Suppose $x = 1 - e + ge$, where $g \in G$ and e is a minimal idempotent of KG. Then $x \in G$ if and only if $g\mathrm{Ker}\, e \cap \mathrm{Ker}\,(1-e) \neq \emptyset$.*

Proof. Let $x \in G$. Then $x = 1 - e + ge$ implies $xe = ge$ and $x(1-e) = 1-e$. We obtain, from the second equality, that $g^{-1}x \in \mathrm{Ker}\, e$, i.e. $x \in g\mathrm{Ker}\, e$. The equality $x(1-e) = 1-e$ implies $x \in \mathrm{Ker}\, e(1-e)$. In this way $x \in g\mathrm{Ker}\, e \cap \mathrm{Ker}\,(1-e)$, i.e. the last cross-section is not empty.

Conversely, let $h \in g\mathrm{Ker}\, e \cap \mathrm{Ker}\,(1-e)$. Therefore, $hg^{-1} \in \mathrm{Ker}\, e$, i.e. $hg^{-1}e = e$, which is eguivalent to $ge = he$. Besides $h(1-e) = 1-e$. Then $x = 1 - e + ge = h(1-e) + he = h \in G$. The proof is completed. $\square$

We denote $G[p^n] = \{g \in G | g^{p^n} = 1\}$, $n \in \mathbb{N}$.

The following lemma is a light modification of Lemma 1.1 of the paper [M6] of T. Mollov.

Lemma 2.6. *If G is a finite abelian p-group and K is a field of the first kind with respect to p, then for every $i \in s_p(K)$ the number of the idempotents e of KG such that $KGe \cong K(\varepsilon_i)$ is*

$$\delta_i = \begin{cases} \left(\left|G[p^i]\right| - \left|G[p^{i'}]\right|\right)/(K(\varepsilon_i) : K), & \text{if} \quad i \neq i_o \qquad \text{and} \\ \left|G[p^{i_o}]\right|, & \text{if} \quad i = i_o, \end{cases}$$

where i_o is the minimal number of $s_p(K)$ and i' is the maximal number of $s_p(K)$ such that $i^{'} < i$.

The proof of the following lemma is obtained directly by Lemma 2.6.

Lemma 2.7. *Let abelian p-group G be a cyclic group of order p^n, K be a field of the first kind with respect to p and $s_p(K) \supseteq \mathbb{N}$. Then the number δ_i of the minimal idempotents e of KG such that $KGe \cong K(\varepsilon_i)$, $i = 0, 1, 2, ..., n$, is the following:*

(i) if $K(\varepsilon_1) \neq K$, then $\delta_o = 1$ and $\delta_i = (p-1)/d$, $d = (K(\varepsilon_1) : K)$, i=1,2,...,n and

(ii) if $K(\varepsilon_1) = K$, then $\delta_o = \delta_1 = p$ and if $n > 1$, then $\delta_i = p - 1$ for $i = 2, ..., n$.

Lemma 2.8. *If G is a cyclic group of order p, K is a field of the first kind with respect to p, $(K(\varepsilon_1) : K) = p-1$ and $s_p(K) \supseteq \mathbb{N}$, then $S(KG) = G$.*

Proof. Lemma 2.7 implies that there are exactly two minimal idempotents e_0 and e_1 of KG such that $KGe_o \cong K$ and $KGe_1 \cong K(\varepsilon_1)$. Therefore, $\mathrm{Ker}\, e_0 = G$ and $\mathrm{Ker}\, e_1 = 1$. Any element $x \in S(KG)$, by Lemma 2.4, has the form $x = e_o + ge_1$, $g \in G$. Since $\mathrm{Ker}\, e_o = G$, then $x = ge_0 + ge_1 = g \in G$. Therefore, $S(KG) = G$. □

Theorem 2.9. *Let G be an abelian p-group, let K be a field of the first kind with respect to p and let $N \subseteq s_p(K)$. Then*

(i) $\exp(S(KG)/G) = \exp(G) - 1$, *if* $(K(\varepsilon_1) : K) = p - 1$ *and the group G is a cyclic and*

$\exp(S(KG)/G) = \exp(G)$ *otherwise.*

(ii) $\exp(S(KG)/G) = \exp(G)$ *otherwise.*

Proof. (a) *Let G be a cyclic p-group and $(K(\varepsilon_1){:}K) = p{-}1$.* Suppose $G = \langle a \rangle$ is a cyclic group of order p^n. If $n = 1$, then, by Lemma 2.8, $S(KG) = G$, $\exp(S(KG)/G) = 0 = \exp(G) - 1$ and the assertion is true.

Let $n \geq 2$. There exists exactly one minimal idempotent e_i of KG, such that $KGe_i \cong K(\varepsilon_i)$. Let xG be an arbitrary coset of the group $S(KG)/G$. Then the element x has a representation (2.1). Since $\mathrm{Ker}\, e_i$ is a cyclic group of order p^{n-i}, $i=0,1,...,n$, then $(g_ie_i)^{p^{n-i}}=e_i$ for every $i=0,1,...,n-1$. Therefore,

$$x^{p^{n-1}}=e_0+e_1+...+e_{n-1}+a^{kp^{n-1}}e_n=$$
$$a^{kp^{n-1}}e_0+a^{kp^{n-1}}e_1+...+a^{kp^{n-1}}e_{n-1}+a^{kp^{n-1}}e_n=a^{kp^{n-1}}\in G.$$

Consequently, $(xG)^{p^{n-1}}=G$, i.e. $\exp(S(KG)/G)\le n-1=\exp(G)-1$.

Let now $y=1-e+ae$, where e is a minimal idempotent of KG such that $KGe\cong K(\varepsilon_n)$. It is obviously, that $y\in S(KG)$. Then $y^{p^{n-1}}=1-e+a^{p^{n-1}}e$. For the idempotents e and $1-e$ we have $\operatorname{Ker} e=1$ and $\operatorname{Ker}(1-e)=\left\langle a^{p^{n-1}}\right\rangle$. Therefore, $a^{p^{n-2}}\operatorname{Ker} e\cap \operatorname{Ker}(1-e)=\emptyset$, since otherwise we obtain $a^{p^{n-2}}\in \operatorname{Ker}(1-e)=\left\langle a^{p^{n-1}}\right\rangle$ which is a contradiction. Consequently, by Lemma 2.5, $y^{p^{n-2}}\notin G$ and $(yG)^{p^{n-2}}\ne G$, i.e. $\exp(S(KG)/G)\ge n-1$. We finally obtain that $\exp(S(KG)/G)=\exp(G)-1$, i.e. it holds case (i).

We will prove in all other cases that $\exp(S(KG)/G)=\exp(G)$.

(b) *Let the group G is not cyclic and G has a finite exponent n.* Then G can be represented in the form $G=\langle a\rangle\times H$, where $O(a)=p^n$ and $\exp(H)\le n$. Since G is not cyclic, then $H\ne 1$. Therefore, $H=\langle b\rangle\times F$ where the order of the cyclic group $\langle b\rangle$ is at most p^n. Let e be a minimal idempotent of $K\langle b\rangle$ with $\operatorname{Ker} e=1$. We set $x=1-e+ae$. Then $x^{p^n}=1$,

$$x^{p^{n-1}}=1-e+a^{p^{n-1}}e \tag{2.2}$$

and $x^{p^{n-1}}\notin G$. Consequently, $\exp(S(KG)/G)\ge n=\exp(G)$. Since $\exp(G)=n$, then Lemma 2.4 implies $\exp(S(KG))/G)\le n$. Hence $\exp(S(KG)/G)=n=\exp(G)$.

(c) *Let the exponent of G is infinity.* Suppose n is an arbitrary natural. Then there exists element a in G such that the order of a is p^{n+1}. Let e be a minimal idempotent of $K\langle a\rangle$ with $\operatorname{Ker} e=1$. We set $x=1-e+ae$. Then the equality (2.2) holds and Lemma 2.5 implies $x^{p^{n-1}}\notin G$. Therefore, $O(xG)\ge p^n$. In this way we obtain that in the group $S(KG)/G$ there are elements of arbitrary large orders. Consequently, $\exp(S(KG)/G)=\infty$, i.e. $\exp(S(KG)/G)=\exp(G)$.

(d) *Let the group G be cyclic and $(K(\varepsilon_1):K)=d<p-1$.* We note that in this case $p=2$ is impossible. Let the order of $G=\langle a\rangle$ be p^n and let e be a minimal idempotent of KG such that $KGe\cong K(\varepsilon_n)$. Then $\operatorname{Ker} e=1$. The number δ_n of the minimal orthogonal idempotents e of KG such that $KGe\cong K(\varepsilon_n)$ is, by Lemma 2.7,

1) $\delta_n=(p-1)/d$, if $d>1$;

2) $\delta_n = p$, if $d = 1$ and $n = 1$ and
3) $\delta_n = p - 1$, if $d = 1$ and $n > 1$.
Since $d < p - 1$ and $p > 2$, then $\delta_n > 1$, i.e. there is still at least one minimal idempotent of KG with a kernel 1. Therefore, $\mathrm{Ker}\,(1 - e) = 1$. Let we set now $x = 1 - e + ae$. Then the equality (2.2) holds and Lemma 2.5 implies $x^{p^{n-1}} \notin G$. Therefore, $\exp(S(KG/G) \geq n = \exp(G)$. The inverse inequality follows from Lemma 2.4. □

Remark. *Theorem 2.9 is a correction of the result (i) of Section 1,* i.e. this theorem corrects the formulation and the proof of Proposition 2 of [D4] and case (i) of this theorem gives series of counterexamples on this proposition.

We continue *a commentary of the result* (v) *noted in Section 1* which is a main result in [D2]. In the proofs of the preliminary results, namely of the Lemma (Purity) [D2, p. 894] or Lemma 2 of [D4] and of Proposition (Structure) [D2, p. 895] P. Danchev uses the following lemma of W. May [M0].

"**Lemma** (May, W., 1979, Lemma 2). *Let R be a commutative ring with identity and let G be an abelian group. Suppose that there exists a group B such that G is isomorphic to a direct factor of $U(RB)$. Then G is a direct factor of $U(RG)$.*"

This lemma holds for $U(RG)$. However, P. Danchev uses it for $S(KG)$ which is incorrectly. If we must apply this lemma for $S(KG)$, then we must separately prove a subsidiary result for $S(KG)$ which is analogously to Lemma 2 of May. We could not accept this fact without a proof.

The above marked Lemma (Purity) and Proposition (Structure) of P. Danchev are used in the proof of (v), i.e. in the proof of main result Theorem 1 of [D2].

The result (v) noted in section 1, i.e. Theorem 1 of [D2], has additionally the following lapses and mistakes (a)-(f).

(a) From the representation of the element x_b [D2, p. 897, line 9 from above] we see that KB' contains only two minimal orthogonal idempotents e_1 and e_2. This is possible only at some special conditions for the group B' and the field K but such conditions are not met in the proof and in the formulation of the theorem. This is the most essential mistake in the proof.

(b) The author states incorrectly, that the equality on page 896, line 10 from above, is equivalent to the equality on page 896, line 11 from above. We note that the first equality is equivalent to

$$\bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} [S^{p^m}(KG)GS(KB_n)] = GS(KB).$$

The author chooses an arbitrary element x from the left-hand side of the equality on p. 896, line 11 from above. However he should choose the element x from the right-hand side of this equality and should prove that this element belongs to $GS(KB)$. In this way all reasonings in case 2, remain useless, i.e. case 2 is not considered in the proof of Theorem 1.

(c) The author incorrect writes (p. 896, line 18-19 from above): "From our initial discussion in the introduction we with no harm of generality assume that height$(g_1) \geq m, ...$" and this fact remains unchecked, since it is not considered in the introduction.

(d) In fact the author does not prove case 3 in the proof of Theorem 1 when the group B is uncountable and he only writes " The assertion follows by standard transfinite induction...". But this is inadmissible at the indicated defects of countable case 2 and since a transfinite induction with cardinal numbers can not lead here.

(e) The decomposition of $S(KF)$ (p. 894, line 19 from above) is incorrect. Namely, in this decomposition $(K_0)_p$ should not be participate.

(f) The author writes that the results can be proved " without the restriction on the spectrum on the first kind field K." (page 898, line 23 from below) which, because of the note in (a), is untrue.

In this way *Theorem 1 (i.e. assertion* (v) *of Section 1) and therefore Corollaries 1-3 and the Global Theorem of [D2] remain unproved.* So that almost all of the results of the last article are incorrect.

When K is a field of the second kind with respect to p, then in [D2] the author proves Theorem 2, but it is a trivial corollary of a result of [M5].

Lemma 2.10. *If G is an abelian p-group and $|G| \geq p^2$, then $S(KG) \neq G$.*

Proof. We will consider two cases (a) and (b).

(a) Let G contains a cyclic subgroup $H = \langle a \rangle$, $O(a) = p^2$. We will prove that $S(KH) \neq H$. Let e be a minimal idempotent of KH such that $\operatorname{Ker} e = 1$. Then, by Lemma 2.4, the element

$$x = (1 - e + ae) \in S(KH). \tag{2.3}$$

It is not hard to see, using Lemma 2.7, that the equality

$$a\operatorname{Ker} e \cap \operatorname{Ker}(1 - e) = \emptyset \tag{2.4}$$

holds since $\operatorname{Ker}(1 - e) \subseteq \langle a^p \rangle$. Then Lemma 2.5 implies $x \notin H$. Hence $x \in S(KH) \setminus G$. In this way $S(KG) \neq G$.

(b) Let G be an elementary abelian p-group. Suppose H is a subgroup of G and $H = \langle a \rangle \times \langle b \rangle$ such that $O(a) = O(b) = p$. Let e be again a minimal idempotent of KH with $\operatorname{Ker} e = 1$. Then for the element x in (2.3) the equality (2.4) is fulfilled since KH has an idempotent which kernel is $\langle ab^{-1}\rangle$. Therefore, by Lemma 2.5, $x \in S(KG) \setminus G$. □

Lemma 2.11. *If B is an infinite coproduct of cyclic p-groups and K is a field of the first kind with respect to p, then $|S(KB)/B| = |B|$.*

Proof. The equality $|S(KB)| = |B|$, by [M5, Theorem 12] holds. We will prove $|S(KB)/B| \geq |B|$ which will complete the proof. We can accept, by a suitable grouping of the factors in the direct decomposition of B in a coproduct $B = \coprod_{i\in I} B_i$ of finite groups such that $|B_i| \geq p^2$. Then $|I| = |B|$. Since $|B_i| \geq p^2$, then, by Lemma 2.10, $S(KB_i) \neq B_i$. Let $x_i \in S(KB_i) \setminus B_i$ is an arbitrary element of this set, $i \in I$. Then $x_iB \neq x_jB$ for every $i \neq j$, $i, j \in I$ (hence $|S(KB)/B| \geq |B|$). Otherwise $x_i = x_jb$, $b \in B$. Obviously, $b \in S(K(B_i \times B_j)) \cap B = B_i \times B_j$. Obviosly, $b = b_i^{-1}b_j$, $b_i \in B_i$ and $b_j \in B_j$. Then $x_ib_i = x_jb_j$ implies

$$S(KB_i) = x_ib_iS(KB_i) = x_jb_jS(KB_i),$$

i.e. $x_jb_jS(KB_i) = S(KB_i)$. Therefore, $x_jb_j \in S(KB_i) \cap S(KB_j) = 1$. Consequently, $x_j \in B_j$, which is a contradiction. □

We will comment the results (iii) and (iv) *of [D1] noted in Section 1.*

The result (iii), i.e. Theorem 7 (Direct Factor) of [D1] which is identical to Proposition (Structure) (∘∘) of [D2] *remains unproved* because of the following reasons. In the end of the proof of Theorem 7 is used Lemma (Purity), which is Proposition (Structure) (∘∘) of [D2]. These assertions remain unproved because of an incorrect using of the mentioned Lemma 2 of May [M0].

P. Danchev tries to prove the assertion (iv) of Section 1, i.e. Proposition 16 (a) of [D1] or Lemma 3 of [D4]. The formulation of this assertion is the following.

"**Proposition 16** *(a) If A is separable, the quotient group $S(KA)/A$ is separable or equivalently A is nice in $S(KA)$.*"

In connection with the proof of (iv) the author writes "On the other hand F should be a direct factor of A" [D1, p. 42, line t from above]. This fact is untrue. Further he writes "Therefore, consuming (2), $S(KF)/F$ is finite whence separable. On the other hand F should be a direct factor of A. Therefore, as we previously have seen above, $S(KF)A/A$ must be a direct factor of $S(KA)/A$" [D1, p.42, lines 8-9 from above]. This assertion is absolutely ungrounded. Consequently, *(iv) remains unproved.*

The proof of (iv), (*noted in Section 1), i.e. of Proposition 16 (a) of [D1] could be completed by the following way.* Preliminary we prove the following assertion.

Lemma 2.12. *If G is an abelian group, A and B are subgroups of G and*

$$G = A \times B, C = C_1 \times C_2, C_1 \leq A, C_2 \leq B,$$

then $G/C = AC/C \times BC/C$.

Proof. It is sufficiently to prove that $AC/C \cap BC/C = C$. Namely, let xC belongs to the indicated cross-section. Then $xC = aC = bC$, $a \in A$, $b \in B$. Hence $a = bc$, $c = c_1c_2$, $c_i \in C_i$ and $ac_1^{-1} = bc_2 = 1$ because of $A \cap B = 1$. Therefore, $a = c_1 \in C$ and $xC = aC = C$. □

Proof of Proposition 16(a). Let $xA \in S(KA)/A$. We will prove that the height $h(xA)$ of xA in $S(KA)/A$ is finite. Namely, we can suppose that $x \in S(KF)$ where F is a finite direct factor of A. Consequently, $A = F \times A_1$, $A_1 \leq A$. It is well known that

$$S(KA) = S(KF) \times T, T = S(KA, A_1) = S(KA) \cap [1 + I(KA, A_1)], \tag{2.5}$$

where $I(KA, A_1)$ is the ideal of KA generated by the elements $a_1 - 1$, $a_1 \in A_1$. Obviously $A_1 \leq T$ holds. Since $F \leq S(KF)$ and $A_1 \leq T$, then (2.5) and Lemma 2.12 imply $S(KA)/A = S(KF)A/A \times TA/A$.

Therefore, xA belongs to the finite direct factor $S(KF)A/A$ of $S(KA)/A$. Consequently, the height $h(xA)$ is finite, i.e. $S(KA)/A$ is separable. □

Danchev [D4, Theorem 1] formulates the following result:

"**Theorem 1.** *Let G be an abelian p-group with G/G^1 infinite and let $s_p(K) \supseteq \mathbb{N}$. Then, for all $i \geq 0$,*

$$f_i(S(KG)/G) = \begin{cases} |B|, & i < \exp(G/G^1); \\ 0, & i \geq \exp(G/G^1). \end{cases}$$

If G/G^1 is finite, then

$$f_i(S(KG)/G) = f_i(S(K(G/G^1))) - f_i(G/G^1) = f_i(S(KG)) - f_i(G)."$$

We note that *this result must be formulated in the following way* (here ω is the first infinite ordinal).

Theorem 2.13. *Let G be an abelian p-group and let K be a field of the first kind with respect to p. Suppose that $s_p(K) \supseteq \mathbb{N}$ and α is an arbitrary ordinal. Then*

(i) if $\alpha \geq \omega$, then $f_\alpha(S(KG)/G) = 0$;

(ii) if $\alpha < \omega$ and $\exp(G/G^1) = \infty$, then $f_\alpha(S(KG)/G) = |B|$, where B is a basic subgroup of G;

(iii) if $\alpha < \omega$, G/G^1 is infinite and $\exp(G/G^1) < \infty$*, then*

$$f_\alpha(S(KG)/G) = \begin{cases} |B|, & \text{if} \quad \alpha < \exp(G/G^1) \quad \text{and} \\ 0, & \text{if} \quad \alpha \geq \exp(G/G^1)\ ; \end{cases}$$

(iv) if $\alpha < \omega$ and G/G^1 is finite, then $f_\alpha(S(KG)/G) = f_\alpha(S(KG) - f_\alpha(G)$.

We note also that *the result (ii), noted in Section 1, i.e. the main result of [D4], namely Theorem 1, remains in general unproved because of the following reasons.*

(a) In the proof of Theorem 1 of [D4, p.153, lines 11-13 from above] the author writes absolutely ungrounded

"Therefore, $f_i(S(KB)/B)=|B|$ via exploiting the facts that $|S(KB)/B|= |B|$ and that the cyclic factors of $S(KB)/B$ of order p are precisely $|B|$, which follows analogously to [M1,Theorem 12].", i.e. analogously to [M5,Theorem 12]. We prove in Lemma 2.11, that $|S(KB)/B| = |B|$. However the second fact that "the cyclic factors of $S(KB)/B$ of order p are precisely $|B|$, which follows analogously to [M1, Theorem 12]" is not proved. More precisely the proof of this fact is not analogously to "[M1, Theorem 12]". It must be additionally proved and it is not made in the paper [D4].

(b) For the proof of Theorem 1 of [D4] P. Danchev uses Lemmas 2 and 3 of the same paper and, as we noted above, these lemmas remain unproved.

We note also that it Theorem 2.13 which is a correct formulation of Theorem 1 of [D4] remains unproved because of the reasons (a), and (b).

We can make still two notes (c) and (d) of the paper [D4]. Namely,

(c) the author writes (p.151, lines 12-11 from below) "Thus it is quite possible that either $i \in s_p(K)$ but $i+1 \notin s_p(K)$, or $i \notin s_p(K)$ but $i+1 \in s_p(K)$". It is easy to see that this is not true, since $i \in s_p(K)$ or $i \notin s_p(K)$ can imply, in the indicated two cases, either $i + 1 \in s_p(K)$ or $i + 1 \notin s_p(K)$.

(d) The author of [D4] notes (p.154, in Remarks) that the main theorem of Mollov [M7] is incorrect. This remark is superfluous, since T. Mollov [M8] in 2004 year corrected a technical mistake in the formulation of this theorem. Namely, in Theorem 6 of [M7] the word "infinite group" is replaced with "unbounded group". Besides, T. Mollov indicates [M8, p.6, lines 3-1 from below] that "when the group G is bounded the calculation of Ulm-Kaplansky invariants $f_i(S)$ is not necessary" since Theorem 12 of [M5] and Proposition 11 of [M6] give the exact description of $S(KG)$. In spite of this P. Danchev [D4, p.154] gives the following invariants of $S(KG)$:

$$"f_i(S(KG)) = \begin{cases} |B|, & i+1 \in s_p(K) \quad \text{but} \\ & i = \exp(G) \notin s_p(K) \quad \text{or} \quad i < \exp(G) \in s_p(K); \\ 0, & i+1 \notin s_p(K) \quad \text{or} \quad i \geq \exp(G) \in s_p(K) \quad \text{or} \\ & s_p(K) \not\ni \exp(G) < i = \mathrm{const}_p(K)." \end{cases}$$

Here an inaccuracy has also since in the first line of this citing, when $f_i(S(KG)) = |B|$, the case when $\exp(G) = \infty$ is omitted.

The results (vii) and (viii) noted in Section 1, i.e. the main results of [D5] and [D6] remain in general unproved because of the following reasons. For the proof of the result (viii) the author uses the result (vii). However, the last result remains unproved in [D5] since in its proof the case $(F(\eta_q) : F) \neq q-1$ is not considered (η_q is the primitive q-th root of unity) (only the case "$q-1 = (F(\eta_q) : F)$" is considered in p. 142, line 7 from below). For example, the case $(F(\eta_q) : F) \neq q-1$ arises when $F = \mathbb{Z}_{19}$ and $q = 5$. Namely, it is not hard to see that $(\mathbb{Z}_{19}(\eta_5) : \mathbb{Z}_{19}) = 2 \neq 4 = q-1$.

3. On the unit groups of modular group algebras

Let G be an abelian group and let R be a commutative ring with identity of prime characteristic p. We will use the following well known formula

$$S^{p^\alpha}(RG) = S(R^{p^\alpha} G^{p^\alpha}).$$

We recall that the ring R is called perfect if $R^p = R$. If G is an abelian p-group and α is an arbitrary ordinal then we denote by $f_\alpha(G)$ the α-th Ulm-Kaplansky invariant of the group G.

Our main aim in this section is to correct the result (vi) *of Section 1 (Theorem 6 (i) of [D3]), i.e. to compute the α-th Ulm-Kaplansky invariant of the group $S(RG)/G_p$ (α is an ordinal), when G^{p^α} and R are finite and to indicate that this result is given in the paper [D3] incompletely and ambiguously.* For any ordinal α we define G^{p^α} and R^{p^α} inductively by the following way:

$$G^{p^0} = G, \quad G^{p^{\alpha+1}} = (G^{p^\alpha})^p \quad \textit{and} \quad G^{p^\alpha} = \bigcap_{\beta<\alpha} G^{p^\beta}$$

if α is a limit ordinal. Analogously,

$$R^{p^0} = R, \quad R^p = \{r^p | r \in R\}, R^{p^{\alpha+1}} = (R^{p^\alpha})^p \quad \textit{and} \quad R^{p^\alpha} = \bigcap_{\beta<\alpha} R^{p^\beta}$$

if α is a limit ordinal.

We note that *the following lemma in [D3] is not proved* (the ring R is perfect without nilpotent elements).

"*Lemma 1. The subgroup G_p is balanced in $S(RG)$...*"

Namely the author does not prove that $g_j \in G_p$ [D3, p.135, line 1 from below]. After superfluous reasonings in the end of this lemma he concludes that $x \in G_p S(R^{p^\delta} G^{p^\delta})$ (p. 136, line 1 from above), which is given in fact since x is chosen in $\bigcap_{\tau<\alpha}[S^{p^\tau}(RG)G_p]$ (p. 135, line 3 from the proof of Lemma 1). Therefore, there is obtained a vicious circle. There is a vicious circle in advance since in page 135, line 3 from below, the author accepts $\delta = \alpha$, i.e. this which he must prove. Here he could accept only $\delta < \alpha$.

We will denote the support of $x \in V(RG)$ by $\operatorname{supp}(x)$ and by $h(x)$ the p-height of x in $V(RG)$.

Let $g \in G$ and $h_G(g)$ be the p-height of g in G.

Remark. If $g \in G$, then $h_G(g) = h_{V(RG)}(g)$, since G is p-isotype subgroup of $V(RG)$. Therefore, without more precise definition we will write $h(g)$.

Lemma 3.1. *If G is an abelian group and R is a commutative perfect ring with identity of prime characteristic p, then G is p-balanced in $V(RG)$.*

Proof. It is well known that G is p-isotype subgroup of $V(RG)$. We will prove that G is p-nice in $V(RG)$. It is obviously that if we consider an arbitrary coset xG of $V(RG)$ mod G, then we can choose the element x such that $1 \in \operatorname{supp}(x)$. We will prove that for any element $y \in xG$ the inequality $h(y) \le h(x)$ holds which will complete the proof. Indeed, $y = gx$, $g \in G$.

(i) If $h(g) \ne h(x)$, then

$$h(y) = h(gx) = \min(h(g), h(x)) \le h(x), \quad i.e. \quad h(y) \le h(x).$$

(ii) Let $h(g) = h(x)$. Let

$$x = \alpha_0 + \sum_{i=1}^{n} \alpha_i g_i, \alpha_i \in R, g_i \in G.$$

Then $y = gx = \alpha_0 g + \sum_{i=1}^{n} \alpha_i g g_i$ and

$$h(y) = \min(h(g), h(gg_1), ..., h(gg_n)) \le h(g) = h(x), i.e. h(y) \le h(x).$$

$\square$

Lemma 3.2. *If G is an abelian group and R is a commutative perfect ring with identity of prime characteristic p, then the subgroup G_p is balanced in $S(RG)$.*

Proof. It is sufficiently to prove that G_p is nice in $S(RG)$. Namely, let $x \in S(RG)$. We will prove that in supp (x) there exists at least one element of G_p. We may write $x = y + z$ where supp $(y) \subseteq G_p$ and supp $(z) \cap G_p = \emptyset$. For some k we have $1 = x^{p^k} = y^{p^k} + z^{p^k}$. Since supp $(z^{p^k}) \cap G_p$=$\emptyset$, then $z^{p^k} = 0$, $y^{p^k} = 1$ and there exists $g_p \in$ supp (y). Hence $g_p \in$ supp $(x) \cap G_p$. Therefore, $1 \in$ supp $(g_p^{-1}x)$. Further the proof is the same as in Lemma 3.1.

Second proof (when R is without nilpotent elements.) Obviously, Lemma 4 of May's paper [May] can be expanded for a perfect ring of characteristic p. Then, in the notations of this lemma, we set $H = G_p$. Since, obviously, G_p is a isotype subgroup of G, then, by the notations of mentioned Lemma 4, G_p is a balanced subgroup in $K(G_p) = 1 + I(RG, G_p) = S(RG)$, i.e. G_p is a balanced subgroup of $S(RG)$, where $I(RG, G_p)$ is the relative fundamental ideal of RG which is generated by the subgroup G_p. □

The proof of Lemma 3.1 and the first proof of Lemma 3.2 were sent to the authors of the present paper from Warren U. May be e-mail, when the ring R is a perfect field of prime characteristic p.

The proof of the last lemma is a correct and short proof of [D3, Lemma 1]. In the following proposition *we will denote the group* $S(RG)$ *by* S.

Proposition 3.3. *Let G be an abelian group and let R be a finite commutative ring with identity of prime characteristic p without nilpotent elements. If α is any ordinal and G^{p^α} is finite, then*

$$f_\alpha(S/G_p)) = f_\alpha(S) - f_\alpha(G_p), \tag{3.1}$$

$$f_\alpha(S) = (|G^{p^\alpha}| - 2|G^{p^{\alpha+1}}| + |G^{p^{\alpha+2}}|)log_p|R|\,.$$

Proof. Since R is a finite commutative ring of characteristic p without nilpotent elements, then R is a direct sum of finite number of finite fields of characteristic p. Therefore, the ring R is perfect. Since, by Lemma 3.2, the subgroup G_p is balanced in $S(RG)$, then

$$(S(RG)/G_p)^{p^\alpha} = S(RG^{p^\alpha})G_p/G_p \cong S(RG^{p^\alpha})/{G_p}^{p^\alpha}.$$

Hence, taking into account that $f_\alpha(G) = f_0(G^{p^\alpha})$, we obtain

$$f_\alpha(S(RG)/G_p) = f_0(S(RG^{p^\alpha})/{G_p}^{p^\alpha}). \tag{3.2}$$

Since ${G_p}^{p^\alpha}$ is finite and, by Lemma 3.2, a pure subgroup of $S(RG^{p^\alpha})$, then, by [F, Theorem 27.5, p. 140], ${G_p}^{p^\alpha}$ is a direct factor of $S(RG^{p^\alpha})$, i.e.

$$S(RG^{p^\alpha}) = {G_p}^{p^\alpha} \times T, \quad T \cong S(RG^{p^\alpha})/{G_p}^{p^\alpha}.$$

Therefore, $f_0(S(RG^{p^\alpha})/{G_p}^{p^\alpha}) = f_0(S(RG^{p^\alpha})) - f_0({G_p}^{p^\alpha})$. Then, by a using of (3.2) and the last equality we obtain the first equality of (3.1).

For the value $f_\alpha(S)$ the result of N. Nachev [N3, Theorem 2.2] holds. We will consider the cases (a) and (b).

(a) Let $G_p^{p^\alpha} \neq 1$. Then obviously $G^{p^\alpha} \neq G^{p^{\alpha+1}}$. Therefore, case 1) of Theorem 2.2 of [N3] holds and we obtain the second equality of (3.1).

(b) Let $G_p^{p^\alpha} = 1$. Obviously $G^{p^\alpha} = G^{p^{\alpha+1}}$. Consequently, case 4) of Theorem 2.2 of [N3] holds and we obtain $f_\alpha(S) = 0$. Hence $f_\alpha(S/G_p) = 0$. Obviously, this value of $f_\alpha(S/G_p)$ can be obtained by a using of the second equality of (3.1) since $|G^{p^\alpha}| = |G^{p^{\alpha+1}}| = |G^{p^{\alpha+2}}|$. $\square$

We note that *Proposition 3.3 gives a complete answer for the Ulm-Kaplansky invariants $f_\alpha(S/G_p)$ in case (i) of Theorem 6 of [D3]* and, as we mentioned, *case (i) of this theorem is not completed and not ambiguous.*

In order to ground our assertion we will give an original formulation of case (i) of Theorem 6 of [D3].

"**Theorem 6.** *Suppose $1 \neq G$ is an abelian group and R is an unitary perfect commutative ring without nilpotent elements in prime characteristic p. Then*

(i) if $|R| < \aleph_0$ and $|G^{p^\sigma}| < \aleph_0$ for any ordinal σ,

$$f_\sigma(S(RG)/G_p) = \begin{cases} (|G^{p^\sigma}| - 2|G^{p^{\sigma+1}}| + |G^{p^{\sigma+2}}|)log_p|R| - f_\sigma(G_p) \\ \quad \text{when } G_p^{p^\sigma} \neq 1 \text{ and } |G^{p^\sigma}| \neq |G^{p^\sigma}[p]| \neq 2 \text{ or } |R| \neq 2; \\ 0 \text{ when } G_p^{p^\sigma} = 1 \text{ or } |G^{p^\sigma}| = |G^{p^\sigma}[p]| = 2 \text{ and } |R| = 2." \end{cases}$$

In detail, *we will make three remarks 1), 2) and 3) on [D3, Theorem 6, case(i)].*

1) *Case (i) of Theorem 6 [D3] has not an ambiguous interpretation* and it is interpreted in four different ways. Indeed, let we set $\sigma = 0$. Let A be the assertion "$G_p \neq 1$", B be the assertion "$|G| \neq |G[p]| \neq 2$", C be "$|R| \neq 2$" and let D be "$|G| = |G[p]| = 2$". Denote by $\overline{A}$ the negative of the assertion A. Then case (i) of Theorem 6, by $\sigma = 0$, can be formulated in the following way which is equivalent to the original.

(i) If $|R| < \aleph_0$ and $|G| < \aleph_0$, then (3.2) holds when
(a) A and B or C is fulfilled, i.e. $A \wedge B \vee C$
and $f_i(S/G_p) = 0$ when
(b) $\overline{A}$ or D and $\overline{C}$ is fulfilled, i.e. $\overline{A} \vee D \wedge \overline{C}$.

It is obviously that we can interpret the assertion(a) of (i) in two ways: either as $(A \wedge B) \vee C$ or as $A \wedge (B \vee C)$. We note that the author does not mention how the brackets must be put. Analogously, the assertion (b), i.e. $\overline{A} \vee D \wedge \overline{C}$ can be interpreted also in two ways. Consequently case (i) of Theorem 6 of [D3] is not ambiguous and it has four interpretations.

2) *Case (i) of Theorem 6 of [D3] is not complete.* It is not hard to see that in order case (i) to be completed (by $\sigma = 0$) it should be $\overline{D} = B$ fulfilled which obviously is not realized.

3) *Case (i) of Theorem 6 of [D3] is complicated.* Namely, it is not necessary to be considered two different subcases, as it is done in the paper [D3] (see, for example, Proposition 3.3).

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НЯКОИ БЕЛЕЖКИ ВЪРХУ МУЛТИПЛИКАТИВНИТЕ ГРУПИ НА КОМУТАТИВНИ ГРУПОВИ АЛГЕБРИ

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Резюме. Нека G е абелева p-група, K е поле от първи род спрямо p, с характеристика различна от p, и спектърът $s_p(K)$ на полето K спрямо p съдържа $\mathbb{N}$. Означаваме с KG груповата алгебра на G над K и с $S(KG)$ p-компонента на групата от нормираните единици на KG. В тази статия изчисляваме $\exp(S(KG)/G)$ и доказваме, че ако G е сепарабелна група, то $S(KG)/G$ е сепарабелна, т.е. G е хубава подгрупа на $S(KG)$.

Нека G е крайна абелева група, G_p е p-компонентата на G и R е краен комутативен пръстен с единица и проста характеристика без нилпотентни елементи. В тази статия изчисляваме инвариантите на Улм-Каплански на групата $S(RG)/G_p$.

Посочените резултати коригират съществени неточности и непълноти във формулировките и доказателствата на резултати в това направление на П. В. Данчев (2005 (Zbl. 1107.16030), 2003 (Zbl. 1035.16025) и 2004 (Zbl. 1067.16054)). Ние отбелязваме също, че, поради известни пропуски, някои резултати на същия автор (2003 (Zbl. 1035.16025), 2004 (Zbl. 1080.16022), 2005 (Zbl. 1107.16030 и Zbl. 1097.16007) и 2008 (Zbl. pre 05375552)) остават недоказани.