INTERSECTION OF POLYHEDRONS AND A PLANE WITH GEOGEBRA

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**Abstract**

*Using GeoGebra, infinite points and the swap of finite and infinite points we present an innovative method for teaching of the topic "Intersection of polyhedrons with a plane ". This method optimizes considerably the sketch work and it allows a possibility to generate solutions of a whole set of problems, by solving just one of them and using the applet Swap finite and infinite points. At any stage of the solution, presented step by step, this applet can be used.Thanks to the powerful potential of GeoGebra this approach expands training of "geometry with computer" and introduces a create component in it.*

Keywords: dynamic geometry software; GeoGebra; Infinite point; Intersection of a plane and a polyhedron.

## 1 Introduction

Nowadays the concept of mathematical competence includes the ability for dealing with modern software systems. The students develop their programming skills and algorithmic thinking and so their digital competence goes beyond that of the traditionally passive user of information technologies. On the other hand this deepens the mathematical knowledge and conceptual understanding of mathematics. We support the efforts done worldwide [1, 2, 6, 12, 17, 22] to start teaching mathematics with the help of mathematical software. A contribution in this direction in Bulgaria for teaching of the students from the specialty "Pre–service teachers" is presented in [21]. A specialized dynamic software *Sam* [7, 8] is used for this purpose. A new function *Swap finite and infinite points* for the dynamic geometry software is included in the menu of *Sam*, which easies its usage. All experiments with the usage of Sam in teaching geometry with students in different grades [14, 15, 21], different cultures (bilingual students) [5] and with different abilities (Model High School of Mathematics "Academic Kiril Popov"–Plovdiv and ordinary schools) show an increased interest towards mathematics. This function optimizes considerably the sketch work and supports the development of a creative stile in teaching geometry. It challenges the teachers and the students quickly to discover new connections between the investigated objects, to summarize certain tasks (both elementary and from international competitions), to create new tasks [8, 16, 19].

While Sam is a specialized DGS, GeoGebra is a very well developed universal, freely available DGS with an abundance of various functions. Particularly useful advantage of GeoGebra for this work is the preservation of the structures after changes in the definitions of the objects in the window "Object properties". In [18] the author combines the rich possibilities of GeoGebra with the function *Swap finite and infinite points*, simulating its effect with the help of the tool *Check Box*. Later this idea is developed with students from fifth grade [14].

The introduction of the notion of infinite point, defined intuitively ([3], p.98) by the definition *We call that two lines intersect at an infinite point, if they are parallel* is accepted from all the students in the secondary school. This vitally important concept ([3], p.4) is due to the great German astronomer Kepler (1571-1631). So with every set of parallel lines an infinite point $U_\infty$ is associated which indicates their direction. That is why we present the infinite point on the drawing by a vector no matter of its length.

In this article we would like to present with the help of *GeoGebra* and the infinite points, one method for teaching of the theme "Intersection of polyhedrons with a plane". This method will introduce a creative component in it, will optimize the sketch work and will expand training of "geometry with computer".

We consider three basic tasks from the theme "Intersection of polyhedrons with a plane" with their solutions and with the applet *Swap finite and infinite points* we demonstrate the possibilities to generate a set of new tasks with their solutions. More detailed comments on the creative phases of these implementations are made in the second task.

All figures in the paper can be downloaded from: http://fmi-plovdiv.org/GetResource?id=2068

## 2 Preliminaries

The function *Swap finite and infinite points*, defined in DGS *Sam* [8], allows the visualization of the wide variety of quadrangular and triangular prisms, pyramids and truncated pyramids with just one click of a button [8, 20, 21]. The basic construction for the proper work of the function *Swap finite and infinite points* is the so called *universal parallelepiped*. The construction of the *universal parallelepiped* is completely described in [8]. Just for a completeness of the presentation we will describe it again (Figure 1).

`Line[A,B] Line[A,B]`

### 2.1 Universal Parallelepiped

The free objects in its construction are: the points $A$, $U$, $V$, $W$, $U_\infty$, $V_\infty$, $W_\infty$. We define two Boolean variables $a$ and $b$ with a captions "Swap $U$ and $U_\infty$" and "Swap $V$ and $V_\infty$", respectively. We construct the lines $c = If[a, Line[A, U], Line[A, U_\infty]]$ and $d = If[b, Line[A, V], Line[A, V_\infty]]$. If the Check box "Swap $U$ and $U_\infty$" is not checked we get the $Line[A, U_\infty]$ and if it is checked we get the $Line[A, U]$. The same holds for the other Check box with a Boolean variable $b$. With the commands $B = Point[c]$ and $D = Point[d]$ we choose the points $B$ and $D$ arbitrary on the lines $c$ and $d$, respectively. We define the lines $e = If[b, Line[B, V], Line[B, V_\infty]]$ and $f = If[a, Line[D, U], Line[D, U_\infty]]$. The vertex $C$ is the intersection point of the lines $e$ and $f$ and it is defined by $C = Intersect[e, f]$. The edges are constructed as segments. The figure $ABCD$, constructed in this way, we will consider as a lower base of the parallelepiped $ABCDA'B'C'D'$ and we call it *universal parallelogram* [8]. We add the third Boolean variable $i$ with a captions "Swap $W$ and $W_\infty$" to construct the surrounding edges of the prism, which lay on the lines $h = If[i, Line[A, W], Line[A, W_\infty]]$, $g = If[i, Line[B, W], Line[B, W_\infty]]$, $j = If[i, Line[C, W], Line[C, W_\infty]]$ and $k = If[i, Line[D, W], Line[D, W_\infty]]$. The choice of the point $A'$ on the line $h$ defines the length of the surrounding edges. The edges of the upper base $A'B'C'D'$ and the remaining vertices are constructed with the help of the tools *parallel line* and *intersection of two lines*. The above described construction of the *universal parallelepiped* creates an obligated

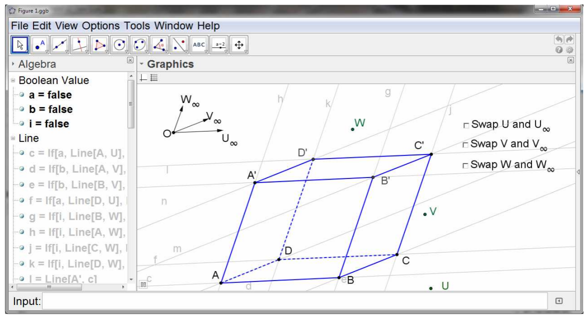


Figure 1: "Universal Parallelepiped"

status of its faces $ABCD$ and $A'B'C'D'$ to be its bases. The rotation of the infinite points $U_\infty$ and $V_\infty$ changes the angle of the parallelogram and the rotation of $W_\infty$ transforms the parallelepiped from sloping to right angled parallelepiped. The limited freedom of the points $B \in AU_\infty$, $D \in AV_\infty$ and $A' \in AW_\infty$ allows us to change the length of the edges of the parallelepiped.

### 2.2 Connected Figures

We call *connected figures* all the figures that are obtained from one another with the help of the function *Swap finite and infinite points* [8]. For example the figures 1, 2 and 3 are connected. A

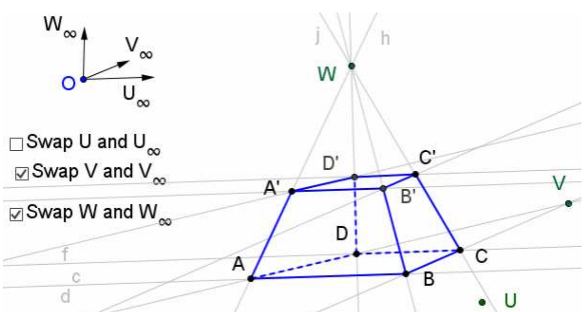


Figure 2: Truncated Pyramid

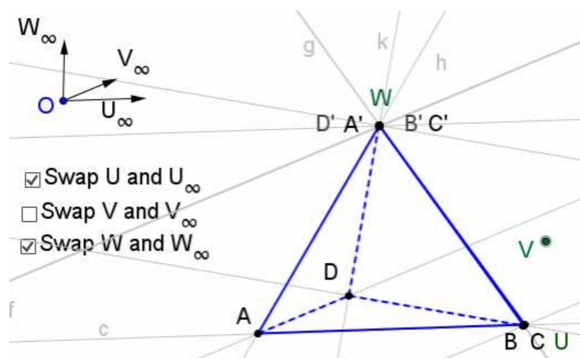


Figure 3: Triangular Pyramid

truncated pyramid with a base the trapezoid $ABCD$ is obtained from the *universal parallelepiped*, by swapping $V_\infty$ and $V$; $W_\infty$ and $W$ (Figure 2). A triangular pyramid is obtained from the *universal parallelepiped* by swapping $U$ and $U_\infty$; $W$ and $W_\infty$, positioning the point $B$ (and $C$) to coincide with $U$ and positioning the points $A'$, ($B'$, $C'$ and $D'$) to coincide with $W$.

**Remark 1** *A new sketch is displayed after checking the Check box "Swap $V$ and $V_\infty$". It may be needed to move the point $D$ to carry out the relation $D/AV$ (point $D$ is between the points $A$ and $V$). After checking the Check box "Swap $W$ and $W_\infty$" it may be needed to move the point $A'$ to carry out* *the relation $A'/AW$ (Figure 2). After checking the Check box "Swap $U$ and $U_\infty$" it may be needed to move the point $B$ to carry out the relation $B/AU$.*

### 2.3 Basic Constructions for finding the intersection of polyhedrons with a plane

Just to simplify the reading we will denote everywhere the plane of the intersection by $\alpha$. The basic constructions in the process of solving the problem of finding the intersection of $\alpha$ and a polyhedron are:

A) Finding the piercing point $G$ of a line $g$, generated by two vertexes of the polyhedron and the plane $\alpha$. To find the point $G = g \cap \alpha$ we do the following constructions:

- $A_1$) We construct an auxiliary plane $\beta$, which is incident with the line $g$;
- $A_2$) We construct the line $s = \alpha \cap \beta$;
- $A_3$) We locate the point $G = g \cap s = g \cap \alpha$ (Figure 4).

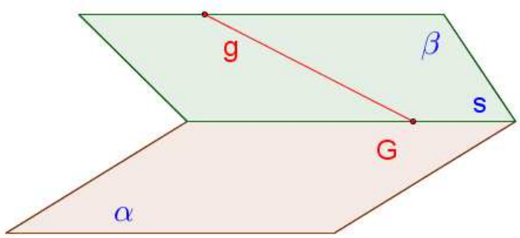


Figure 4: Construction A)

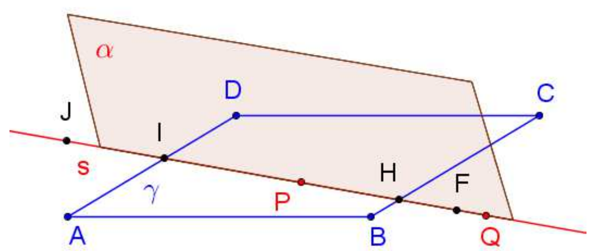


Figure 5: Construction C)

B) We locate the intersection line $s$ of the plane $\alpha$ with the plane $\gamma$, containing a face of the polyhedron. To find $s$ it is enough to locate two common points $P$ and $Q$ for the planes $\alpha$ and $\gamma$. Then the intersection line is $s = \alpha \cap \gamma = PQ$.

C) The problem of finding the piercing points of the edges of the polyhedron, that are located in one of its faces with $\alpha$ is simplified, because these edges have a common auxiliary plane $\beta = \gamma$ – the plane of the face (Figure 5). Then the intersection points of the line $s$ with all the lines incident with the edges, that belong to the face $\gamma$, are the piercing points of these lines with $\alpha$. From Figure 5 we see that the edges $BC$ and $AD$ have common points $H$ and $I$ with $\alpha$, respectively while there will be no vertices from the polygon, which is the intersection of the polyhedron with $\alpha$, that lay on the edges $AB$ and $CD$.

## 3 Intersection of polyhedrons with a plane

The theme "Intersection of polyhedrons with a plane" is one of the most difficult themes for the students in school. It is one of the reasons that this theme is not included in the course of all secondary schools in Bulgaria. There are different presentations of the theme in various textbooks in Bulgaria [4, 9, 10, 11, 13].

We would like to present a new kind of strategy for the construction of the intersection of polyhedrons with a plane with *GeoGebra* and the help of a new applet *Swap finite and infinite points*. This approach introduces a creative component in the process of studying this topic. The idea is based on the projective property – a preservation of the incidence of the basic objects (points, lines and planes) in the central projection. We refuse to use the affine property that the parallel faces of the parallelepiped intersect the plane of the section in parallel lines, because after the transition into prisms with bases different from parallelograms or into pyramids (truncated or not) this property is not reserved (except for the bases).

The finding of the intersection lines of the planes of two faces of the polyhedron with $\alpha$ ensures the availability of two common intersection points of the planes of any of the rest faces of the polyhedron with $\alpha$. Thereafter we can determine all the piercing points of the edges of the polyhedron with $\alpha$. Without loss of generality we can choose one of the first intersection lines to lie on a base of the polyhedron.

Let the plane of the intersection $\alpha$ be determined by the non collinear points $P$, $Q$, $R$.

Let us accept the notations $(A_1A_2\dots A_n)$ for the face $A_1A_2\dots A_n$ of the polyhedron and $\gamma(A_1A_2\dots A_n)$ for the plane of this face.

We will start the introduction of our method for determining the intersection of a polyhedron and a plane with the following classical problem:

**Problem 1** *Find the intersection of the parallelepiped $ABCDA'B'C'D'$ with the plane $\alpha(P,Q,R)$, where $P \in BC$, $Q \in AA'$, $R \in C'D'$.*

*Solution:* At first we will find the intersection line $s$ of the plane $\alpha$ and the plane $\gamma(ABCD)$ (Figure 6). As far as we have one point $P$, which belongs to $s$, it is enough to find the piercing point $F$ of the line $RQ$ with the plane $\gamma(ABCD)$. We consider the auxiliary plane $\beta(Q,R,W_\infty)$. To find the intersection line $t=\beta\cap\gamma$, we determine two incident with it points $R_1 = RW_\infty \cap \gamma = RW_\infty \cap CD$ and $Q_1 = QW_\infty \cap \gamma = A$. They are the projections of $R$ and $Q$ on the plane $\gamma(ABCD)$ along $W_\infty$. This step is usually called "method of projections". Consequently there holds the relation $F = RQ \cap \gamma = RQ \cap t = RQ \cap R_1Q_1$ and $s = \alpha \cap \gamma = PF$. From $C$) the points $G = AB \cap s$, $I = AD \cap s$ and $H = CD \cap s$ are the piercing points of the lines $AB$, $AD$ and $CD$ with $\alpha$, respectively. With the help of the lines $IQ = \alpha \cap \delta(ADD'A')$ and $HR = \alpha \cap \epsilon(CDD'C')$ we determine the vertexes of the intersection $J = IQ \cap A'D'$ and $L = HR \cap CC'$.

We will illustrate how with the applet *Swap finite and infinite points* we can generate a set of new problems with their solutions in *GeoGebra*. We will swap the points $P$ and $P_\infty$. For the correct functioning of the above construction it is necessary the point $P$ to be swapped with an arbitrary infinite point $P_\infty$, which is incident with the plane $\gamma(ABCD)$. The position of the point $P_\infty$ can be

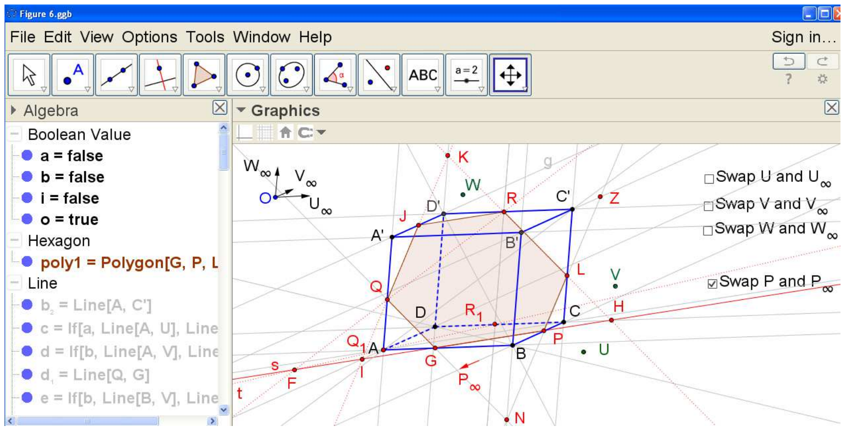


Figure 6: Problem 1

changed in the window "Object properties". We will present several intersections, which are generated from Figure 6 by changing the point $P_\infty$. When the points Q and R are not fixed, for each new drawing we adjust the type of polygon depending on our wish and manually outline it.

If $P_\infty \in BC$, then the plane $\alpha$ is parallel to the line $BC$ (Figure 7); If $P_\infty \in BD$, then the plane

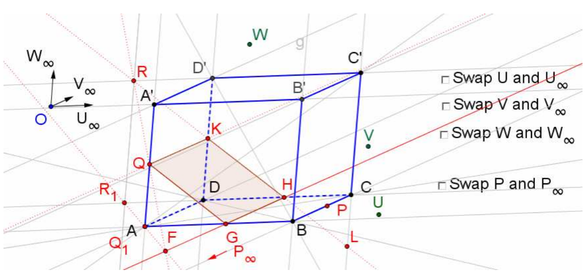


Figure 7: $P_\infty \in BC$

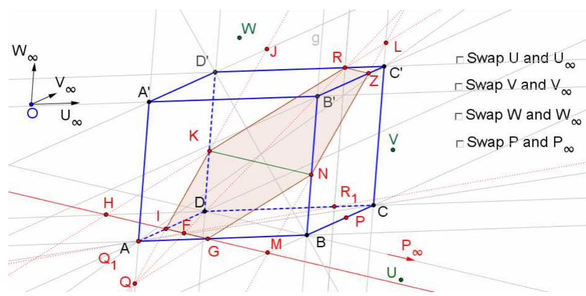


Figure 8: $P_\infty \in BD$

$\alpha$ is parallel to the line $BD$ (Figure 8);

If $P_\infty \in AC$, the plane $\alpha$ is parallel to the line $AC$ (Figure 9). Besides, we have swapped $U$ with $U_\infty$ and $W$ with $W_\infty$ in Figure 9 and thus we have produced an intersection of $\alpha$ with a truncated pyramid with bases, which are trapeziums. The Figures 6, 7, 8 and 9 are connected figures. The user of *GeoGebra* (teacher or student) can generate a basic problem of his own and to produce a set of new problems with their solutions. The construction of the solutions can be presented step by step with the help of a slider for the basic problem and its satellites. For example let us consider the following problem.

**Problem 2** *Find the intersection of the parallelepiped $ABCDA'B'C'D'$ with the plane $\alpha(P, Q, R)$, provided that two of the points $P$, $Q$, $R$ lay on the plane of one and the same face of the parallelepiped $ABCDA'B'C'D'$ and the third point is arbitrary.*

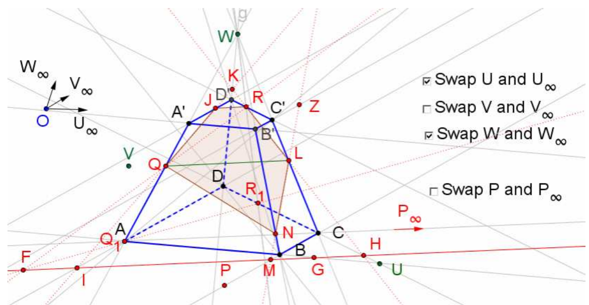


Figure 9: $P_\infty \in AC$

*Solution:* Without loss of generality we may assume that the points $P, Q \in \gamma(ABCD)$ and $R$ be an arbitrary point (Figure 10). Nevertheless that $R$ is an arbitrary point we need to determine its position in the space, because the universal parallelepiped $ABCDA'B'C'D'$ can be considered as a main axonometric image of an arbitrary parallelepiped in general axonometric view, where the three groups of parallel lines are parallel to the virtual axonometric axes. Let us point out that the projections of the vertexes $A$ and $A'$ on the plane $\gamma(ABCD)$ along $W_\infty$ are $A_1 = A'_1 = A$, on the plane $\delta(ADD'A')$ along $V_\infty$ are $A_2 = D$, $A'_2 = D'$. The projections of the other vertexes can be seen easily. That is why we chose and denote by $R_1$, the projection of $R$ on the plane $\gamma(ABCD)$ along $W_\infty$. It is obvious that $s = PQ = \alpha \cap \gamma$. According to Remark 1 the points $K = s \cap AB$, $T = s \cap AD$,

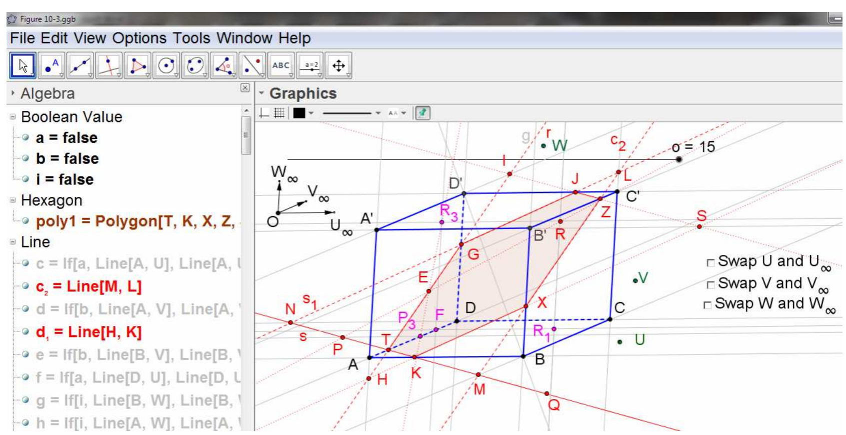


Figure 10: Problem 2

$M = s \cap BC$ and $N = s \cap CD$ are the piercing points of $\alpha$ with the lines $AB$, $AD$, $BC$ and $CD$, respectively. Any of these points belong to the plane $\gamma(ABCD)$ and belong to the plane of one more of the faces of the parallelepiped. We will find the intersection line of $\alpha$ with the plane of another face of the parallelepiped. Without loss of generality we may search for the intersection line with the plane $\delta(ADD'A')$. We need a second point from this line, because already $T \in AD$ is a point incident with it. It is enough to find the piercing point of the line $PR$ (or the line $QR$) with the plane $\delta$. For this purpose we will use the auxiliary plane $\lambda(P, R, U_\infty)$. After projecting the points $P$ and $R$ on the plane $\delta(ADD'A')$ along $U_\infty$ we get their projections $P_3 = AD \cap PU_\infty$ and $R_3 = RU_\infty \cap FW_\infty$, where $F = R_1U_\infty \cap AD$. Consequently $\delta \cap \lambda = P_3R_3$ and the point $E = PR \cap P_3R_3$ is the piercing point of the line $PR$ with the plane $\delta(ADD'A')$. Then the line $r = TE$ is the intersection line of $\alpha$ with the plane $\delta(ADD'A')$.

According to $C$) from Section 2 we determine the piercing points $H = r \cap AA'$, $G = r \cap DD'$, $I = r \cap A'D'$. We construct the line $NG$, which is the intersection line of the plane $\alpha$ with the plane $\epsilon(CDD'C')$ and we find the piercing points $L = CC' \cap \alpha = CC' \cap NG$, $J = C'D' \cap NG$. We construct the line $ML$, which is the intersection line of the plane $\alpha$ with the plane $\rho(BCC'B')$ and thus we find the piercing points $X = BB' \cap \alpha = BB' \cap ML$ and $Z = B'C' \cap \alpha = B'C' \cap ML$. At the end let us note the piercing point $S = A'B' \cap \alpha = d_1 \cap A'B'$, where $d_1 = HK$.

There hold the relations $TK \parallel JZ$, $TG \parallel XZ$, $GJ \parallel KX$, because there hold $\gamma(ABCD) \parallel \mu(A'B'C'D')$, $\delta(ADD'A') \parallel \rho(BCC'B')$, $\epsilon(CDD'C') \parallel \sigma(ABB'A')$, respectively.

Following [8] we will accept the definition: *The problems, which are generated from a problem A with the help of the applet "Swap finite and infinite points" will be called satellites of problem A and be denoted in the paper by A.1, A.2,... etc.*

The applet *Swap finite and infinite points* in *GeoGebra* allows us to produce a set of satellites of Problem 2 with their solutions. We will illustrate some of these possibilities and we will comment the creative stages of their realization.

**Problem 2.1** *Find the intersection of a truncated pyramid $ABCDA'B'C'D'$ which bases are parallelograms and the surrounding edge $AA'$ is orthogonal to the bases, with the plane $\alpha(P, Q, R)$, where $P, Q \in \gamma(ABCD)$ and $R \in \rho(BCC'B')$.*

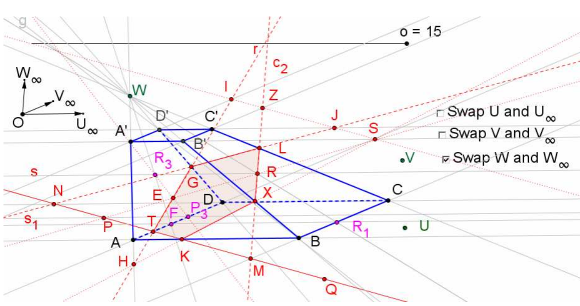


Figure 11: Problem 2.1

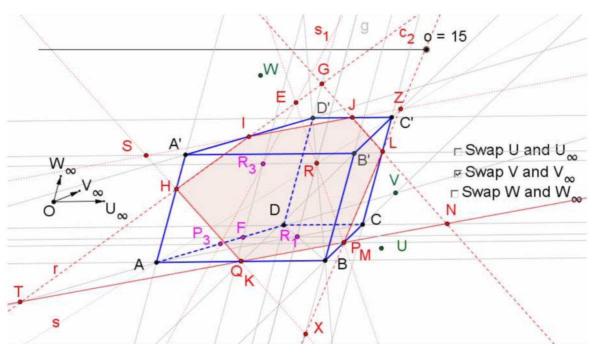


Figure 12: Problem 2.2

*Solution:* It is enough to change the definition of the point $R_1$ from $Point[i_1]$ into $Intersect[i_1, e]$, where $i_1 = RW_\infty$ and $e = BV_\infty$ in the window "Object properties" (Figure 11).This ensures that $R \in \rho(BCB'C')$. With the help of the Check box "Swap $W$ and $W_\infty$" we change the Boolean value of $i$ from "false" to "true". After applying the dynamics on the point $A'$ we realize the relation $A'/AW$.

The truncated pyramid with its intersection with the plane $\alpha$ is displayed on the monitor. By using dynamics of the point $W$ we can provide the condition $AA' \perp AB$, which means $AA' \perp \gamma(ABCD)$ in the parameters of the cabinet projection (usually used in the secondary school).

**Problem 2.2** *Find the intersection of the oblique prism $ABCDA'B'C'D'$ which bases are trapeziums ($AB \parallel CD$) with the plane $\alpha(P, Q, R)$, where the points $P$ and $Q$ are the midpoints of the segments $BC$ and $AB$, respectively and $R$ is the intersection point of the diagonals $AC'$ and $BD'$.*

*Solution:* It is enough to change in the window "Object properties" the definitions of the points: $R = Intersect[q_1, b_2]$, where $q_1 = BD'$, $b_2 = AC'$, $P = Midpoint[B, C]$ and $Q = Midpoint[A, B]$ (Figure 12). By the Check box "Swap $V$ and $V_\infty$" we change the Boolean value of $b$ from "false" to "true" and thus the lines $BC$ and $AD$ transform from parallel lines to intersecting into the finite point $V$. After rotation of $W_\infty$ we determine the direction of the surrounding edges of the prism.

**Remark 2** *The conditions $R \in BD'$ and $R_3 \in AD'$ are equivalent because the projection of the line $BD'$ on the plane $\delta(ADD'A')$ along the point $U$ is the line $AD'$ and also the central projecting (regardless from a final or an infinite center) preserves the incidence.*

The last Remark simplifies the solutions of the following two satellite problems.

**Problem 2.3** *Find the intersection of a truncated quadrangular pyramid $ABCDA'B'C'D'$ with a plane $\alpha$, which passes through the midpoints of the edges $AB$ and $BC$ and $\alpha$ is parallel to the diagonal $BD'$.*

*Solution:* We use the dynamic sketch Figure 12, but now $P = Midpoint[A, B]$, $Q = Midpoint[B, C]$ (Figure 13). With the help of the Check boxes "Swap $U$ and $U_\infty$" and "Swap $W$ and $W_\infty$" we transform the trapezium $ABCD$ into an arbitrary quadrangular and the parallelepiped into a truncated pyramid. We define a new Boolean variable $o$ with a caption "Swap $R$ and $R_\infty$", where the point $R_\infty \in BD'$. According to Remark 2 the point $R_3$ as and the point $(R_\infty)_3$ belong to $AD'$ because of which $E = P_3(R_\infty)_3 \cap PR_\infty = AD' \cap PR_\infty$.

After the automatic generation of the sketch we have drawn the segment $SG$, just to emphasize the presence of the relation $SG \parallel BD'$, because $\alpha \parallel BD'$ and $SG = \alpha \cap \pi(BDD'B')$.

**Problem 2.4** *Find the intersection of a quadrangular pyramid $ABCDW$ with a plane $\alpha$, which passes through the midpoints of the edges $AB$ and $BC$ and $\alpha$ is parallel to the surrounding edge $WB$.*

*Solution:* The solution is generated from Figure 13, by moving the point $A'$ to coincide with $W$ (Figure 14).

**Problem 2.5** *Find the intersection of a parallelepiped $ABCDA'B'C'D'$ with a plane $\alpha(P, Q, R)$, where the point $P$ coincides with a vertex of a base, $Q$ lays in the plane $\gamma(ABCD)$ and $R$ is an arbitrary point.*

*Consider the following cases:*

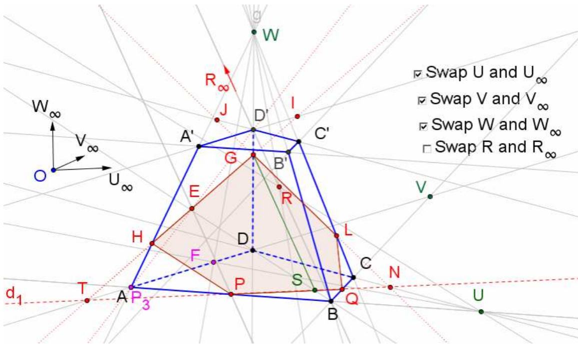


Figure 13: Problem 2.3

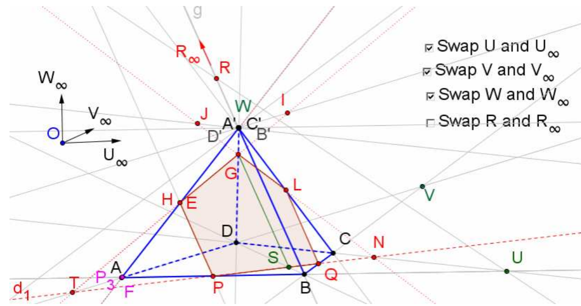


Figure 14: Problem 2.4

*a) the plane $\alpha$ is parallel to a diagonal of the base and*

*$a_1$) $ABCDA'B'C'D'$ is a prism with a base an arbitrary quadrangle;*

*$a_2$) $ABCDA'B'C'D'$ is a pyramid;*

*b) the point $R$ lays on a diagonal, that connects two vertexes, which are not from one and the same face of the parallelepiped $ABCDA'B'C'D'$.*

*Solution:* We will use the sketch of the Problem 2 (Figure 10). Let chose the point $P$ to coincide with the vertex $B$. It means that the definition of the point $P$ have to be changed from free point to an $Intersect[c, e]$, where $c = AU_\infty$ and $e = BV_\infty$.

a) From the choice of the point $P$ it follows that it is possible to consider the case $\alpha$ to be parallel to the diagonal $AC$. Therefore the lines $AC$ and $PQ$ will be parallel, i.e. $Q_\infty \in AC$. We define a new Boolean variable $o$ with a caption "Swap $Q$ and $Q_\infty$" (Figure 15).

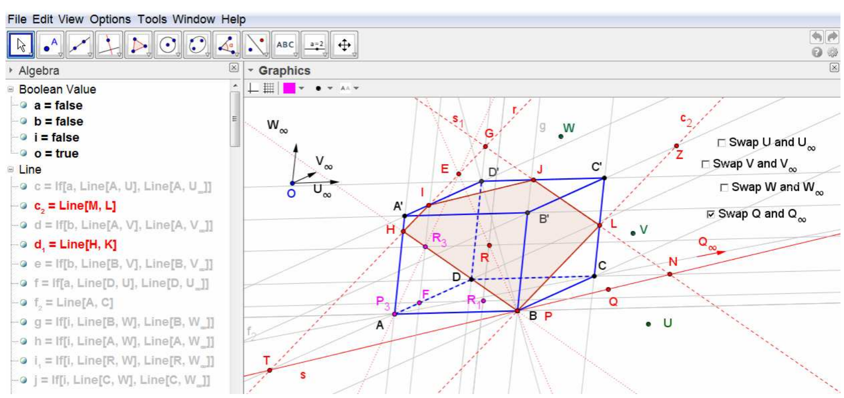


Figure 15: Problem 2.5 $a$)

$a_1$) It is enough to change the Boolean values of $a$, $b$ to "true" and the values of $i$, $o$ to "false" to generate the solution (Figure 16).

$a_2$) We set the value "false" for $a$, $b$, $o$ and "true" for $i$ to generate Figure 17. We have drawn afterwards the segment $HL$ just to emphasize that there holds the relation $HL \parallel AC$.

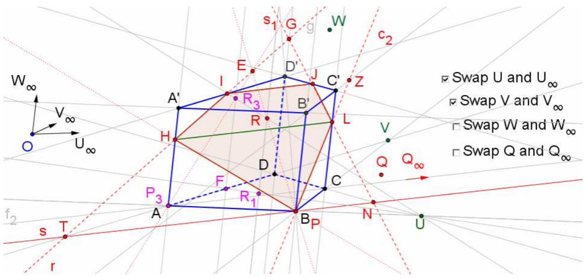


Figure 16: Problem 2.5 $a_1$)

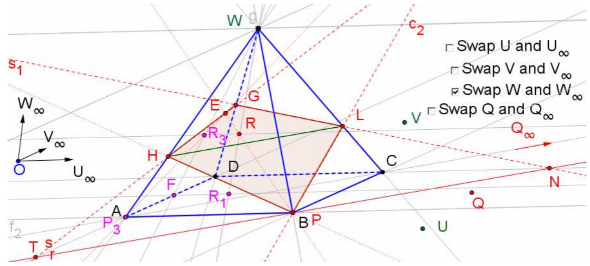


Figure 17: Problem 2.5 $a_2$)

$b$) The requirement $R$ to be incident with a diagonal, which is not lying in a face of the parallelepiped, generates two cases:

$b_1$) $R \in BD'$. We define the points $R$ and $R_1$ : $R = Point[q_1]$, $R_1 = Intersect[i_1, g_2]$, where $q_1 = BD'$, $i_1 = RW$, $g_2 = BD$ (Figure 18). Let $ABCDA'B'C'D'$ be a truncated pyramid. The intersection in this case is always a quadrangle with a vertex $D' = PR \cap \mu(A'B'C'D') = E = G$.

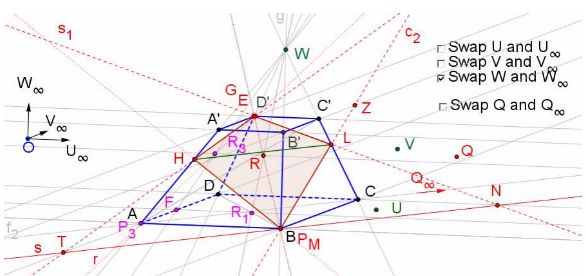


Figure 18: Problem 2.5 $b_1$)

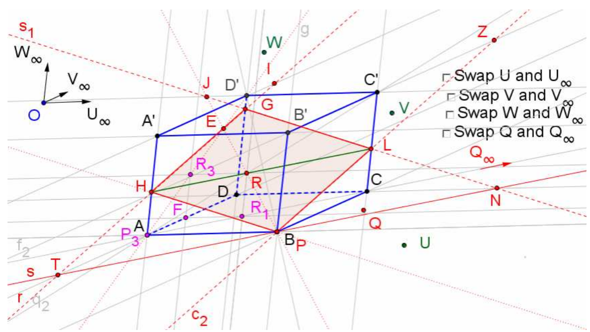


Figure 19: Problem 2.5 $b_2$)

$b_2$) $R \in AC'$. We define the points $R$ and $R_1$ : $R = Point[q_2]$, $R_1 = Intersect[i_1, f_2]$, where $q_2 = AC'$, $i_1 = RW$, $f_2 = AC$ (Figure 19). Let $ABCDA'B'C'D'$ be a parallelepiped. The intersection could be either pentagon or parallelogram and the point $G$ is not necessary to coincide with $D'$.

**Remark 3** *We have not included the infinite line in the problems, as it was done in [5], because the investigation is intended mainly to teachers and students in the secondary schools. This reduces the number of different problems that can be generated from one basic problem.*

In the solutions of the above proposed two basic problems the preference for the first intersection was for the plane of the lower base of the parallelepiped. While in Problem 1 both bases had equivalent conditions to be selected, in Problem 2 this selection was recommended. The solution of the next problem is an illustration of the possibility the upper base of the parallelepiped to be our choice.

**Problem 3** *Find the intersection of the parallelepiped $ABCDA'B'C'D'$ with the plane $\alpha$, which passes through the point $P$, belonging to the plane $\sigma(ABB'A')$, through the point $R$, belonging to the plane $\delta(ADD'A')$ and $\alpha$ is parallel to the line $BD$.*

*Solution:* (Figure 20) Let us at first to find the intersection $s$ of $\alpha$ with the plane $\mu(A'B'C'D')$, creating a structure that will allow us to generate decisions and to a set of other tasks. For this purpose we

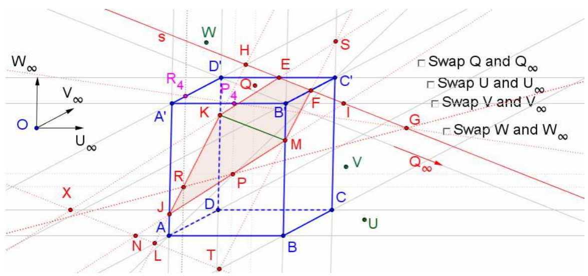


Figure 20: Problem 3

find pierce $G$ of the line $PR$ with the plane $\mu(A'B'C'D')$: $G = PR \cap P_4R_4$, where $P_4$ and $R_4$ are the projections of $P$ and $R$, respectively along $W_\infty$ on the plane $\mu(A'B'C'D')$. Choose an arbitrary point $Q$ of the plane $\mu$ and set point $Q_\infty \in B'D'(BD)$. We create Boolean value with a caption "Swap $Q$ and $Q_\infty$", recording the conditional operator: $If[v, Line[G, Q], Line[G, Q_\infty]]$.

Let us construct the line $s = GQ = \alpha \cap \mu$. By the Check box "Swap $Q$ and $Q_\infty$" we choose a Boolean value "false" and the line $s = GQ$ is replaced with the line $s = GQ_\infty$. Further the decision continues as like as in Problem 2 and it is shown on Figure 20. We define the points: $E = s \cap C'D'$, $F = s \cap B'C'$, $I = s \cap A'B'$, $H = s \cap A'D'$ and then the lines $RH = \alpha \cap \delta(ADD'A')$ and $PI = \alpha \cap \sigma(ABB'A')$. The points $M = PI \cap BB'$, $N = PI \cap AB$, $J = PI \cap AA'$; $K = RH \cap DD'$, $L = RH \cap AD$; $S = MF \cap CC'$, $T = MF \cap BC$, $X = KE \cap CD$ and $E$, $F$, $I$, $H$ are potential vertices of the polygon – section of the plane $\alpha$ with the polyhedron.

We suggest the following four satellites problems:

**Problem 3.1** *Find the intersection of the truncated quadrangular pyramid $ABCDA'B'C'D'$ which bases are parallelograms with the plane $\alpha$, which passes through the point $P$, belonging to the plane $\sigma(ABB'A')$, through the point $R$, belonging to the plane $\delta(ADD'A')$ and $\alpha$ is parallel to the line $BD$.*

**Problem 3.2** *Find the intersection of the truncated quadrangular pyramid $ABCDA'B'C'D'$ which bases are trapezoids ($AD \parallel BC$) with the plane $\alpha$, which passes through the point $P$, belonging to the plane $\sigma(ABB'A')$, through the point $A$ and $\alpha$ is parallel to the line $BD$.*

**Problem 3.3** *Find the intersection of the pyramid $ABCDA'B'C'D'$ which base is trapezoid ($AB \parallel CD$) with the plane $\alpha$, which passes through the point $P$, belonging to the plane $\sigma(ABB'A')$, through the point $R$, belonging to the plane $\delta(ADD'A')$ and $\alpha$ is parallel to the line $BD$.*

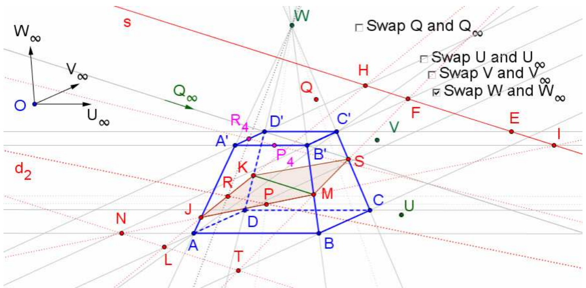


Figure 21: Problem 3.1

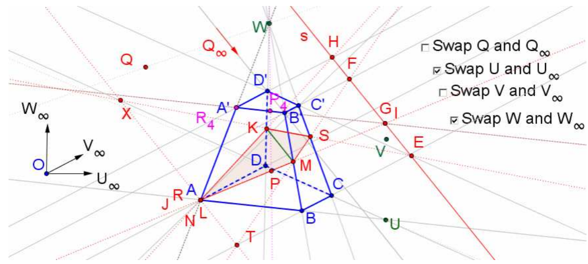


Figure 22: Problem 3.2

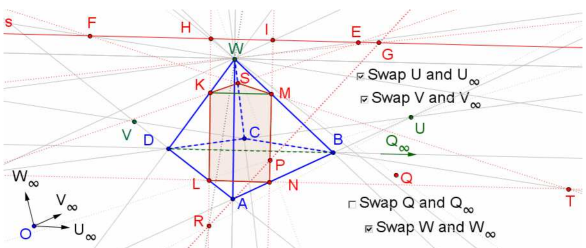


Figure 23: Problem 3.3

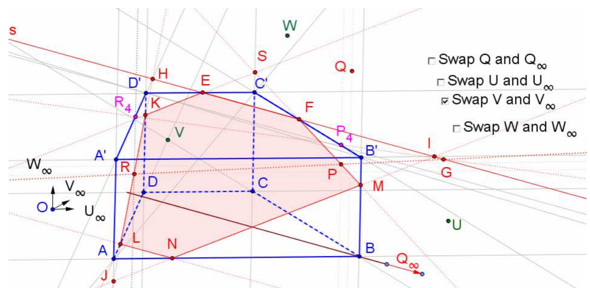


Figure 24: Problem 3.4

**Problem 3.4** *Find the intersection of the quadrangular prism $ABCDA'B'C'D'$ which bases are trapezoids ($AB \parallel CD$) with the plane $\alpha$, which passes through the point $P$, belonging to the plane $\rho(BCC'B')$, through the point $R$, belonging to the plane $\delta(ADD'A')$ and $\alpha$ is parallel to the bisector of the angle $ABC$.*

# 4 Conclusion

Convinced that the presence of heuristic moments and DGS enhances teaching of school geometry, we offer an innovative method for studying the topic "Intersection of polyhedrons with a plane" in the dynamic environment of GeoGebra. We have shown how the user of *GeoGebra* (teacher or student) can generate a basic problem of his own and to produce a set of new problems with their solutions with the help of the applet *Swap finite and infinite points*.

With this paper we would like not only to share with the readers of the journal a new approach to the subject "Intersection of polyhedrons with a plane", but and to pay attention to *GeoGebra* developers on the function *Swap finite and infinite points*, and if they appreciate its benefits to include it in *GeoGebra* menu, which will facilitate its application.

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