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SYLVESTER'S POLYNOMIALS

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Abstract. In this paper we give some results on the distribution of the zeros of the polynomials of the type $S(f;z) = \sum_{k=0}^{n} f^{(k)}(z)$, where $f(z)$ is a polynomial of degree n with real coefficients. For example, we prove that if the polynomial $S(f;z)$ has only real and simple zeros, then the polynomial $f(z)$ has also only real and simple zeros and the zeros of $f(z)$ separate the zeros of $S(f;z)$.

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Let $f(z)$ be a polynomial of degree n with real coefficients. Then the polynomial

$$S(f;z) = \sum_{k=0}^{n} f^{(k)}(z) \tag{1}$$

will be called (Bojorov's idea) Sylvester's polynomial. If $f(z) = \frac{z^n}{n!}$, then we get the polynomial

$$S(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!}, \tag{2}$$

for which the following result of Sylvester is known ([1], p.17): the polynomial $S(z)$ has no more than one real zero. Obviously, if $S(z)$ has a real zero, it is a negative one. From (1) we can easily get the equation

$$S(f;z) = S^{'}(f;z) + f(z) \tag{3}$$

or the more general equation

$$S^{(p-1)}(f;z) = S^{(p)}(f;z) + f^{(p-1)}(z), \quad p = 1, 2, \ldots, n. \tag{4}$$

Bojorov has stated without a proof the following unpublished theorem.

Theorem 1. *Every zero of the polynomial $S(f;z)$ of order r is a zero of the polynomial $f(z)$ of order $r-1$.*

Proof. Let $z = \alpha$ be a zero of the polynomial $S(f;z)$ of order r. This means

$$S(f;\alpha) = S'(f;\alpha) = \ldots = S^{(r-1)}(f;\alpha) = 0\,, \quad S^{(r)}(f;\alpha) \neq 0\,.$$

From (4) when $p = 1, 2, \ldots, r$ we obtain

$$f(\alpha) = f'(\alpha) = \ldots = f^{(r-2)}(\alpha) = 0\,, \quad f^{(r-1)}(\alpha) = -S^{(r)}(f;\alpha) \neq 0\,,$$

i.g. the point $z = \alpha$ is a zero of the polynomial $f(z)$ of order $r-1$. □

Corrolary 1. *The simple zeros of the polynomial $S(f;z)$ are not zeros of the polynomial $f(z)$.*

Theorem 2. *In every open interval, not containing zeros of $f(z)$, the polynomial $S(f;z)$ can have no more than one zero.*

Proof. Suppose that the polynomial $f(z)$ has no zeros in the interval (a,b). Assume that the polynomial $S(f;z)$ has more than one zero in the interval (a,b) and let α and β be two consecutive zeros of $S(f;z)$ such that $a < \alpha < \beta < b$. The zeros α and β are simple zeros of the polynomial $S(f;z)$. Otherwise, according to Theorem 1, they would be zeros of the polynomial $f(z)$, but as we know $f(z)$ does not have any zeros in the interval (a,b). From (3) we obtain

$$S'(f;\alpha) = -f(\alpha)\,, \quad S'(f;\beta) = -f(\beta)$$

and then

$$S'(f;\alpha)S'(f;\beta) = f(\alpha)f(\beta).$$

Since the polynomial $f(z)$ does not have zeros in the interval (a,b), the numbers $f(\alpha)$ and $f(\beta)$ have the same sign. Therefore,

$$S'(f;\alpha)\,S'(f;\beta) > 0,$$

from which we conclude that $S^{'}(f;z)$ has an even number of zeros in the interval (α, β). On the other hand, according to Roll's theorem, the polynomial $S(f;z)$ has an odd number of zeros in the interval (α, β), which gives the desired contradiction. Thus, the Sylvester's polynomial $S(f;z)$ can have no more than one zero in the interval (a, b). Moreover, if the polynomial $S(f;z)$ has a zero in this interval, it is a simple one. □

Corrolary 2. *If the polynomial $f(z)$ does not have real zeros, then the polynomial $S(f;z)$ does not have real zeros either.*

Theorem 3. *If the real zeros of the polynomial $S(f;z)$ are simple, then*

$$Z_R(S(f;z)) \le Z_R(f(z)), \tag{5}$$

where $Z_R(f(z))$ denotes the number of real zeros of the polynomial $f(z)$, counting multiplicities.

Proof. Let $Z_R(f(z)) = m$ ($m \le n$) and let $x_1, x_2, \dots, x_r$ be the distinct real zeros of the polynomial $f(z)$ ($r \le m$) and $m_1, m_2, \dots, m_r$—their orders respectively, $\sum_{k=0}^{r} m_k = m$. A real zero of the polynomial $f(z)$ cannot be a zero of the polynomial $S(f;z)$, because all the real zeros of $S(f;z)$ are simple by Corollary 1. In each of the intervals

$$(-\infty, x_1),\ (x_r, +\infty),\ (x_k, x_{k+1}), \qquad k = 1, 2, \dots, r-1,$$

the polynomial $S(f;z)$ can have no more than one real (simple) zero. Thus,

$$Z_R(S(f;z)) \le r + 1 \le m + 1 .$$

If $deg f(z)$ is an even number, then m is also an even number and therefore we cannot have $Z_R(S(f;z)) = m + 1$, because $deg S(f;z)$ is an even number as well. If $deg f(z)$ is an odd number, then m is also an odd number and then $Z_R(S(f;z)) = m + 1$ is again impossible, because $deg S(f;z)$ is an odd number. Therefore, $Z_R(S(f;z)) \le m$. □

Theorem 4. *If the polynomial $S(f;z))$ has only real and simple zeros, then the polynomial $f(z)$ has also only real and simple zeros and the zeros of $f(z)$ separate the zeros of $S(f;z)$.*

Proof. Suppose that the polynomial $S(f;z)$ has only real and simple zeros, say $x_1, x_2, \ldots, x_n$ with $x_1 < x_2 < \ldots < x_n$. From (5) we have $Z_R(f(z)) \geq n$ and since $deg f(z) = n$, it follows that $Z_R(f(z)) = n$, i.g. the polynomial $f(z)$ has only real zeros. According to Corollary 1, the zeros of the polynomial $f(z)$ are different from the zeros of the polynomial $S(f;z)$. Consider either of the intervals (x_k, x_{k+1}), $k = 1, 2, \ldots, n-1$. We have

$$S(f;x_k) = S(f;x_{k+1}) = 0, \quad S^{'}(f;x_k) \neq 0, \quad S^{'}(f;x_{k+1}) \neq 0 .$$

According to Roll's theorem, the polynomial $S^{'}(f;z)$ has an odd number of zeros in the interval (x_k, x_{k+1}) and then

$$S^{'}(f;x_k)S^{'}(f;x_{k+1}) < 0 . \tag{6}$$

From (3) we conclude that

$$f(x_k) = -S^{'}(f;x_k), \quad f(x_{k+1}) = -S^{'}(f;x_{k+1}) . \tag{7}$$

Now from (6) and (7) we get that $f(x_k)f(x_{k+1}) < 0$. Therefore the polynomial $f(z)$ has an odd number of zeros in the interval (x_k, x_{k+1}). Thus in each of the intervals (x_k, x_{k+1}), $k = 1, 2, \ldots, n-1$, the polynomial $f(z)$ has at least one zero. In fact it has at least $n-1$ real zeros. Since $deg f(z) = n$, it follows that in each of the intervals (x_k, x_{k+1}), $k = 1, 2, \ldots, n-1$, the polynomial $f(z)$ has exactly one zero and one zero in the interval $(-\infty, x_1)$, or in the interval $(x_n, +\infty)$. This completes the proof of the theorem. □

Now, using the theorems we have proven so far, we state the following result about the zeros of the polynomials of the type

$$f_{\nu+1}(z) = \sum_{k=0}^{n} \binom{k+\nu}{\nu} f^{(k)}(z), \quad \nu = 0, 1, 2, \ldots, \tag{8}$$

where

$$f_1(z) = S(f;z)\,, \; f_2(z) = S(f_1;z)\,, \ldots .$$

Corrolary 3. *If the polynomial $f(z)$ does not have real zeros, then none of the polynomials* (8) *has real zeros.*

Finally, considering the polynomials

$$f_{\nu+1}(z) = \sum_{k=0}^{n} \binom{n-k+\nu}{\nu} \frac{z^k}{k!}\,, \quad \nu = 0, 1, 2, \dots, \tag{9}$$

where

$$f_1(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}\,,$$

$$f_2(z) = S(f_1; z)\,,\ f_3(z) = S(f_2; z)\,, \dots,$$

we also establish the following statement.

Corrolary 4. *Each of the polynomials* (9) *can have at most one real zero.*

References

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ПОЛИНОМИ НА СИЛВЕСТЪР

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Резюме. В работата са получени някои резултати за разпределението на нулите на полиномите от вида $S(f;z) = \sum_{k=0}^{n} f^{(k)}(z)$, където $f(z)$ е полином от степен n с реални коефициенти. Доказано е, че ако полиномът $S(f;z)$ има само реални и прости нули, то полиномът $f(z)$ има също само реални и прости нули и нулите на $f(z)$ разделят нулите на $S(f;z)$.