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ON A CLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper we obtain coefficient estimates, distortion and covering theorems for the class $T_p^*(A,\alpha)$ of analitic and p-valent functions in the unit disk.

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***Key words*:** analytic functions, multi-valent functions.

1. Introduction and Definitions

Let $S(p)$ $(p \geq 1)$ denote the class of functions of the form $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ which are analytic and p-valent in the unit disk $E = \{z : |z| < 1\}$. For A, $-1 \leq A < 1$ and $0 \leq \alpha < p$ we say that $f \in S_p^*(A,\ \alpha)$ if and only if

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{\dfrac{zf'(z)}{f(z)} - Ap - (1-A)\alpha} \right| < 1, \quad z \in E.$$

Let T_p denote the subclass of S_p consisting of functions analytic and p-valent which can be expressed in the form $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|\, z^{p+n}$ and let

$$T_p^*(A,\ \alpha) = S_p^*(A,\ \alpha) \cap T_p.$$

We note that the class $T_p^*(-1,\ 0)$ was studied by Goel and Sohi [1]. The class $T_1^*(-1,\ \alpha)$ was studied by Silverman [2].

In this paper we obtain coefficient estimates, distortion and covering theorems for the class $T_p^*(A,\ \alpha)$.

2. Coefficient Inequalities

Theorem 1. *A function* $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|\, z^{p+n}$ *is in* $T_p^*(A, \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty} [2n + (1-A)(p-\alpha)]\, |a_{p+n}| \le (1-A)(p-\alpha). \tag{1}$$

The result is sharp.

Proof. Let $|z| = 1$, then

$$\begin{aligned} &|zf'(z) - pf(z)| - |zf'(z) - [Ap + (1-A)\alpha]\, f(z)| = \\ = \quad &\left| \sum_{n=1}^{\infty} -n\, |a_{p+n}|\, z^{p+n} \right| - \\ - \quad &\left| (1-A)(p-\alpha) z^p - \sum_{n=1}^{\infty} [n + (1-A)(p-\alpha)]\, |a_{p+n}|\, z^{p+n} \right| \le \\ \le \quad &\sum_{n=1}^{\infty} [2n + (1-A)(p-\alpha)]\, |a_{p+n}| - (1-A)(p-\alpha) \le 0. \end{aligned}$$

Hence by the principle of maximum modulus $f(z) \in T_p^*(A, \alpha)$.

Conversely, suppose that

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-p}{\dfrac{zf'(z)}{f(z)}-[Ap+(1-A)\alpha]}\right| =$$

$$= \left|\frac{\sum_{n=1}^{\infty} n\,|a_{p+n}|\,z^{p+n}}{(1-A)(p-\alpha)z^p-\sum_{n=1}^{\infty}[n+(1-A)(p-\alpha)]\,|a_{p+n}|\,z^{p+n}}\right| < 1,$$

$z \in E$.

Since $|\mathrm{Re}\ z| \le |z|$ for all z we have

$$\text{(2)}\qquad \mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty} n\,|a_{p+n}|\,z^{p+n}}{(1-A)(p-\alpha)z^p-\sum_{n=1}^{\infty}[n+(1-A)(p-\alpha)]\,|a_{p+n}|\,z^{p+n}}\right\} < 1.$$

Choose values of z on the real axis so that $\dfrac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2) and letting $z \to 1$ through real values we obtain

$$\sum_{n=1}^{\infty} n\,|a_{p+n}| \le \left\{(1-A)(p-\alpha)-\sum_{n=1}^{\infty}[n+(1-A)(p-\alpha)]\,|a_{p+n}|\right\}$$

which implies that

$$\sum_{n=1}^{\infty}[2n+(1-A)(p-\alpha)]\,|a_{p+n}| \le (1-A)(p-\alpha).$$

□

The function

$$f(z) = z^p - \frac{(1-A)(p-\alpha)}{2n+(1-A)(p-\alpha)}z^{p+n}$$

is an extremal function.

Corollary 1. *If $f(z) \in T_p^*(A, \alpha)$, then*

$$|a_{p+n}| \leq \frac{(1-A)(p-\alpha)}{2n+(1-A)(p-\alpha)}$$

with equality only for functions of the form

$$f(z) = z^p - \frac{(1-A)(p-\alpha)}{2n+(1-A)(p-\alpha)} z^{p+n}$$

3. Distortion and Covering Theorems for the Class $T_p^*(A, \alpha)$

Theorem 2. *If $f(z) \in T_p^*(A, \alpha)$, then*

$$\begin{aligned} &r^p - \frac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} r^{p+1} \leq |f(z)| \leq \\ \leq\ &r^p + \frac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} r^{p+1}, \quad |z| = r \end{aligned} \tag{3}$$

with equality for $f(z) = z^p - \dfrac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} z^{p+1}$, $z = \pm r$.

Proof. From Theorem 1 we have

$$\begin{aligned} &[2+(1-A)(p-\alpha)] \sum_{n=1}^{\infty} |a_{p+n}| \leq \\ \leq\ &\sum_{n=1}^{\infty} [2n+(1-A)(p-\alpha)]\, |a_{p+n}| \leq (1-A)(p-\alpha). \end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} |a_{p+n}| \leq \frac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)}. \tag{4}$$

Thus

$$\begin{aligned} &|f(z)| \leq |z|^p + \sum_{n=1}^{\infty} |a_{p+n}|\, |z|^{p+n} \leq r^p \left(1 + r \sum_{n=1}^{\infty} |a_{p+n}|\right) \leq \\ \leq\ &r^p + \frac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} r^{p+1}. \end{aligned}$$

Similarly

$$
\begin{aligned}
|f(z)| &\geq |z|^p - \sum_{n=1}^{\infty} |a_{p+n}|\, |z|^{p+n} \geq r^p \left(1 - r \sum_{n=1}^{\infty} |a_{p+n}|\right) \geq \\
&\geq r^p - \frac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} r^{p+1}.
\end{aligned}
$$

□

Theorem 3. *The disc $|z| < 1$ is mapped onto a domain that contains the disc $|w| < \dfrac{2}{2+(1-A)(p-\alpha)}$ by any $f(z) \in T_p^*(A, \alpha)$. The theorem is sharp with extremal function*

$$
f(z) = z^p - \frac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} z^{p+1}.
$$

Proof. The result follows upon letting $r \to 1$ in Theorem 2. □

Theorem 4. *If $f(z) \in T_p^*(A, \alpha)$, then*

$$
\begin{aligned}
& pr^{p-1} - \frac{(p+1)(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} r^p \leq \left|f'(z)\right| \leq \\
\leq \; & pr^{p-1} + \frac{(p+1)(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} r^p, \quad |z| = r.
\end{aligned}
$$

Equality holds for

$$
f(z) = z^p - \frac{(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)} z^{p+1}, \quad z = \pm r.
$$

Proof. We have

$$
\begin{aligned}
\left|f'(z)\right| &\leq pr^{p-1} + \sum_{n=1}^{\infty} (p+n)\, |a_{p+n}|\, r^{p+n-1} \leq \\
&\leq pr^{p-1} + r^p \sum_{n=1}^{\infty} (p+n)\, |a_{p+n}| = \\
&= r^{p-1} \left[p + r \sum_{n=1}^{\infty} (p+n)\, |a_{p+n}|\right].
\end{aligned} \tag{5}
$$

In view of Theorem 1

$$\sum_{n=1}^{\infty} 2\left[n+p-\frac{2p+(A-1)(p-\alpha)}{2}\right]|a_{p+n}| \leq (1-A)(p-\alpha)$$

or

$$(6)\qquad \begin{aligned} &\sum_{n=1}^{\infty} 2(n+p)\,|a_{p+n}| \leq (1-A)(p-\alpha)+ \\ + \quad &[2p+(A-1)(p-\alpha)]\sum_{n=1}^{\infty}|a_{p+n}|\,. \end{aligned}$$

Now (6) with the help of (4) implies that

$$(7)\qquad \sum_{n=1}^{\infty}(n+p)\,|a_{n+p}| \leq \frac{(p+1)(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)}.$$

A substitution of (7) into (5) yields the right-hand inequality.

On the other hand,

$$\begin{aligned} \left|f'(z)\right| &\geq r^{p-1}\left[p-r\sum_{n=1}^{\infty}(p+n)\,|a_{p+n}|\right] \geq \\ \geq \quad & pr^{p-1}-\frac{(p+1)(1-A)(p-\alpha)}{2+(1-A)(p-\alpha)}r^{p}. \end{aligned}$$

This completes the proof. □

Remarks: Putting $\alpha = 0$ in the above theorems we get the results obtained by R. M. Goel and N. S. Sohi [1]. Putting $p = 1$ and taking $A = -1$ in the above theorems, we get the results obtained by Silverman [2].

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ВЪРХУ ЕДИН КЛАС МНОГОЛИСТНИ ФУНКЦИИ С ОТРИЦАТЕЛНИ КОЕФИЦИЕНТИ

Донка Желева Пашкулева, Климент Василев Василев

Резюме. В работата са получени коефициентни оценки и свойства за класа $T_p^*(A,\alpha)$ от функции, аналитични и p-листни в единичния кръг.