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ON A FAMILY CURVES OF THE SECOND CLASS

BOYAN ZLATANOV

ABSTRACT. There are a number of tasks from school course in geometry, in which are classified (may be and for the author) very interesting new relations between the objects or a possibility to generate new geometric figures. With the help of DGS, Pappus and Desargues Theorems we discover a set of curves of the second class (conics) connected with the conditions of one school task.

1. INTRODUCTION

There are a number of tasks from school course in geometry, in which are classified (may be and for the author) very interesting new relations between the objects or a possibility to generate new geometric figures. An impressive example is a small task [7], which was developed in [11]. With the help of Dynamic Geometry Softwares (DGS), Pappus and Desargues Theorems we discover a set of curves of the second class connected with the conditions of one school task.

2. PRELIMINARY

This work represents one aspect of the hidden potential of the following task ([6], Problem 83):

Problem 1. ([4, 6]) Let $ABCD$ be a parallelogram and K be a point in its interior. Let us construct the lines p and q through the point K, such that $p \parallel AD$ and $q \parallel AB$. Let us denote: $R = p \cap AB$, $S = p \cap CD$, $M = q \cap AD$, $N = q \cap BC$, $T = AN \cap CR$, $Q = BS \cap DN$ $G = CM \cap AS$, $P = DR \cap BM$. Prove that the following triads of points (P, K, C); (T, K, D), Figure 1; (Q, K, A); (G, K, B) are collinear.

The authors of [6] offer a solution for the triad of points (T, K, D), using Thales' Theorem and properties of homothety. A simple solution with a single application of Pappus Theorem, including the infinite points of the sides parallelogram $ABCD$ is presented in [4, 5]. This approach allows not only quickly to prove the collinearity of the triads of points (D, K, T), (A, K, Q), (B, K, G) and (C, K, P), but also to prove the validity of the same statement when $ABCD$ is a trapeze or any quadrangle and the lines p and q pass through the points $U = AB \cap CD$ and $V = BC \cap AD$, respectively, assuming that parallel lines intersect at an infinite point. It is proved in [4] that the collinearity of any one of the above mentioned four triads of points is

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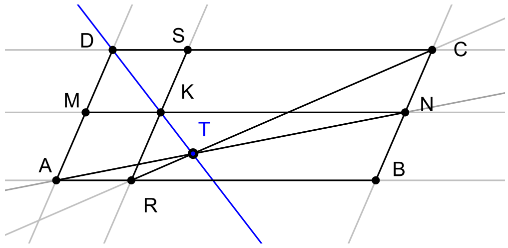

Figure 1:

equivalent to the collinearity of other three triads of points. An essential moment in the generalizations is the concept of infinite point and application of the function ”Swap finite and infinite point”. Firstly this function was introduced in the specialized DGS *Sam* [3,5]. This function optimizes the drawing work and support creativity of teachers and students [4]. The applet ”Swap finite and infinite point” was introduced and in GeoGebra [11] and further exploited in [8,9].

We have found the opportunity to generate new geometric figures relating to the conditions of Problem 1. So we illustrate an evolution of the idea implemented in this small school problem. Just to simplify the reading we will recall three fundamental theorems of projective geometry used in this work.

Theorem 1. *(The fundamental theorem of projective geometry) A projectivity is determined when three collinear points and the corresponding three collinear points are given.([2], p.34)*

Theorem 2. *(Pappuss) Let be given two lines g and g' . If $A, B, C \in g$ and $A', B', C' \in g$, then the points $P = BC' \cap CB'$, $Q = AC' \cap CA'$, $R = AB' \cap BA'$ are collinear (Figure 2).*

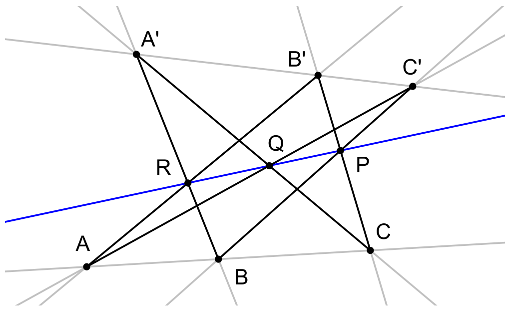

Figure 2:

The line o, which is incidental with the points P, Q, R plays an essential role for the projectivity φ, where $\varphi(A, B, C) = A', B', C'$. The line o is called axis of φ and participates in the simplest structure for finding the point $X' = \varphi(X)$, because the point $XA' \cap X'A = S$ lies on o (Figure 3).

Definition. (Steiner) If points X and X' vary on fixed lines g and g' in such a way that $X \overline{\wedge} X'$ but not $X \overline{\overline{\wedge}} X'$, then the set of lines XX' defines a curve of the second class $c(\varphi : g \longrightarrow g')$ (Figure 4).

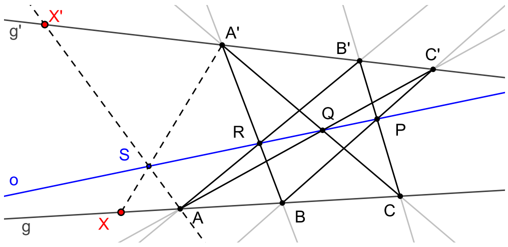

Figure 3:

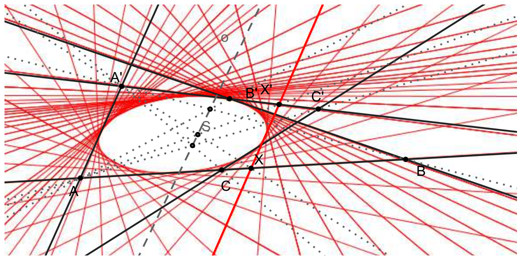

Figure 4:

The lines g and g' belong to $c(\varphi : g \longrightarrow g')$.

Definition. A triangle is called the set of three noncollinear points and their three joining lines.

Theorem 3. *(Desargues) The connecting lines of the couples of corresponding vertices of two triangles ABC and $A'B'C'$ are intersecting at a point S if and only if the intersection points of the couples of corresponding sides $P = BC \cap B'C'$, $Q = AC \cap A'C'$, $R = AB \cap A'B'$ lie on a line s (Figure 5).*

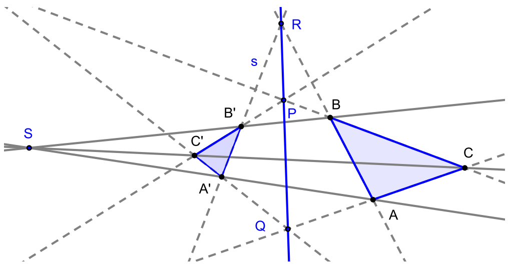

Figure 5:

Two triangles that satisfied the conditions of Theorem 2 are called perspective. The point S is called perspective center and the line s is called a perspective axis.

Definition. A complete quadrangle is called the set of four points A, B, C, D, of which no three are collinear, and the lines AB, BC, CD, AD, AC, BD. The intersections of the opposite sides $U = AB \cap CD$, $V = AD \cap BC$, $W = AC \cap BD$ are called diagonal points.

Following [2] we will accept to use the name range for the set of all points on a line and pencil for the set of all lines that lie in a plane and pass through a point.

3. Main result

Theorem 4. *Let $ABCD$ be a complete quadrangle with diagonal points $U = AB \cap CD$ and $V = BC \cap AD$ and the lines u and v pass through the points U and V, respectively. Let denote:*

$$\begin{array}{l} v \cap AB = R,\ v \cap CD = S,\ u \cap AD = M,\ u \cap BC = N; \\ DN \cap BS = Q,\ CM \cap AS = G,\ DR \cap BM = P,\ AN \cap CR = T. \end{array} \tag{3.1}$$

1) When the line v describes the pencil with a center point V, then the lines RQ, RG, SP and ST describe curves of the second class c_1, c_2, c_3 and c_4, respectively, which contain the lines BC and AD.

2) When the line u describes the pencil with a center point U, then the lines MT, MQ, NG and NP, describe curves of the second class c_5, c_6, c_7 and c_8, respectively, which contain lines AB and CD.

Remark. The assertion is true in specific cases when $ABCD$ is a trapeze or parallelogram. Using applet "Swap finite and infinite point" optimizes drawing work when we illustrate the allegations in specific cases.

Lemma 5. *Let us have the construction in Theorem 4 and let us denote*

$$\begin{array}{l} E = DN \cap AB,\ E_1 = CM \cap AB,\ W_1 = BD \cap VE,\ W_2 = AC \cap VE_1; \\ L = BM \cap CD,\ L_1 = AN \cap CD,\ W_3 = BD \cap VL,\ W_4 = AC \cap VL_1. \end{array} \tag{3.2}$$

Then the triads of points U, W_1, W_2 and U, W_3, W_4 are collinear.

Proof. (Figure 6) The triangles ACM and BDN are perspective with a perspective center U, because by the assumptions the lines AB, CD, MN pass through the point U.

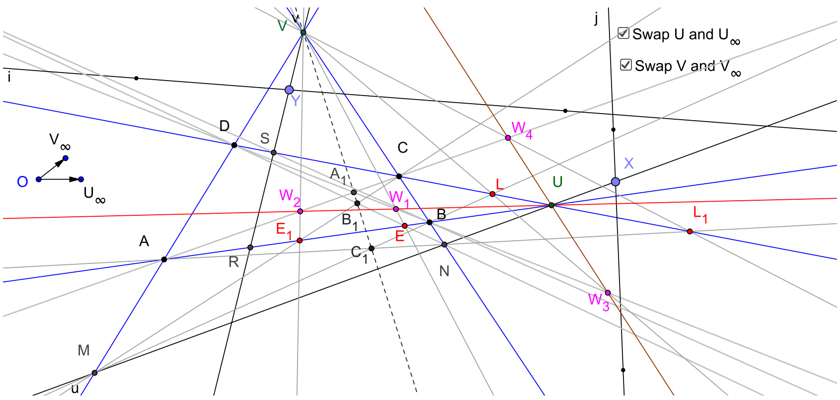

Figure 6:

Therefore

$$A_1 = AC \cap BD, \quad B_1 = CM \cap DN, \quad V = AM \cap BN \quad \text{are collinear points.} \tag{3.3}$$

Applying Theorem 2 for the triads collinear points A, D, M and B, C, N we obtain that

$$A_1 = AC \cap BD, \quad B_1 = CM \cap DN, \quad C_1 = AN \cap BM \quad \text{are collinear points.} \tag{3.4}$$

From (3.3) and (3.4) it follows

$$A_1,\ B_1,\ C_1,\ V \text{ are collinear points.} \tag{3.5}$$

Let's consider the triangles EW_1D and E_1W_2C. According to (3.2) and (3.3) the points $EW_1 \cap E_1W_2 = VE \cap VE_1 = V$, $W_1D \cap W_2C = BD \cap AC = A_1$ and $ED \cap E_1C = DN \cap CM = B_1$ are collinear. Then applying Theorem 3 we obtain that the lines EE_1, W_1W_2, DC are concurrent, which is equivalent to the condition the line W_1W_2 pass through the point $EE_1 \cap DC = AB \cap DC = U$.

Similar observation can be done and for the triangles LW_3B and L_1W_4A. According to (3.5) the points $LW_3 \cap L_1W_4 = VL \cap VL_1 = V$, $W_3B \cap W_4A = BD \cap AC = A_1$ and $LB \cap L_1A = BM \cap AN = C_1$ are collinear. Then applying Theorem 3 we obtain that the lines LL_1, W_3W_4, BA are concurrent, which is equivalent to the condition the line W_3W_4 pass through the point $LL_1 \cap BA = DC \cap AB = U$.

In the specific cases i.e. when U is U_∞ or we have at the same time U_∞ and V_∞ we get the Figures 7, 8. □

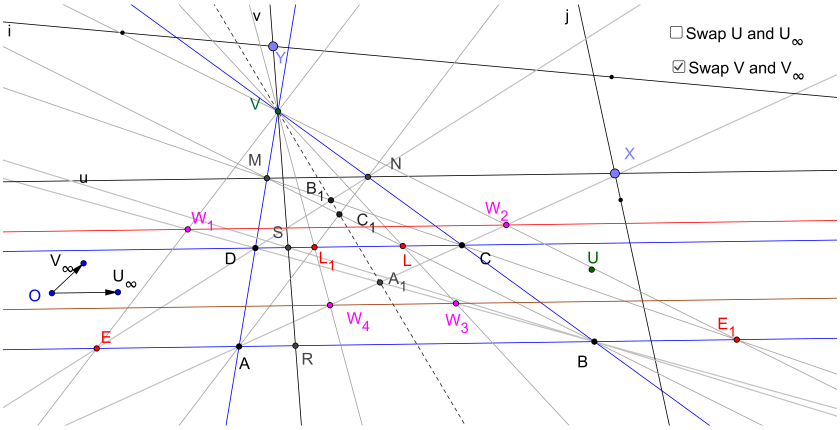

Figure 7:

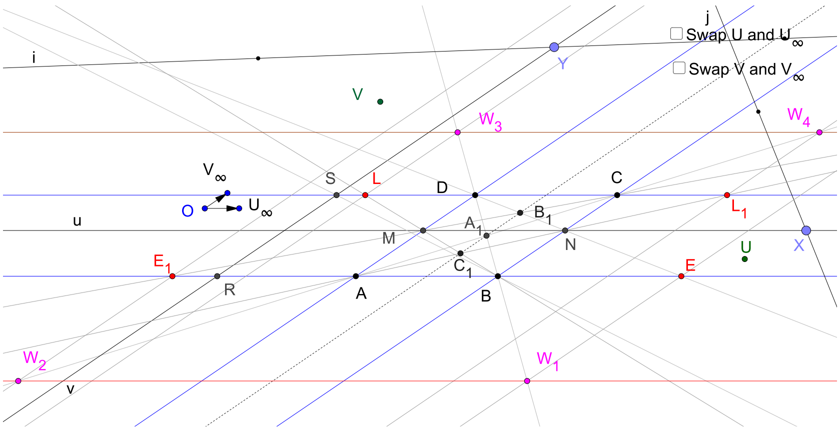

Figure 8:

Lemma 6. *Let us have the construction in Theorem 4 and let us denote*

$$I = CR \cap AD,\ I_1 = BS \cap AD,\ W_5 = AC \cap UI,\ W_6 = BD \cap UI_1;$$
$$F = AS \cap BC,\ F_1 = DR \cap BC,\ W_7 = AC \cap UF,\ W_8 = BD \cap UF_1. \tag{3.6}$$

Then the triads points V, W_5, W_6 and V, W_7, W_8 are collinear.

Proof. (Figure 9) The triangles ACR and DBS are perspective with a perspective center V, because by condition the lines AD, CB, RS pass through the point V.

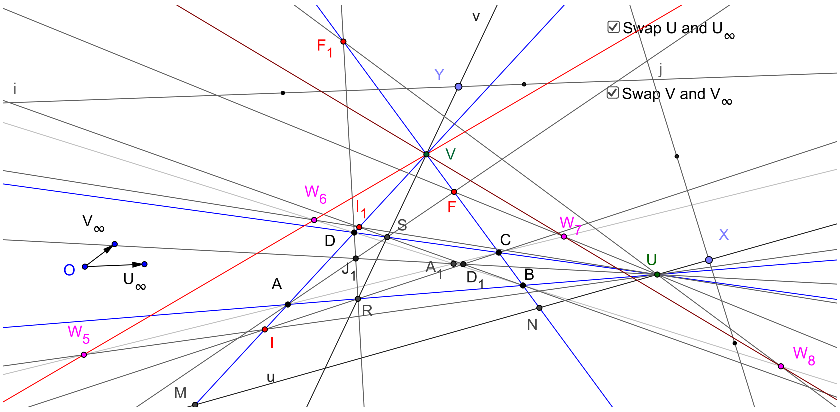

Figure 9:

Therefore

$$A_1 = AC \cap DB, \quad D_1 = CR \cap BS, \quad U = AR \cap DS \quad \text{are collinear points.} \tag{3.7}$$

Applying Theorem 2 for the triads collinear points A, B, R and D, C, S we obtain that

$$A_1 = AC \cap BD, \quad D_1 = CR \cap BS, \quad J_1 = RD \cap AS \quad \text{are collinear points.} \tag{3.8}$$

From (3.7) and (3.8) it follows

$$A_1,\ D_1,\ J_1,\ U \text{ are collinear points.} \tag{3.9}$$

Let's consider the triangles IW_5C and I_1W_6B. According to (3.6) and (3.7) the points $IW_5 \cap I_1W_6 = UI \cap UI_1 = U$, $W_5C \cap W_6B = AC \cap BD = A_1$ and $IC \cap I_1B = CR \cap BS = D_1$ are collinear. Then applying Theorem 3 we obtain that the lines II_1, W_5W_6, BC are concurrent, which is equivalent to the condition the line W_5W_6 pass through the point $II_1 \cap BC = AD \cap BC = V$.

Similar observation can be done and for the triangles FW_7A and F_1W_8D. According to (3.9) the points $FW_7 \cap F_1W_8 = UF \cap UF_1 = U$, $W_7A \cap W_8D = AC \cap BD = A_1$ and $FA \cap F_1D = AS \cap DR = J_1$ are collinear. Then applying Theorem 3 we obtain that the lines FF_1, W_7W_8, AD are concurrent, which is equivalent to the condition the line W_7W_8 pass through the point $FF_1 \cap AD = BC \cap AD = V$.

In the specific cases i.e. when V is V_∞ or when simultaneously V is V_∞ and U is U_∞ we get the Figures 10, 11. □

Now we shall proceed to the proof of Theorem 4.

Proof of Theorem 4*:* 1) Let us note that when the line v describes pencil with center V , then the points R and S describe the lines AB and CD, respectively.

Remark. The movement of the line v is carried out by movement of the point $Y \in v$ on an arbitrary line i.

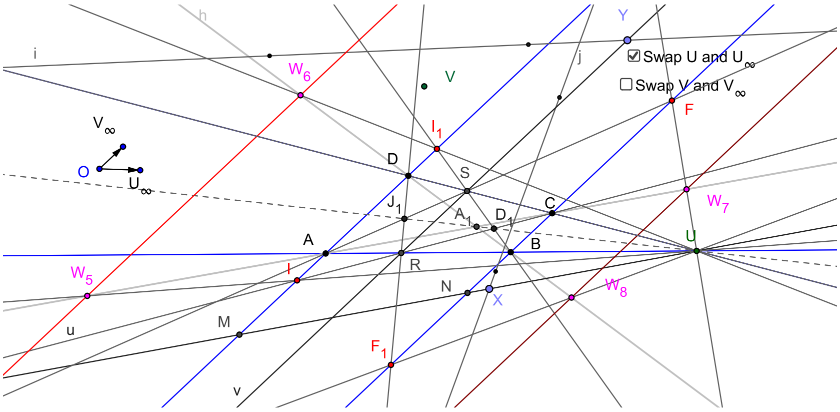

Figure 10:

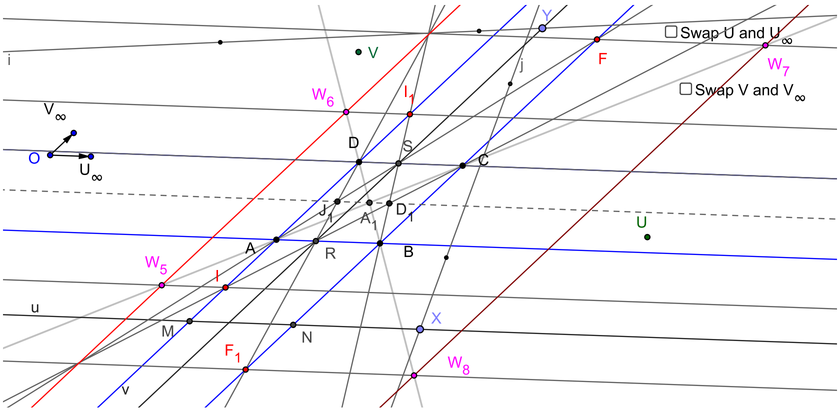

Figure 11:

1.1) Let φ_1 is a projectivity of the range AB into the range AD, where $\varphi_1(B, E, U) = V, D, A$.

According to [1] and (3.2) the axis of φ_1 is the line $o_1 = (EA \cap DU)(BD \cap VE) = (AB \cap DU)(BD \cap VE) = UW_1$. (Figure 12)

Let $\varphi_1(R) = R'$. Hence

$$RS \cap R'B = RV \cap R'B = Z_1 \in o_1, \; RD \cap R'E = H_1 \in o_1. \tag{3.10}$$

From (3.2), (3.10) and $SD \cap BE = CD \cap AB = U$ it follows that the triangles RSD and $R'BE$ are perspective with a perspective axis o_1. Then according to Theorem 3 the lines RR', SB, DE are concurrent, i.e. $SB \cap DE = SB \cap DN = Q$ and $Q \in RR'$. Therefore when the point R describes the range AB, then the line $RQ = RR'$ describes curve of the second class $c_1(\varphi_1; AB \longrightarrow AD)$. It contains the lines AB, AD, BC, DN. According to ([1], 9.11, p.81) U and $T_1 = o_1 \cap AD$ are the points of contact of AB and AD, respectively with the conic c_1.

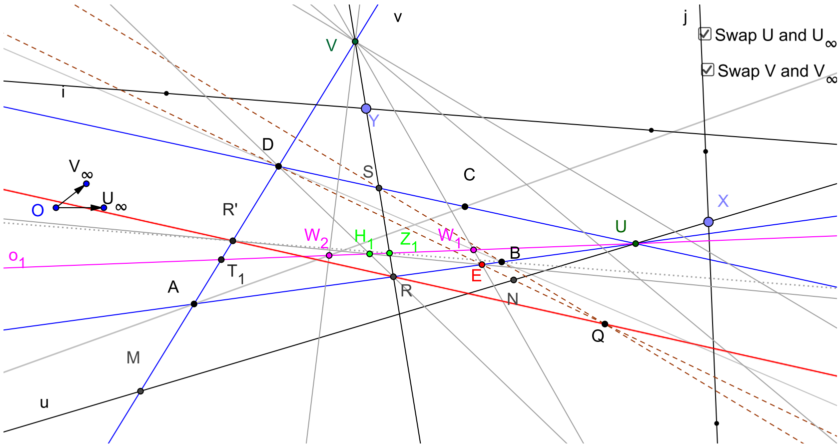

Figure 12:

We will illustrate curves of the second class in the case when V and U are finite points (Figure 13) and in the case when V and U are simultaneously infinity points i.e. V_∞, U_∞ (Figure 14).

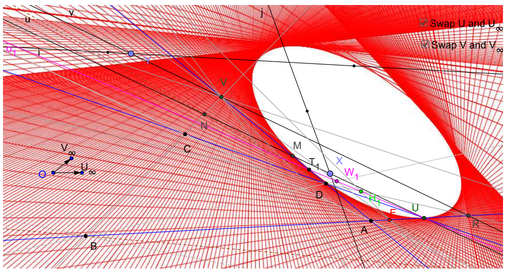

Figure 13:

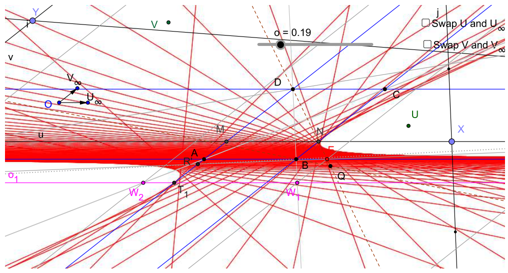

Figure 14:

1.2) Let φ_2 is a projectivity of the range AB into the range BC, where $\varphi_2(A, E_1, U) = V, C, B$.

According to [1] and (3.2) the axis of φ_2 is the line $o_2 = (E_1B \cap CU)(AC \cap VE_1) = (AB \cap CU)(AC \cap VE_1) = UW_2$ (Figure 15).

Using Lemma 5 we establish that o_2 coincides with o_1, i.e. $o_1 \equiv o_2$.

Let $\varphi_2(R) = R''$. Hence

$$RS \cap R''A = RV \cap R''A = Z_2 = Z_1 \in o_1,\ RC \cap R''E_1 = H_2 \in o_1. \tag{3.11}$$

From (3.2), (3.11) and $SC \cap AE_1 = CD \cap AB = U$ it follows that the triangles RSC and $R''AE_1$ are perspective with a perspective axis o_1. Then according to Theorem 3 the lines RR'', SA, CE_1 are concurrent, i.e. $SA \cap CE_1 = SA \cap CM = G$ and $G \in RR''$. Therefore when the point R describes the range AB, then the line $RG = RR''$ describes curve of the second class $c_2(\varphi_2; AB \longrightarrow BC)$. It contains the lines AB, AD, BC, CM. According to ([1], 9.11, p.81) U and $T_2 = o_1 \cap BC$ are the points of contact of AB and BC, respectively with the conic c_2.

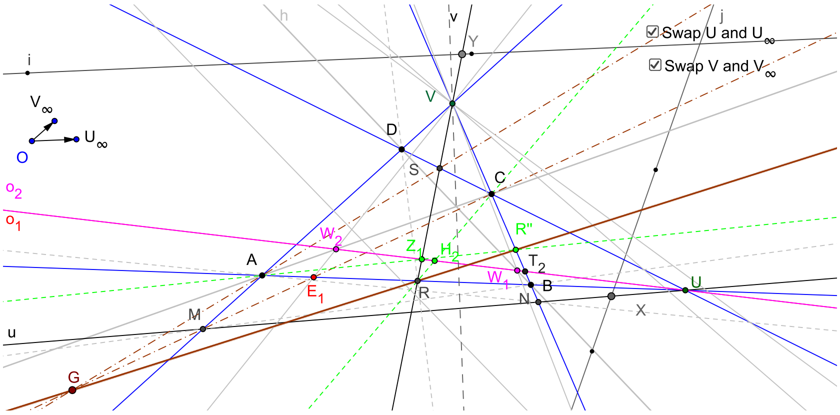

Figure 15:

1.3) Let φ_3 is a projectivity of the range CD into the range BC, where $\varphi_3(D, L, U) = V, B, C$.
According to [1] and (3.2) the axis of φ_3 is the line $o_3 = (LC \cap BU)(BD \cap VL) = (CD \cap BU)(BD \cap VL) = UW_3$. (Figure 16)
Let $\varphi_3(S) = S'$. Hence

$$SR \cap S'D = SV \cap S'D = Z_3 \in o_3, \; SB \cap S'L = H_3 \in o_3. \tag{3.12}$$

From (3.2), (3.12) and $RB \cap DL = AB \cap CD = U$ it follows that the triangles SRB and $S'DL$ are perspective with a perspective axis o_3. Then according to Theorem 3 the lines SS', RD, BL are concurrent, i.e. $RD \cap BL = RD \cap BM = P$ and $P \in SS'$. Therefore when the point S describes the range CD, then the line $SP = SS'$ describes curve of the second class $c_3(\varphi_3; CD \longrightarrow BC)$. It contains the lines CD, AD, BC, BM. According to ([1], 9.11, p.81) U and $T_3 = o_3 \cap BC$ are the points of contact of CD and BC, respectively with the conic c_3.

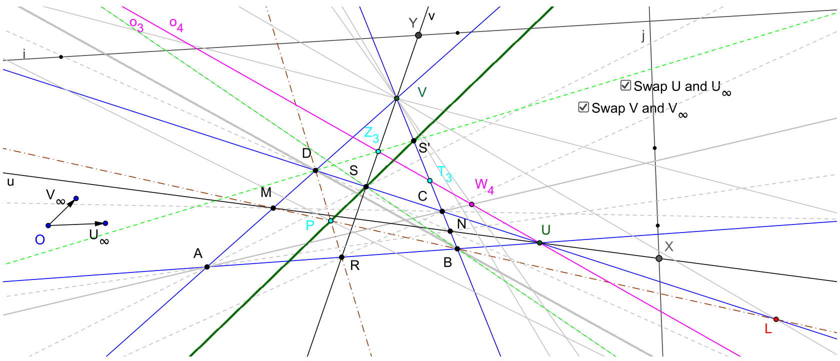

Figure 16:

We will illustrate curves of the second class in the case when V is V_∞ and U is a finite point (Figure 17) and in the case when V and U are simultaneously infinity points i.e. V_∞, U_∞ (Figure 18).

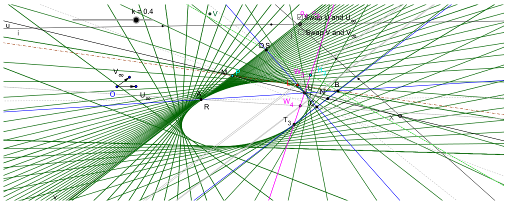

Figure 17:

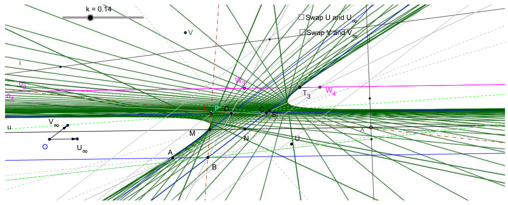

Figure 18:

1.4) Let φ_4 is a projectivity of the range CD into the range AD, where $\varphi_4(C, L_1, U) = V, A, D$.

According to [1] and (3.2) the axis of φ_4 is the line $o_4 = (L_1D \cap AU)(AC \cap VL_1) = (CD \cap AU)(AC \cap VL_1) = UW_4$. Using Lemma 5 we establish that o_4 coincides with o_3, i.e. $o_4 \equiv o_3$.

Let $\varphi_4(S) = S''$. Hence

$$SR \cap S''C = SV \cap S''C = Z_4 = Z_3 \in o_3,\ SA \cap S''L_1 = H_4 \in o_3. \tag{3.13}$$

From (3.2), (3.13) and $RA \cap CL_1 = AB \cap CD = U$ it follows that the triangles SRA and $S''CL_1$ are perspective with a perspective axis o_3. Then according to Theorem 3 the lines SS'', RC, AL_1 are concurrent, i.e. $RC \cap AL_1 = RC \cap AN = T$ and $T \in SS''$. Therefore when the point S describes the range CD, then the line $ST = SS''$ describes curve of the second class $c_4(\varphi_4; CD \longrightarrow AD)$. It contains the lines CD, AD, BC, AN. According to ([1], 9.11, p.81) U and $T_4 = o_3 \cap AD$ are the points of contact of CD and AD, respectively with the conic c_4.

2) Let us note that when the line u describes pencil with center U, then the points M and N describe the lines AD and BC, respectively.

Remark. The movement of the line u is carried out by movement of the point $X \in u$ on an arbitrary line j.

2.1) Let φ_5 is a projectivity of the range AD into the range CD, where $\varphi_5(A, I, V) = U, C, D$.

According to [1] and (3.6) the axis of φ_5 is the line $o_5 = (DI \cap CV)(AC \cap UI) = (AD \cap CV)(AC \cap UI) = VW_5$ (Figure 19) .

Let $\varphi_5(M) = M'$. Hence

$$MN \cap M'A = MU \cap M'A = Z_5 \in o_5,\ MC \cap M'I = H_5 \in o_5. \tag{3.14}$$

From (3.6), (3.14) and $CN \cap IA = BC \cap AD = V$ it follows that the triangles MNC and $M'AI$ are perspective with a perspective axis o_5. Then according to Theorem 3 the lines MM', NA, CI are concurrent, i.e. $NA \cap CI = NA \cap CR = T$ and $T \in MM'$. Therefore when the point M describes the range AD, then the line $MT = MM'$ describes curve of the second class $c_5(\varphi_5; AD \longrightarrow CD)$.

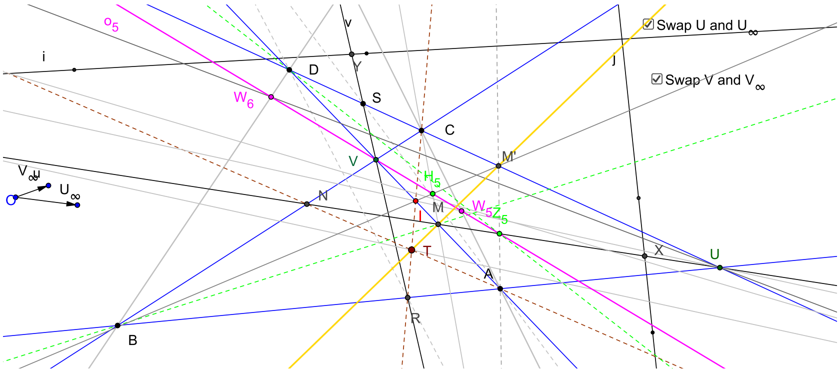

Figure 19:

It contains the lines AD, AB, CD, CR. According to ([1], 9.11, p.81) V and $T_5 = o_5 \cap CD$ are the points of contact of AD and CD, respectively with the conic c_5.

We illustrate the curve of second class, described in this case by Figure 20.

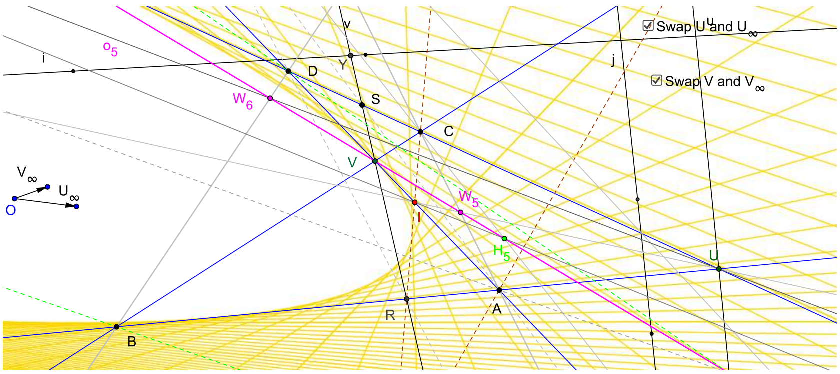

Figure 20:

2.2) Let φ_6 is a projectivity of the range AD into the range AB, where $\varphi_6(D, I_1, V) = U, B, A$ (Figure 21). (Figure 21). According to [1] and (3.6) the axis of φ_6 is the line $o_6 = (I_1A \cap BV)(BD \cap UI_1) = (AD \cap BV)(BD \cap UI_1) = VW_6$. Using Lemma 6 we establish that o_6 coincides with o_5, i.e. $o_6 \equiv o_5$.

Let $\varphi_6(M) = M''$. Hence

$$MN \cap M''D = MU \cap M''D = Z_6 = Z_5 \in o_5,\ MB \cap M''I_1 = H_6 \in o_6. \tag{3.15}$$

From (3.6), (3.15) and $BN \cap DI_1 = BC \cap AD = V$ it follows that the triangles MBN and $M''I_1D$ are perspective with a perspective axis o_5. Then according to Theorem 3 the lines MM'', ND, BI_1 are concurrent, i.e. $ND \cap BI_1 = ND \cap BS = Q$ and $Q \in MM''$. Therefore when the point M describes the range AD, then the line $MQ = MM''$ describes curve of the second class $c_6(\varphi_6; AD \longrightarrow AB)$. It contains the lines AD, AB, CD, BS. According to ([1], 9.11, p.81) V and $T_6 = o_5 \cap BC$ are the points of contact of AD and BC, respectively with the conic c_6.

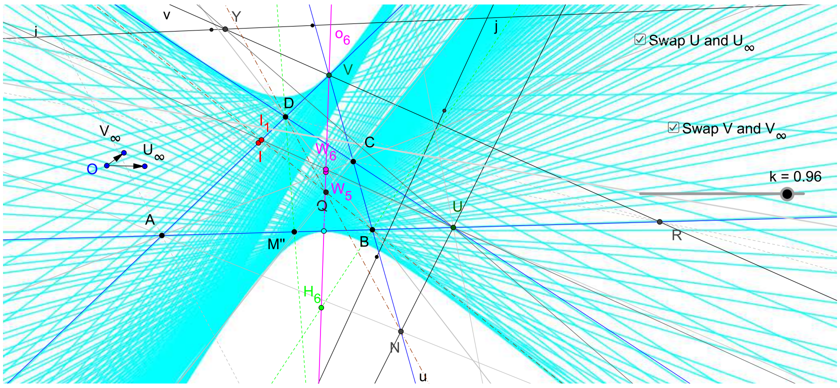

Figure 21:

2.3) Let φ_7 is a projectivity of the range BC into the range AB, where $\varphi_7(C,F,V) = U,A,B$.

According to [1] and (3.6) the axis of φ_7 is the line $o_7 = (BF \cap AV)(CA \cap UF) = (BC \cap AD)(AC \cap UF) = VW_7$ (Figure 22).

Let $\varphi_7(N) = N'$. Hence

$$MN \cap CN' = NU \cap CN' = Z_7 \in o_7,\ NA \cap N'F = H_7 \in o_7. \tag{3.16}$$

From (3.6), (3.16) and $AM \cap FC = AD \cap BC = V$ it follows that the triangles NAM and $N'FC$ are perspective with a perspective axis o_7. Then according to Theorem 3 the lines NN', AF, MC are concurrent, i.e. $AF \cap MC = AS \cap MC = G$ and $G \in NN'$. Therefore when the point N describes the range BC, then the line $NG = NN'$ describes curve of the second class $c_7(\varphi_7; BC \longrightarrow AB)$. It contains the lines BC, AB, CD, AS. According to ([1], 9.11, p.81) V and $T_7 = o_7 \cap AB$ are the points of contact of BC and AB, respectively with the conic c_7.

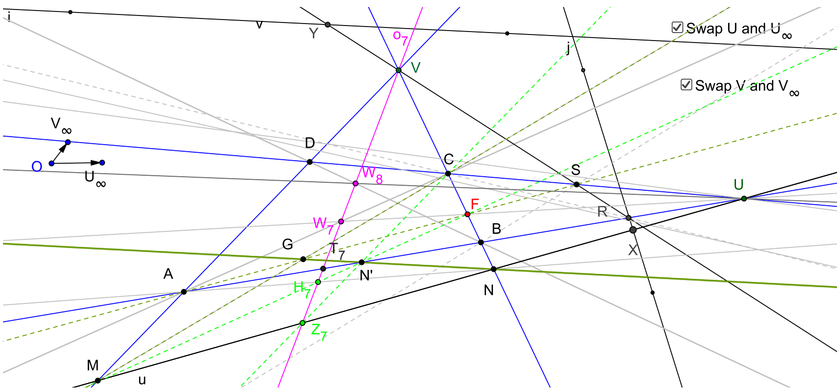

Figure 22:

2.4) Let φ_8 is a projectivity of the range BC into the range CD, where $\varphi_8(B, F_1, V) = U, D, C$. (Figure 23)

According to [1] and (3.6) the axis of φ_8 is the line $o_8 = (F_1C \cap DV)(BD \cap UF_1) = (BC \cap AD)(BD \cap UF_1) = VW_8$. Using Lemma 6 we establish that o_8 coincides with o_7, i.e. $o_8 \equiv o_7$.

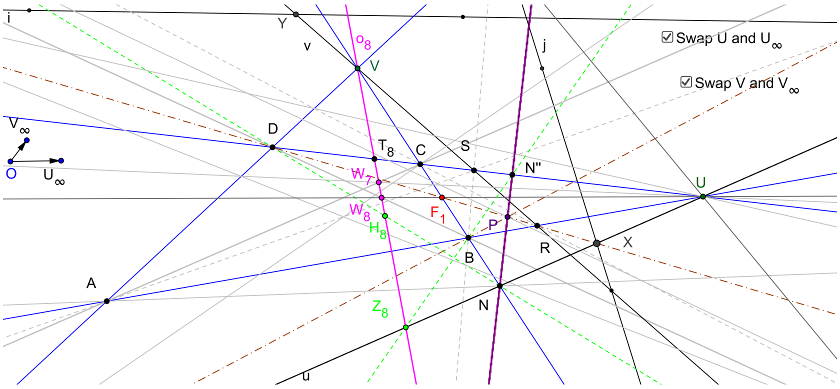

Figure 23:

Let $\varphi_8(N) = N''$. Hence

$$NM \cap N''B = NU \cap N''B = Z_8 = Z_7 \in o_7,\ ND \cap N''F_1 = H_8 \in o_7. \qquad (3.17)$$

From (3.6), (3.17) and $DM \cap BF_1 = AD \cap BC = V$ it follows that the triangles NDM and $N''F_1B$ are perspective with a perspective axis o_7. Then according to Theorem 3 the lines NN'', MB, DF_1 are concurrent, i.e. $MB \cap DF_1 = MB \cap DR = P$ and $P \in NN''$. Therefore when the point N describes the range BC, then the line $NP = NN''$ describes curve of the second class $c_4(\varphi_8; BC \longrightarrow CD)$. It contains the lines BC, AB, CD, DR. According to ([1], 9.11, p.81) V and $T_8 = o_7 \cap CD$ are the points of contact of BC and CD, respectively with the conic c_8.

At the end let us present the curve of the second class generated by the line NP in the case when the point V is finite but the point U is infinite (Figure 24).

Corollary 7. *The line AB touches the conics c_1 and c_2 at U. The line CD touches the conics c_3 and c_4 at U. The line AD touches the conics c_5 and c_6 at V. The line BC touches the conics c_7 and c_8 at V.*

Taking advantage of opportunities in DGS Cinderella we offer the reader and curves of the second degree, related to the above described curves of the second class (Figures 25, 26).

Corollary 8. *Since the lines AD and BC touch the conics c_1, c_2, c_3 and c_4, then the polar points P_1, P_2, P_3, P_4 of the line v regarding c_1, c_2, c_3 and c_4, respectively lie on the line g, where the harmonic set $H(AD\ BC,\ v\ g)$ is true.*

Corollary 9. *Since the lines AB and CD touch the conics c_5, c_6, c_7 and c_8, then the polar points P_5, P_6, P_7, P_8 of the line u regarding c_5, c_6, c_7 and c_8, respectively lie on the line h, where the harmonic set $H(AB\ CD,\ u\ g')$ is true.*

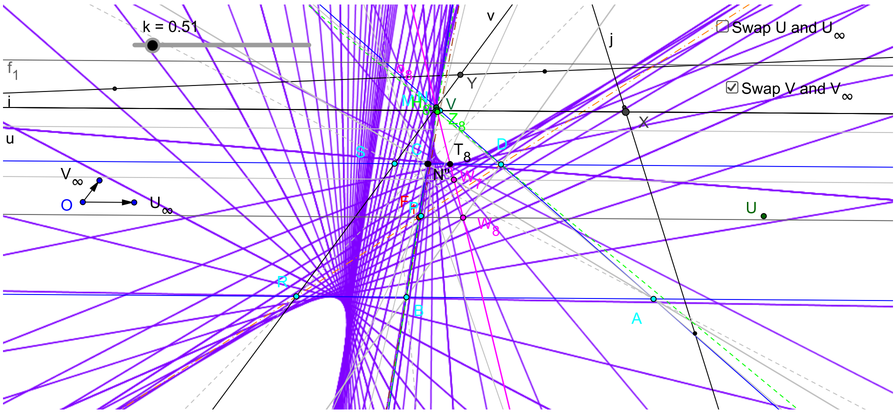

Figure 24:

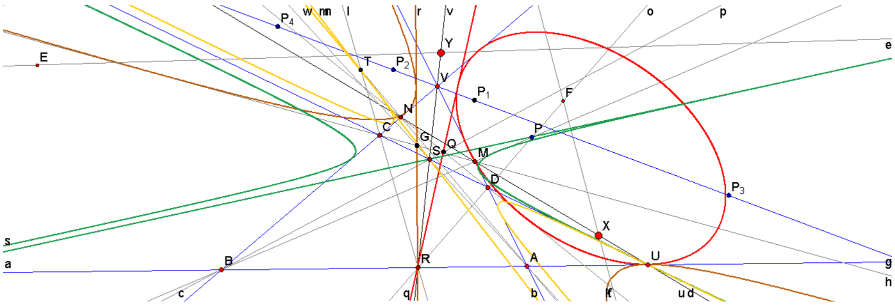

Figure 25:

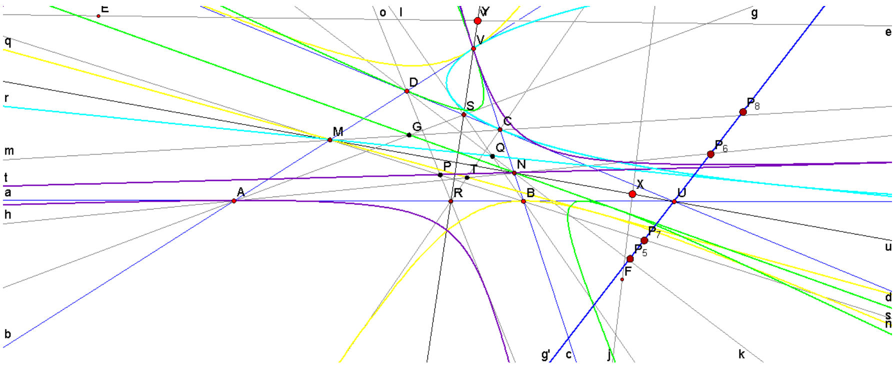

Figure 26:

The dynamic sketches can be downloaded from http://fmi-plovdiv.org/GetResource?id=2530.

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BOYAN ZLATANOV: UNIVERSITY OF PLOVDIV "PAISII HILENDARSKI", FACULTY OF MATHEMATICS AND INFORMATICS, 24 "TSAR ASSEN" STR., PLOVDIV 4000, BULGARIA
E-mail address: `bzlatanov@gmail.com`