Юбилейна международна научна конференция „Компютърни технологии и приложения“, 15-17 септември 2021 г., Пампорово, България
Anniversary International Scientific Conference “Computer Technologies and Applications”, 15-17 September 2021, Pamporovo, Bulgaria

A NOTE ON THE ODD LOMAX-G INVERSE WEIBULL CUMULATIVE DISTRIBUTION FUNCTION

Maria Vasileva[1,*], Olga Rahneva[2], Georgi Spasov[3], Anna Malinova[4]

[1, 3 ,4] Faculty of Mathematics and Informatics, University of Plovdiv “Paisii Hilendarski”, 24 Tzar Asen Str., 4000 Plovdiv, Bulgaria
[2] Faculty of Economy and Social Sciences, University of Plovdiv “Paisii Hilendarski”, 24 Tzar Asen Str., 4000 Plovdiv, Bulgaria
Emails: mariavasileva@uni-plovdiv.bg, georgi.spasov@uni-plovdiv.bg, malinova@uni-plovdiv.bg, orahneva@uni-plovdiv.bg
** Corresponding author: mariavasileva@uni-plovdiv.bg*

Abstract. In this note we study properties of a new modern generalization of the inverse Weibull and odd Lomax distribution named odd Lomax-G inverse Weibull (OLIW) distribution. More precisely, we prove estimates for the “saturation” - d about Hausdorff metric. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: *Odd Lomax-G Inverse Weibull, Hausdorff distance, Heaviside step function, Upper and lower bounds.*

Introduction

Lifetime distributions are very important in modeling phenomena and pandemics also for other fields, such as industry, engineering, reliability, and medical research. Many researchers shows that the inverted distributions have a great importance due to their applicability in sciences areas such as biological, life test problems, medical and others.

A new superior distribution named odd Lomax-G inverse Weibull (OLIW) distribution with four parameters was introduced from Almetwally [1]. This new distribution is a combination of inverse Weibull distribution and the odd Lomax-G family.

Definition 1. *The odd Lomax-G inverse Weibull (OLIW) distribution is associated with the cdf given as*

$$F(t;\alpha,\beta,\lambda,\theta) = 1 - \beta^{\alpha}\left(\beta + \frac{e^{-\left(\frac{\theta}{t}\right)^{-\lambda}}}{1-e^{-\left(\frac{\theta}{t}\right)^{-\lambda}}}\right)^{-\alpha}, \quad t > 0, \alpha,\beta,\lambda,\theta > 0. \qquad (1)$$

Definition 2. *The shifted Heaviside step function is defined by*

$$h_{t_0}(t) = \begin{cases} 0, & \text{if } t < t_0 \\ [0,1], & \text{if } t = t_0 \\ 1, & \text{if } t > t_0 \end{cases}.$$

Definition 3. *[2][3] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \qquad (2)$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e.g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

In the next lemma, we present one technical result.

Lemma 1. *The following inequality holds*

$$F_0(t;\alpha,\beta,\lambda,\theta) \le F(t;\alpha,\beta,\lambda,\theta) \le F_{00}(t;\alpha,\beta,\lambda,\theta),$$

where

$$F_0(t;\alpha,\beta,\lambda,\theta) = 1 - \beta^{\alpha}\left(\beta + \frac{1-\left(\frac{\theta}{t}\right)^{\lambda}}{\left(\frac{\theta}{t}\right)^{\lambda}}\right)^{-\alpha} \quad and \quad F_{00}(t;\alpha,\beta,\lambda,\theta) = 1 - \beta^{\alpha}\left(\beta + \frac{1}{\left(\frac{\theta}{t}\right)^{\lambda}}\right)^{-\alpha}.$$

Proof. The proof follows immediately from the following well know inequalities

$$1 - x \le e^{-x} \le \frac{1}{x+1}$$

that holds true for every $x > -1$. The desired result is a consequence of the following inequalities

$$\frac{1-x}{x} \le \frac{e^{-x}}{1-e^{-x}} \le \frac{1}{x}.$$

This completes the proof.

□

Figure 1. Confidential bounds is a graphical representation of Lemma 1. Note that functions F_0 and F_{00} can be used as "confidential bounds" for CDF function of OLIW distribution.

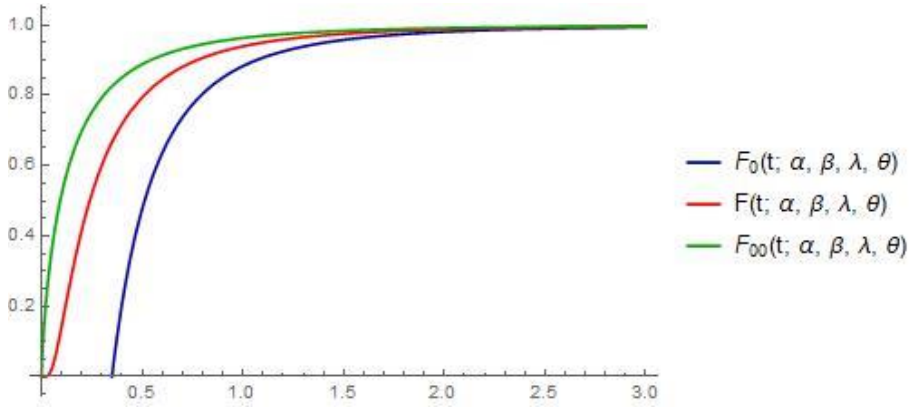

Figure 1. Confidential bounds

The OLIW distribution has a lot of versatility and can be used to model distorted data, so it is commonly used in fields like biomedical studies, biology, reliability, physical engineering, and survival analysis. Also, can be used with success in approximating parameterize data in the field of "virus-theory", insurance mathematics and population dynamics. For some modeling and approximation problems, see [4-18] and references therein.

The main purpose of this study is to present some properties of cumulative function of OLIW distribution and prove estimate for the "saturation" - d about Hausdorff metric. The applicability of the model is proved in simulation study to "COVID-19 data" of France.

Main result

In this section, we investigate the "saturation" - d in the Hausdorff sense to the horizontal asymptote. For the function $F(t; \alpha, \beta, \lambda, \theta)$ defined by (1) we have

$$F(t_0; \alpha, \beta, \lambda, \theta) = \frac{1}{2} \quad \text{with} \quad t_0 = \theta \ \log\left(\frac{1}{2^{1/\alpha}\beta - \beta} + 1\right)^{-\frac{1}{\lambda}} .$$

Then the Hausdorff distance d between $F(t; \alpha, \beta, \lambda, \theta)$ defined by (1) and the Heaviside function $h_{t_0}(t)$ satisfies the following nonlinear equation

$$F(t\_0 + d; \alpha, \beta, \lambda, \theta) = 1 - d. \qquad (3)$$

In the next theorem, we prove upper and lower estimates for the Hausdorff approximation d.

Theorem 1. *Let*

$$A = 1 + \frac{1}{\theta}\left(2^{-\frac{\alpha+1}{\alpha}}\left(2^{\frac{1}{\alpha}}-1\right)\alpha\lambda\left(\left(2^{\frac{1}{\alpha}}-1\right)\beta+1\right)\log\left(\frac{2^{\frac{1}{\alpha}}+\beta^{-1}-1}{2^{1/\alpha}-1}\right)^{\frac{1}{\lambda}+1}\right)$$

and $2.1\,A > e^{1.05}$. *Then for the Hausdorff distance* d *between shifted Heaviside function* $h_{t_0}(t)$ *and the CDF function of OLIW distribution* $F(t;\alpha,\beta,\lambda,\theta)$ *defined by (1) the following inequalities hold true:*

$$d_l = \frac{1}{2.1\,A} < d < \frac{\log(2.1A)}{2.1\,A} = d_r.$$

Proof. Let us consider the function

$$H(d) = F(t_0 + d;\ \alpha,\beta,\lambda,\theta) - 1 + d.$$

It is easy to show that $H'(d) > 0$, so the function $H(d)$ is increasing. We examine the following approximation of $H(d)$ as we use the function

$$G(d) = -\frac{1}{2} + A\,d\,.$$

Indeed from Taylor expansion, we get $G(d) - H(d) = O(d^2)$. This means that $G(d)$ approximates $H(d)$ with $d \to 0$ as $O(d^2)$ (see Figure 2). More over $G'(d) > 0$ and function $G(d)$ is also increasing. Let the following condition $2.1A > e^{1.05}$ holds. Then it is easy to show that

$$G(d_l) = -\frac{1}{2} + A\frac{1}{2.1A} < 0 \text{ and } G(d_r) = -\frac{1}{2} + A\frac{\log(2.1A)}{2.1\,A} > -\frac{1}{2} + \frac{1.05}{2.1} = 0.$$

This completes the proof.

□

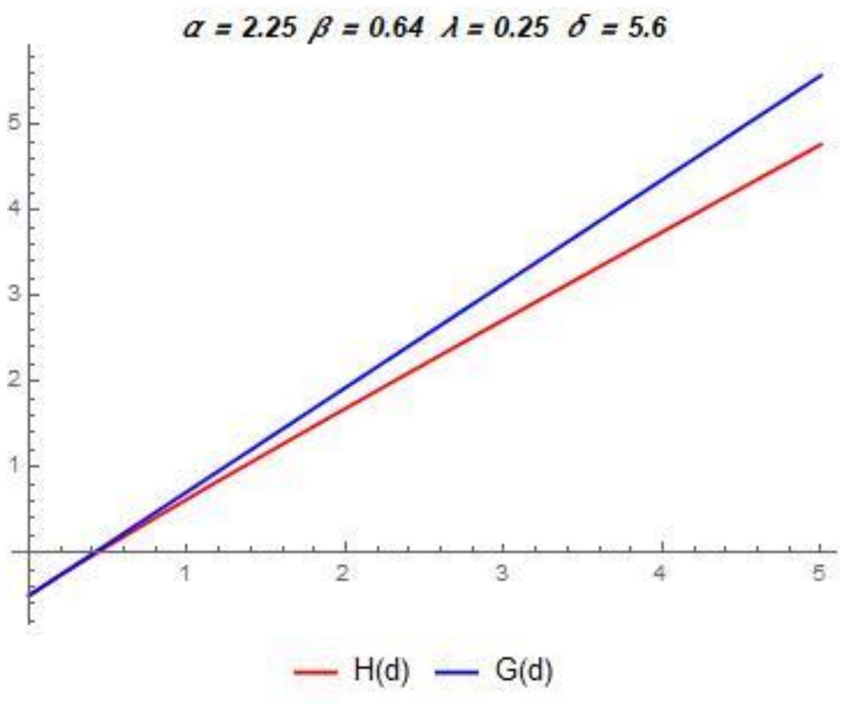

Figure 2. Functions $H(d)$ and $G(d)$

In Table 1 we present some computational examples for different values of parameters α, β, λ and θ. We use Theorem 1 for computation of values of upper and lower estimates d_l and d_r. Several graphical representations are presented in Figure 3 and one can see that the "saturation" is faster.

Table 1. Bounds for Hausdorff distance d computed by Theorem 1

α	β	λ	δ	d_l	d computed by (3)	d_r
3.15	0.31	4.63	0.23	0.0206933	0.041398	0.0802475
8.01	0.31	0.25	1.91	0.0169438	0.068632	0.0690945
7.25	0.18	2.71	1.13	0.0739097	0.100674	0.192528
1.52	0.81	0.95	0.07	0.0573896	0.135953	0.164013
3.71	0.64	1.51	0.35	0.0730497	0.117894	0.191143
2.89	1.83	2.53	0.47	0.123729	0.168403	0.258552

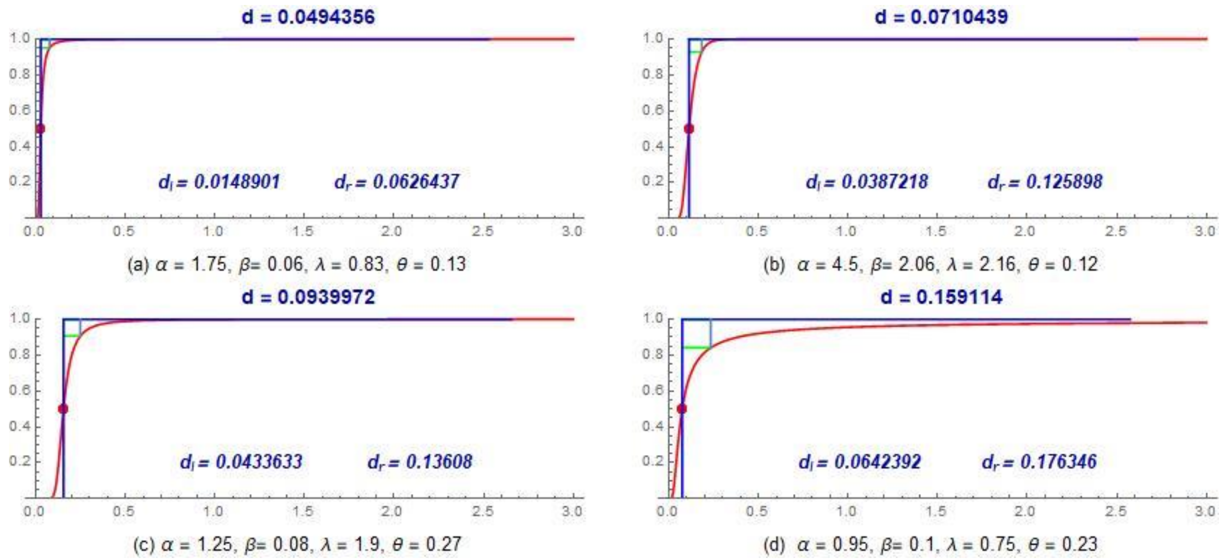

Figure 3. Approximation of CDF function of OLIW Distribution

Some applications

The main motivation of Almetwally [1] to proposes the OLIW distribution, a modern generalization of the inverse Weibull and odd Lomax distribution, is for modeling of mortality rate for the COVID-19 pandemic of France. In Figure 4 we present the results that we obtain with the programming environment *CAS Mathematica* for the analysis of the considered OLIW cumulative distribution function. We present an approximation the Heaviside step function and the CDF function $F(t; \alpha, \beta, \lambda, \theta)$ with parameters $\alpha = 14.95$, $\beta = 19.3$, $\lambda = 1.53$, and $\theta = 0.09$. Namely, we obtain the corresponding values of Hausdorff distance $d = 0.100992$, its upper and lower estimates $d_l = 0.0620456$ and $d_r = 0.17248$, respectively. Also, we get the graphical visualization of the results. Note that in a choice for a model for approximation of cumulative data in a various modeling problems specialists can used upper and lower estimations from Theorem 1 as "confidence bounds".

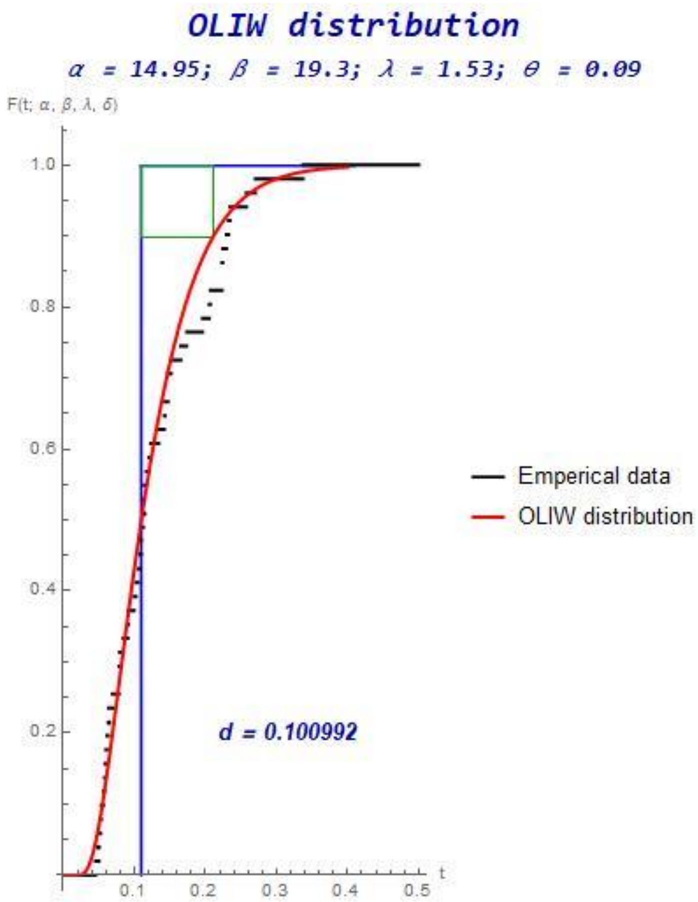

Figure 4. The model (1) for COVID-19 data (normalized) belong to France

Acknowledgments

This work has been accomplished with the financial support by the Project FP21-FMI-002 "Intelligent innovative ICT in research in mathematics, informatics and pedagogy of education", (2021–2022).

References

[1] E. Almetwally, Application of COVID-19 Pandemic by Using Odd Lomax-G Inverse Weibull Distribution, *Math. Sci. Lett.*, Vol. 10, No. 2, 2021, 47–57, ISSN: 2090-9616, DOI: https://doi.org/10.18576/msl/100203.

[2] F. Hausdorff, *Set Theory (2 ed.)*, New York, Chelsea Publ., 1962.

[3] B. Sendov, *Hausdorff Approximations*, Boston, Kluwer, 1990.

[4] N. Kyurkchiev, A. Iliev, S. Markov, *Some Techniques for Recurrence Generating of Activation Functions: Some Modeling and Approximation Aspects*, LAP LAMBERT Academic Publishing, 2017, ISBN: 978-3-330-33143-3.

[5] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, *Nontrivial Models in Debugging Theory (Part 2)*, LAP LAMBERT Academic Publishing, 2018, ISBN: 978-613-9-87794-2.

[6] N. Kyurkchiev, A. Iliev, A. Rahnev, *Some Families of Sigmoid Functions: Applications to Growth Theory*, LAP LAMBERT Academic Publishing, 2019, ISBN: 978-613-9-45608-6.

[7] A. Iliev, N. Kyurkchiev, A. Rahnev, T. Terzieva, *Some models in the theory of computer viruses propagation*, LAP LAMBERT Academic Publishing, 2019, ISBN: 978-620-0-00826-8.

[8] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, *Some New Logistic Differential Models: Properties and Applications*, LAP LAMBERT Academic Publishing, 2019, ISBN: 978-620-0-43442-5.

[9] A. Malinova, O. Rahneva, A. Golev, V. Kyurkchiev, Investigations on the Odd-Burr-III-Weibull Cumulative Sigmoid. Some Applications, *Neural, Parallel Sci. Comput*., Vol. 27, No. 1, 2019, 35–44, ISSN: 1061-5369, DOI: https://doi.org/10.12732/npsc.v27i1.4.

[10] N. Kyurkchiev, *Selected Topics in Mathematical Modeling: Some New Trends (Dedicated to Academician Blagovest Sendov (1932-2020))*, LAP LAMBERT Academic Publishing, 2020, ISBN: 978-620-2-51403-3.

[11] O. Rahneva, A. Malinova, E. Angelova, V. Kyurkchiev, Hausdorff approximation of the Heaviside step function by Unit Gompertz and Complementary Unit Gompertz cumulative functions, *Communications in Applied Analysis*, Vol. 24, No. 1, 2020, 75–87, ISSN: 1083-2564, DOI: https://doi.org/10.12732/caa.v24i1.6.

[12] N. Kyurkchiev, A look at the inverse Weibull “Adaptive function”: Properties and applications, *Int. J. Differ. Equ. Appl.*, Vol. 19, No. 1, 2020, 153–165, ISSN: 311-2872, DOI: https://doi.org/10.12732/ijdea.v19i1.11.

[13] N. Kyurkchiev, Some Intrinsic Properties of Tadmor-Tanner functions. Related Problems and Possible Applications, *Mathematics*, Vol. 8, No. 11, 2020, 1963, ISSN: 2227-7390, DOI: https://doi.org/10.3390/math8111963.

[14] M. Vasileva, A. Iliev, A. Rahnev, N. Kyurkchiev, On the approximation of the Haar scaling function by sigmoidal scaling functions, *Int. J. Differ. Equ. Appl.*, Vol. 20, No. 1, 2021, 1–13, ISSN: 311-2872, DOI: https://doi.org/ 10.12732/ijdea.v20i1.932.

[15] M. Vasileva, O. Rahneva, A. Malinova, V. Arnaudova, The odd Weibull-Topp-Leone-G power series family of distributions, *Int. J. Differ. Equ. Appl.*, Vol. 20, No. 1, 2021, 43–58, ISSN: 311-2872, DOI: https://doi.org/ 10.12732/ijdea.v20i1.4.

[16] N. Kyurkchiev, A note on the Burr - Hatke - exponential model. Some applications, *Compt. rend. Acad. bulg. Sci.*, Vol. 74, No. 4, 2021, 488–495, ISSN: 1310–1331, DOI: https://doi.org/10.7546/CRABS.2021.04.02.

[17] N. Kyurkchiev, *Some intrinsic properties of adaptive functions to piecewise smooth data*, Plovdiv University Press, Plovdiv, 2021, ISBN: 978-619-202-670-7.

[18] N. Kyurkchiev, V. Kyurkchiev, A. Iliev, A. Rahnev, *Another look at SIR/SIRD/SEIR/GSEIR Models: New trends, methodological aspects*, Plovdiv University Press, Plovdiv, 2021, ISBN: 978-619-202-685-1.