GEOMETRIC PROPERTIES OF THE U-LINES IN TORIC COORDINATES [1]

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Abstract. *Toric coordinates have been introduced in* [1]. *We explore some geometrical properties of its* u-*lines, such as finding the image of* u-*line by the Gauss map, finding the conjugated family, etc.*

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1. Movement of a point onto an u-line

We remind the formulae of the toric coordinates:

$$(1)\qquad \left|\begin{array}{ll} x = (a + r\cos(u+v))\cos u & \\ y = (a + r\cos(u+v))\sin u, & \rho = a + r\cos(u+v) \\ z = r\sin(u+v), & u, v \in [0, 2\pi];\ 0 < r < a \end{array}\right. .$$

Let the vector equation of an u-line $C_r(u)$ be $\vec{r} = \vec{r}(u)$, r and v are constants (when we write in bold $\mathbf{r}$ we mean the vector $\vec{r}$). Then the vector $\vec{r}\,''(u)$ is:

$$(2)\qquad \left|\begin{array}{l} x = -a\cos u - 2r\cos(2u+v) \\ y = -a\sin u - 2r\sin(2u+v) \\ z = -r\sin(u+v) \end{array}\right. .$$

Theorem 1.1. The vector $\vec{r} + \vec{r}\,''$ describes a circumference onto the plane xOy.

Proof. We must summarize (1) and (2), and to simplify the result. Then

$$(3)\qquad \vec{r} + \vec{r}\,'' : \left|\begin{array}{l} x = \frac{r}{2}\cos v - \frac{3}{2} r\cos(2u+v) \\ y = -\frac{r}{2}\sin v - \frac{3}{2} r\sin(2u+v) \\ z = 0 \end{array}\right. .$$

The last equation (3) represents a circumference $K(v)$ with a center $D\left(\frac{r}{2}\cos v, -\frac{r}{2}\sin v\right)$ and a radius $\dfrac{3}{2}\, r$. The point M' runs two times through K when M runs once through $C_r(u)$. The initial point is $M_0'(-r\cos v, -2r\sin v)$, $(u = 0)$. We have a family of circumferences, when v varies from 0 to 2π. Its centers D make a phase torsion to an angle $-v$.

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Treating $\mathbf{r}''$ as an acceleration, we can find its tangent component. We have the scalar product

$$\mathbf{r}' . \mathbf{r}'' = -r\sin(u+v)\,.\rho\,. \tag{4}$$

Hence the tangent component of $\mathbf{r}''$ has the opposite sign to $\sin(u+v)$. If we take $v = 0$ and $u \in [0, \pi]$, then the acceleration is minus and the speed decreases. When $u \in [\pi, 2\pi]$ then the speed of M increases.

2. Image of an u-line by the Gauss map

The Gauss map Γ stands to each point M of the torus T - the point of the sphere, whose radius-vector is equal to the unit normal vector $n(M)$ of T.

Let (1) are the coordinates of M, $r = 1$ and $O_M(a\cos u, a\sin u, 0)$ be the center of the cross section of T. Then $n(M) = \overrightarrow{OM}$ has the following coordinates:

$$\Big(\cos(u+v)\cos u,\ \cos(u+u)\sin u,\ \sin(u+v)\Big). \tag{5}$$

The image of the u-line $C_1(v)$ consists of all points (5) on the sphere, when $v = \mathit{const}$. Let we denote this image by $C'(v)$. The projection of the curve $C'(v)$ onto xOy is a circumference

$$K : \left| \begin{array}{l} x = \cos(u+v)\cos u \\ y = \cos(u+v)\sin u \end{array} \right. .$$

The circumferences K passes through O and has a radius $\frac{1}{2}$. The center of K is $O'\Big(\frac{\cos v}{2}, -\frac{\sin v}{2}\Big)$. Particularly, when $v = 0$, $O'\Big(\frac{1}{2}, 0\Big)$. Then O' turns round the origin to the angle $-v$: Thus we state the following theorem:

Theorem 2.1. The image $\Gamma(C_1(v))$ of every u-line is the Viviani's curve.

This image describes the upper part of the curve, when $0 \leq u \leq \pi$ and then goes to its lower part.

3. Differential-geometric properties of the u-lines

First, we represent the coefficients of the first and second quadratic form in toric coordinates (1). They are:

$$E = r^2 + \rho^2, \quad G = F = 1$$

$$L = -\Big(1 + \cos(u+v)\Big)^2, \qquad M = N = -1.$$

where $\rho = a + r\cos(u+v)$ is the distance of M to the axis Oz. We set $r \equiv 1$.

A. Normal curvature is an arbitrary point $M \in C(u)$. We must compute the angle φ between the tangent vector

$$\mathbf{t} = \mathbf{r}'\Big(-a\sin u - r\sin(2u+v),\ a\cos u + r\cos(2u+v),\ r\cos(u+v)\Big)$$

and the unit vector $\mathbf{e_p}(-\sin u, \cos u)$ to the parallel at M. The scalar product

$$\mathbf{t}.\mathbf{e_p} = a + r\cos(u+v) = \rho \quad \text{and} \quad \cos\varphi = \frac{\rho}{\sqrt{r^2+\rho^2}}\,.$$

Then, by the Euler's formula, we get

$$k_n = k_1\cos\varphi + k_2\sin\varphi.$$

The curvatures k_1 and k_2 satisfy the equation

$$\rho\, k^2 - \Big[\rho + \cos(u+v)\Big]\, k + \cos(u+v) = 0.$$

So we obtain

$$k_1 = \frac{\cos(u+v)}{\rho}\,, \quad k_2 = 1\,, \quad k_n = \frac{r+\cos(u+v)}{\sqrt{r^2+\rho^2}}\,. \tag{6}$$

B. The fundamental differential operator $\mathscr{D}$ in a point $M \in C_1(u)$.

As we see, the Gauss map Γ maps an arbitrary point M, given by (1), to the point (5) of the sphere. The differential $\mathscr{D}\,\Gamma$ is an isomorphism of the corresponding tangent spaces. We have

$$\mathscr{D}\,\Gamma(t) = \mathscr{D}\,\Gamma(\partial_u\, r) = \partial_u\vec{n} \quad \text{and} \quad \mathscr{D}\,\Gamma(\partial_u\vec{r}) = \partial_v\vec{n} \tag{7}$$

about the basic vectors. Therefore, when $r = 1$

$$\left|\begin{array}{l} \partial_u\vec{n} = \Big(-\sin(2u+v),\ \cos(2u+v),\ \cos(u+v)\Big) \\ \partial_v\vec{r} = \partial_v\vec{n} = \Big(-\sin(u+v)\cos u,\ -\sin(u+v)\sin u,\ \cos(u+v)\Big) \end{array}\right. . \tag{8}$$

We can find the matrix $\mathscr{D}$ by a well-known formula

$$\mathscr{D} = \begin{pmatrix} k_1 & 0 \\ 1-k_1 & k_2 \end{pmatrix}, \quad \text{(see [1])},$$

where k_1 and k_2 are given by (6).

It could be directly verified that the linear transformation, given by this matrix, satisfies formulae (8). This means that $\mathscr{D}$ is really the fundamental operator $\mathscr{A}$.

C. Finding the conjugated family of the family of the u-lines.

For this aim we should find the differential equation of the family of u-lines. The coordinates of its tangent vector $\mathbf{t}$ are $(1,0)$.

If we denote by du and dv the coordinates of the conjugated direction, the following equation is verified

$$Lt_1\, du + M(t_1 dv + t_2 du) + Nt_2\, dv = 0.$$

In our case we obtain

$$-[1 + cos(u+v)]^{\,2}\, du - dv = 0. \tag{9}$$

In order to solve (9) we set $u + v = U$ and we obtain the equation

$$(1 + \cos U)^{2}\, dU = [(1 + \cos U)^2 - 1]\, dv,$$

which is an equation with separate variables. For the family of its integral curves we obtain that

$$\int \frac{(1 + \cos U)^{2}}{2\cos U + \cos^2 U}\, dU = v + C. \tag{10}$$

This is the family of the conjugated lines of our u-lines.

References

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