GEOMETRY OVER TWO ARBITRARY ARITHMETIC PROGRESSIONS

Grozio Stanilov, Liudmila Filipova

Abstract: *We consider quadrate matrices with elements of the first row members of an arithmetic progression and of the second row members of other arithmetic progression. We prove the set of these matrices is a group. Then we give a parameterization of this group and investigate about some invariants of the corresponding geometry. We find an invariant of any two points and an invariant of any sixth points. All calculations are made by Maple.*

Keywords: arithmetic progression, Klein geometry, invariant, cross product, group

2010 Mathematics Subject Classification: 51N25, 51A45, 20B40

Introduction

We introduce the square real regular matrices of order 2 of the following type:

(1) $$M_2 = \begin{pmatrix} a+d & a+2d \\ b+e & b+2e \end{pmatrix}$$

We have proved the following

Theorem1. The set of these matrices is a group.

To prove this theorem we establish the following two lemmas:

Lemma1. The inverse matrix of any regular matrix of type (1) is such matrix.

Lemma2. The product of any two matrices of type (1) is such matrix.

This group we shall denote by G. In the Klein sense it induces a geometry denoted by Γ.

Remark 1. The elements of the first row $a+d, a+2d$ are arbitrary two consecutive elements of an arithmetic progression $\Pi_1(a,d)$. Same holds for the elements of the second row-they are elements of an arithmetic progression $\Pi_2(b,e)$. All assertions are true if we take $b=kd, e=d$, i.e. the elements of the second row are another consecutive two elements of the progression $\Pi_1(a,d)$. In this case the title of the paper must be: Geometry over an arbitrary arithmetic progression.

Fundamental group of this geometry

Let us take an arbitrary such matrix

(2) $$p_2 = \begin{pmatrix} x+z & x+2z \\ y+u & y+2u \end{pmatrix}$$

If we multiply both matrices (1) and (2) we get the matrix

(3) $$P_2 = \begin{pmatrix} X+Z & X+2Z \\ Y+U & Y+2U \end{pmatrix},$$

which elements are:

(4) $$\begin{aligned} X &= ax+dx+ay+2dy, \\ Y &= bx+ex+by+2ey, \\ Z &= az+dz+au+2du, \\ U &= bz+ez+bu+2eu. \end{aligned}$$

These functions make a fundamental group of the geometry Γ.

So we consider the transformation: (*x*,*y*,*z*,*u*) ->(*X*,*Y*,*Z*,*U*). We want to investigate some facts of the geometry Γ.

Remark 2. The transformations (4) are homogeneous linear. So the group G can be considered as a subgroup of the projective group in the 3- dimensional

projective space. These transformations can be considered as linear transformations induced from linear maps in a linear space. So some time we can speak about points, some time about vectors.

Scalar product

Let us take two vectors take $v_1(x_1, y_1, z_1, u_1), v_2(x_2, y_2, z_2, u_2)$. We define their scalar product

(5) $$v_1 v_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 + u_1 u_2.$$

For the images of these vectors $V_1(X_1, Y_1, Z_1, U_1), V_2(X_2, Y_2, Z_2, U_2)$ under the transformations (4) we have

(6) $$V_1 V_2 = X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + U_1 U_2.$$

For the invariance of the scalar product we consider some special vectors and find the conditions:

$$
(7)\quad
\begin{aligned}
&a^2+b^2+d^2+e^2+2(ad+be)=1,\\
&a^2+b^2+4(d^2+e^2)+4(ad+be)=1,\\
&d^2+e^2+ad+be=-1,\\
&3(a^2+b^2)+10(d^2+e^2)+11(ad+be)=2.
\end{aligned}
$$

These equalities are necessary conditions for the invariance of the scalar product (5). We solve this system using Maple:

```
solve({a^2+d^2+e^2+b^2+2*a*d+2*b*e=1,
a^2+4*d^2+4*e^2+b^2+4*a*d+4*b*e=1,
d^2+e^2+a*d+b*e=-1,
3*a^2+10*d^2+10*e^2+3*b^2+11*a*d+11*b*e=2},{a,b,d,e});
```

All solutions are:

$$\{d=\mathrm{RootOf}(\_Z^2+e^2-2),\ b=\mathrm{RootOf}(5\,e^2-1+6\,\_Z\,e+2\,\_Z^2),\ e=e,$$
$$a=\frac{(\mathrm{RootOf}(5\,e^2-1+6\,\_Z\,e+2\,\_Z^2)\,e+3)\,\mathrm{RootOf}(\_Z^2+e^2-2)}{e^2-2}\},$$
$$\{a=1,\ b=-2,\ e=1,\ d=-1\},\ \{a=-1,\ b=-2,\ d=1,\ e=1\},$$
$$\{a=1,\ b=2,\ e=-1,\ d=-1\},\ \{a=-1,\ b=2,\ d=1,\ e=-1\},$$
$$\{a=\mathrm{RootOf}(2\,\_Z^2-1),\ b=-\frac{3}{2}\mathrm{RootOf}(\_Z^2-2),\ e=\mathrm{RootOf}(\_Z^2-2),\ d=0\}$$

In the first case we have:

I. $d1:=\sqrt{2-e^2}\,,\ b1:=-\frac{3\,e}{2}+\frac{\sqrt{2-e^2}}{2}\,,\ a1:=-\frac{-3\,e^2+e\sqrt{2-e^2}+6}{2\sqrt{2-e^2}}$

or

II. $d2:=-\sqrt{2-e^2}\,,\ b2:=-\frac{3\,e}{2}+\frac{\sqrt{2-e^2}}{2}\,,\ a2:=\frac{-3\,e^2+e\sqrt{2-e^2}+6}{2\sqrt{2-e^2}}$

In the fourth case we have

III. $d3:=\sqrt{2-e^2}\,,\ b3:=-\frac{3\,e}{2}-\frac{\sqrt{2-e^2}}{2}\,,\ a3:=\frac{3\,e^2+e\sqrt{2-e^2}-6}{2\sqrt{2-e^2}}$

or

IV. $d4 := -\sqrt{2-e^2}$, $b4 := -\frac{3\,e}{2} - \frac{\sqrt{2-e^2}}{2}$, $a4 := -\frac{3\,e^2 + e\sqrt{2-e^2} - 6}{2\sqrt{2-e^2}}$

We calculate directly in all these cases the scalar product is invariant. We formulate this result as

Theorem 2. Exactly in the cases I, II, III, IV the scalar product (5) is invariant.

We establish also the following result

Lemma 3. In the cases I, II, III, IV M_2 is an orthogonal matrix.

Considering the representations I, II, III, IV we see they define some curves. In the Euclidean space they represent ellipses.

Cross product of three vectors

If are given 4 vectors:

$v_1(x_1, y_1, z_1, u_1), v_2(x_2, y_2, z_2, u_2), v_3(x_3, y_3, z_3, u_3), v_4(x_4, y_4, z_4, u_4)$ we define

$$m := \begin{bmatrix} x1 & y1 & z1 & u1 \\ x2 & y2 & z2 & u2 \\ x3 & y3 & z3 & u3 \\ x4 & y4 & z4 & u4 \end{bmatrix}$$

and

n1:=(-1)^5*Minor(m,4,1);n2:=(-1)^6*Minor(m,4,2);n3:=(-1)^7*Minor(m,4,3);n4:=(-1)^8*Minor(m,4,4);

$$n1 := -y1\ z2\ u3 + y1\ u2\ z3 - y2\ z3\ u1 + y2\ z1\ u3 - y3\ z1\ u2 + y3\ z2\ u1$$
$$n2 := x1\ z2\ u3 - x1\ u2\ z3 + x2\ z3\ u1 - x2\ z1\ u3 + x3\ z1\ u2 - x3\ z2\ u1$$
$$n3 := -x1\ y2\ u3 + x1\ u2\ y3 - x2\ y3\ u1 + x2\ y1\ u3 - x3\ y1\ u2 + x3\ y2\ u1$$
$$n4 := x1\ y2\ z3 - x1\ z2\ y3 + x2\ y3\ z1 - x2\ y1\ z3 + x3\ y1\ z2 - x3\ y2\ z1$$

The vector $n(n_1, n_2, n_3, n_4)$ we call **Cross product of the first 3 vectors**.

Notation: $n(n_1, n_2, n_3, n_4) = Cross\,\mathrm{Pr}\,oduct(v_1, v_2, v_3)$.

For the images

$V_1(X_1, Y_1, Z_1, U_1), V_2(X_2, Y_2, Z_2, U_2), V_3(X_3, Y_3, Z_3, U_3), V_4(X_4, Y_4, Z_4, U_4)$

of the above vectors we have correspondently

$$M := \begin{bmatrix} X1 & Y1 & Z1 & U1 \\ X2 & Y2 & Z2 & U2 \\ X3 & Y3 & Z3 & U3 \\ X4 & Y4 & Z4 & U4 \end{bmatrix}$$

N1:=(-1)^5*Minor(M,4,1);N2:=(-1)^6*Minor(M,4,2);N3:=(-1)^7*Minor(M,4,3);N4:=(-1)^8*Minor(M,4,4);

$$N1 := -Y1\ Z2\ U3 + Y1\ U2\ Z3 - Y2\ Z3\ U1 + Y2\ Z1\ U3 - Y3\ Z1\ U2 + Y3\ Z2\ U1$$
$$N2 := X1\ Z2\ U3 - X1\ U2\ Z3 + X2\ Z3\ U1 - X2\ Z1\ U3 + X3\ Z1\ U2 - X3\ Z2\ U1$$
$$N3 := -X1\ Y2\ U3 + X1\ U2\ Y3 - X2\ Y3\ U1 + X2\ Y1\ U3 - X3\ Y1\ U2 + X3\ Y2\ U1$$
$$N4 := X1\ Y2\ Z3 - X1\ Z2\ Y3 + X2\ Y3\ Z1 - X2\ Y1\ Z3 + X3\ Y1\ Z2 - X3\ Y2\ Z1$$

$N(N_1, N_2, N_3, N_4) = Cross\Pr oduct(V_1, V_2, V_3)$.

Let us define two matrices: $mn := \begin{bmatrix} n1 + n3 & n1 + 2\ n3 \\ n2 + n4 & n2 + 2\ n4 \end{bmatrix}$.

Its determinant is $n1\ n4 - n3\ n2$ and $mN := \begin{bmatrix} N1 + N3 & N1 + 2\ N3 \\ N2 + N4 & N2 + 2\ N4 \end{bmatrix}$.

Its determinant is $N1\ N4 - N3\ N2$.

We calculate by Maple:

simplify(Determinant(mN)-Determinant(M2)^3*Determinant(mn)); 0

Thus we have proved the following
Theorem 3.

(8) $$\text{Determinant}(mN) = \text{Determinant}(M2)^3\ \text{Determinant}(mn)$$

Some invariants

From the matrix relation

(9) $P_2 = M_2 p_2$

we get

(10) $Det(P_2) = Det(M_2)Det(p_2)$

or

$$XU - YZ = (ae - bd)(xu - yz).$$

We apply this relation for the points $p_1(x_1, y_1, z_1, u_1)$, $p_2(x_2, y_2, z_2, u_2)$:

$X_1U_1 - Y_1Z_1 = (ae - bd)(x_1u_1 - y_1z_1)$, $X_2U_2 - Y_2Z_2 = (ae - bd)(x_2u_2 - y_2z_2)$.

We get the relation $\frac{X_1U_1 - Y_1Z_1}{X_2U_2 - Y_2Z_2} = \frac{x_1u_1 - y_1z_1}{x_2u_2 - y_2z_2}$, which shows that the expression

(11) $I(p_1(x_1, y_1, z_1, u_1), p_2(x_2, y_2, z_2, u_2)) = \frac{x_1u_1 - y_1z_1}{x_2u_2 - y_2z_2}$

is an invariant of any points.

The equation (9) has the form

(12) $N_1N_4 - N_2N_3 = (ae - bd)^3(n_1n_4 - n_2n_3)$

Let us consider another four vectors $v_5(x_5, y_5, z_5, u_5), v_6(x_6, y_6, z_6, u_6), v_7(x_7, y_7, z_7, u_7), v_8(x_8, y_8, z_8, u_8)$ and let

$$m(m_{1,} m_2, m_3, m_4) = Cros\operatorname{Pr}oduct(v_5, v_6, v_7).$$

In the same way as (9) we have

$$M_1M_4 - M_2M_3 = (ae - bd)^3(m_1m_4 - m_2m_3),$$

so that

$$\frac{N_1N_4 - N_2N_3}{M_1M_4 - M_2M_3} = \frac{n_1n_4 - n_2n_3}{m_1m_4 - m_2m_3}$$

which shows that the expression

(13) $I_2(v_1, v_2, v_3, v_5, v_6, v_7) = \frac{n_1n_4 - n_2n_3}{m_1m_4 - m_2m_3}$

is an invariant of any 6 vectors. Of course some of the second triple can coincide with some of the first triple vectors.

Thus we can formulate the following

Theorem 4. The expressions (11) and (13) are invariants in our geometry Γ.

References

[1] Maple 11, Mathematics, Modeling, Simulation, Maplesoft, 2007.

Grozio Stanilov Ivanov,
Bul. J. Boucher 5, Sofia 11,
stanilov@fmi.uni-sofia.bg

Liudmila Stefanova Filipova,
TU-Sofia, filial Plovdiv,
Tzanko Dustabanov 25,
liudmila_filipova@abv.bg