COMPARING ENTIRE PROTOTYPES AT SEMIGROUP HOMOMORPHISMS AND SPECIFICALLY AT REGULAR LANGUAGES SUBSTITUTIONS

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Abstract. *The following statements are proven: A correspondence of a semigroup in another one is a homomorphism if and only if when the entire prototype of the product of images contains (always) the product of their entire prototypes. The Kleene closure of the maximal rewriting of a regular language at a regular language substitution contains in the maximal rewriting of the Kleene closure of the initial regular language at the same substitution. Let the image of the maximal rewriting of a regular language at a regular language substitution covers the entire given regular language. Then the image of any word from the maximal rewriting of the Kleene closure of the initial regular language covers by the image of a set of some words from the Kleene closure of the maximal rewriting of this given regular language everything at the same given regular language substitution. The purposefulness of the first statement is substantiated philosophically and epistemologically connected with the spirit of previous mathematical results of the author. A corollary of its is indicated about the membership problem at a regular substitution.*

Keywords: Hegel's reflexion, Semigroup homomorphism and entire prototypes, Semigroup of regular languages, Regular languages substitution, Maximal rewriting of a regular language at a regular substitution, Membership problem.

2010 Mathematics Subject Classification: 00A30, 03D03, 20M05, 20M15, 20M35, 68Q45, 68Q70, 68R15

1. The necessary preliminary concepts

are recalled from [1] with some not essential but appropriate changes for reading section 3. We adduce especially the definition of regular languages as sets of words but not as automata one for their recognition. Any finite sequence of symbols from an alphabet Σ is *a word* over it. Σ^* denotes both *the set of all words* over Σ (including *the empty word* ϵ) and *the free semigroup generated by it* but Σ^+ denotes *the same ones of all non-empty words. A concatenation* of two words is the word which is received at the rewriting of the second one of them after the first one of them. Any set of words from Σ^* is called *a language over* Σ (and especially $\varnothing$ is *the empty language* which does not contain words).

An union of two languages is their union as sets. *A concatenation "."* of two languages is the set of all words each one of which is a concatenation of a word from the first one of them with a word from the second one of them. *An iteration* L^* (or *a Kleene star*, or *a Kleene closure*) of a language L is the set of all finite concatenation of words of its. *A language is regular* if it can be received from a finite set of finite languages after a finite number unions, concatenations and iterations. *The class of regular languages* over Σ will be denoted by $Reg(\Sigma)$. It is a semigroup $(Reg(\Sigma), .)$ with respect to the operation concatenation of (here only regular) languages. It is playing an important role in theoretical and practical aspects of computer science.

A homomorphism $\varphi : \Delta^+ \to (Reg(\Sigma), .)$ of a finitely generated free semigroup Δ^+ in the semigroup $(Reg(\Sigma), .)$ of all regular languages over Σ is called *a regular language substitution*. Let L be a language in Δ^+. We denote with $\varphi(L) = \{\varphi(w) \ / \ w \in L\}$ the set of regular languages in Σ^* each one of which is an image of some word from L at φ. *The maximal rewriting* of a regular language R in Σ^* with respect to the regular language substitution φ is the set $M_\varphi(R) = \{w \ / \ w \in \Delta^+, \varphi(w) \subseteq R, R \in Reg(\Sigma)\}$ of all words in Δ^+ whose images at φ contain in this regular language. According to **a theorem** from [2] $M_\varphi(R)$ *is a regular language* over Δ. In [1] is proven the following

Theorem 3.1. [1]. *Let* $\varphi : \Delta^+ \to (Reg(\Sigma), .)$ *be a regular language substitution and* w *be a word in* Δ^+. *The membership problem* $[w]$,

$$[w] = \left\{u \ / \ u \in \Delta^+, \varphi(u) = \varphi(w)\right\},$$

is a regular language over Δ.

A connection between sections 2. and 3. is realized by Corollary 1. in section 3. from the Theorem 1. in section 2.

2. Definition of a (semigroup) homomorphism by way of entire prototypes at the correspondence

One of the author's general approaches at his mathematical research was a study of prototypes at structure homomorphisms. It started with [3], passed through [4-9] to its philosophy of science substantiation [10], and continued with [11, 12].

The definition of a semigroup homomorphism origins from the natural subconsciousness requirement for the preserving the properties at imaging. In this case a correspondence of a semigroup in another one is a homomorphism if and only if when it preserves the (semigroup) operation (in the direction of the mapping). So the mathematicians have come to the well known from the student textbooks usual formal **definition:** *A correspondence* φ *of the (multiplicative) semigroup* Π *in the (multiplicative) semigroup* Π_1 *is a homomorphism if and only if when for every two elements* a *and* b *from* Π *we have the equality* $\varphi(ab) = = \varphi(a)\varphi(b)$ *in* Π_1.

This definition does not speak about the prototypes at the correspondence. According to the Hegel's reflexion (please see [10]) the knowledge goes to the original object using consecutive negations. His method exactly was expressed in the cited above previous research of ours and an analogical idea has led us now to the following

Theorem 1. *The correspondence φ of the semigroup Π in the semigroup Π_1 is a homomorphism if and only if when for any elements a_1 and b_1 of Π_1 which are images of elements from Π we have*

$$\varphi^{-1}(a_1b_1) \supseteq \left\{\varphi^{-1}(a_1)\right\}\left\{\varphi^{-1}(b_1)\right\} \text{ in } \Pi.$$

We can express this formal statement in the following way: *a correspondence of a semigroup in another one is a homomorphism if and only if when the entire prototype of the product of images contains (always) the product of their entire prototypes.* That means the preservation of the operation can be expressed in terms of subsets with respect to the operation in the original but not in the image. This difference of principle where an equality of elements in the image is substituted with an inequality of subsets in the original comes from the rising to the next level of knowledge in the Hegel's hierarchy of consecutive negations. It is really closer to the essence of the object which has another gradual formal mathematical expression: the equality of two sets is defined by the their inclusion each into another one. In other words the indicated inequality expresses deeper properties of a correspondence which is a homomorphism. The last notes justify this theorem additionally from an epistemological point of view.

Proof of the theorem. *Necessity.* Let φ be a homomorphism i.e. it preserves the operation in its direction according to the standard definition above. Let a_1 and b_1 from Π_1 be images of corresponding arbitrary elements a and b from Π at φ, $\left\{\varphi^{-1}(a_1)\right\}$ and $\left\{\varphi^{-1}(b_1)\right\}$ be their entire prototypes, and c be an arbitrary element of the product $\left\{\varphi^{-1}(a_1)\right\}\left\{\varphi^{-1}(b_1)\right\}$ in Π i.e. $c = a_0b_0$ for some elements a_0 from $\left\{\varphi^{-1}(a_1)\right\}$ and b_0 from $\left\{\varphi^{-1}(b_1)\right\}$. Then $\varphi(c) = \varphi(a_0b_0) = \varphi(a_0)\varphi(b_0)$ because φ is a homomorphism but the last product is equal to a_1b_1 because a_0 and b_0 are from $\left\{\varphi^{-1}(a_1)\right\}$ and $\left\{\varphi^{-1}(b_1)\right\}$ correspondingly, i.e. $\varphi(c) = a_1b_1$. This arbitrary element c from $\left\{\varphi^{-1}(a_1)\right\}\left\{\varphi^{-1}(b_1)\right\}$ belongs then to $\varphi^{-1}(a_1b_1)$ because it is the entire prototype of a_1b_1 which proves the inclusion $\varphi^{-1}(a_1b_1) \supseteq \left\{\varphi^{-1}(a_1)\right\}\left\{\varphi^{-1}(b_1)\right\}$ in Π if φ is a homomorphism.

Sufficiency. Let a and b be arbitrary elements of Π and the indicated inclusion is valid for for their images $\varphi(a)$ and $\varphi(b)$ in Π_1, i.e. $\varphi^{-1}(\varphi(a)\varphi(b)) \supseteq$ $\supseteq \left\{\varphi^{-1}[\varphi(a)]\right\}.\left\{\varphi^{-1}[\varphi(b)]\right\}$ in Π. The product ab of a and b belongs to the product $\left\{\varphi^{-1}[\varphi(a)]\right\}.\left\{\varphi^{-1}[\varphi(b)]\right\}$ because a is from $\varphi^{-1}[\varphi(a)]$ and b is from $\varphi^{-1}[\varphi(b)]$. Then ab belongs to the entire prototype $\varphi^{-1}(\varphi(a)\varphi(b))$ of $\varphi(a)\varphi(b)$, i.e. $ab \in \varphi^{-1}(\varphi(a)\varphi(b))$. It follows from the last one its image $\varphi(ab)$ coincides with $\varphi(a)\varphi(b)$, i.e. $\varphi(ab) = \varphi(a)\varphi(b)$ in Π_1 which means φ is a homomorphism.

This formal results has another mathematical meaning except the indicated after the its formulation general philosophy of science reflection. The received in it inclusion is valid for any homomorphism. An absolutely justified question rises from it. It is: when is valid an exact equality? The interest to this question is absolutely natural, its justification is not necessary, and therefore any result in this direction as such analogical ones for corresponding subsemigroups from [7-11] is again well-founded. The interest is enforced because an equality is not valid for simple finitely represented semigroups. We can confirm this with the

Example of the natural homomorphism η of the free semigroup $\Pi = \,=< a, b, c >$ over the partially commutative semigroup $\Pi_1 =< a, b, c; ab = ba >$ where η is generated by the natural imaging $\eta(a) = [a]$, $\eta(b) = [b]$, $\eta(c) = [c]$, i.e. every *word* from Π depicts into the class $[word]$ of the equivalent words in Π_1 which contains it.

Let we consider the elements $\alpha_1 = [aca]$ and $\beta_1 = [bcb]$, for which $\eta^{-1}(\alpha_1) = \{aca\}$, $\eta^{-1}(\beta_1) = \{bcb\}$, $\eta^{-1}(\alpha_1) \,.\, \eta^{-1}(\beta_1) = \{acabcb\}$, $\eta^{-1}(\alpha_1\beta_1) = \{acabcb, acbacb\}$. Here $\eta^{-1}(\alpha_1\beta_1) \supset \eta^{-1}(\alpha_1) \,.\, \eta^{-1}(\beta_1)$.

3. Maximal rewriting of a regular language, of its Kleene closure, and the Kleene closure of the first rewriting with respect to a regular language substitution

As an application of Theorem 1. from section 2. to the concepts from this section 3. (please see the last cited theorem in the preliminary section) we receive

Corollary1. *The membership problem for the product of two words with respect to a regular language substitution contains the product of the membership problems of these words with respect to this substitution.*

We simply must keep in mind a regular language substitution is a special homomorphism.

Theorem 2. *The Kleene closure of the maximal rewriting of a regular language at a regular language substitution contains in the maximal rewriting of the Kleene closure of the initial regular language at the same substitution.* (This statement can be expressed symbolically as the inclusion $(M_\varphi(R))^* \subseteq M_\varphi(R^*)$ of regular languages in Δ^+. Please see it below at the proof. In particular for every word w from $\Delta^+ : (M_\varphi(\varphi(w)))^* \subseteq M_\varphi((\varphi(w))^*)$.)

Proof. Let $\varphi : \Delta^+ \to Reg(\Sigma)$ be a regular language substitution, R be a regular language in Σ^*, $M_\varphi(R) = \{w \,/\, w \in \Delta^+, \varphi(w) \subseteq R\}$ is the maximal rewriting of R with respect to φ, $(M_\varphi(R))^*$ is its Kleene closure, and $M_\varphi(R^*) = \{u \,/\, u \in \Delta^+, \varphi(u) \subseteq R^*\}$ is (analogically expressed) the maximal rewriting of the Kleene closure R^* of R with respect to the same substitution.

We have to prove $(M_\varphi(R))^* \subseteq M_\varphi(R^*)$. Really, let the finite product $v = \prod_{i=1}^{n} w_i$, where $w_i \in M_\varphi(R)$, $i = \overline{1-n}$, $n \in N$, is an arbitrary word from $(M_\varphi(R))^*$. Each factor w_i in this product depicts in (some regular language) $\varphi(w_i)$, which contains in R (i.e. $\varphi(w_i) \subseteq R$) according to the definition of the maximal rewriting of R with respect to the substitution φ as it was cited in

the beginning of the proof. Then (the regular language) $\varphi(v) = \varphi\left(\prod_{i=1}^{n} w_i\right) =$
$= \prod_{i=1}^{n} \varphi(w_i)$ is this finite product of (the regular languages) $\varphi(w_i)$, each one of which contains in R. Therefore the product by itself contains in the Kleene closure R^* of R, i.e. $\varphi(v) \subseteq R^*$. The last one means $v \in M_\varphi(R^*)$ according to the definition of the maximal rewriting $M_\varphi(R^*)$ of R^*. Therefore we receive really $(M_\varphi(R))^* \subseteq M_\varphi(R^*)$ what we have had to prove.

According to the cited in preliminaries theorem from [2] the maximal rewriting $M_\varphi(R)$ and $M_\varphi(R^*)$ of the regular languages R and R^* at the regular language substitution φ are regular languages in Δ^+. Therefore both sides in the proven inclusion are regular languages.

Theorem 3. *Let the image of the maximal rewriting of a regular language at a regular language substitution covers the entire given regular language. Then the image of any word from the maximal rewriting of the Kleene closure of the initial regular language covers by the image of a set of some words from the Kleene closure of the maximal rewriting of this given regular language everything at the same given regular language substitution.*

We can express this statement in the following symbolic way at the accepted above designations: *Let* $\varphi : \Delta^+ \to Reg(\Sigma)$ *be a regular language substitution,* R *be a regular language in* Σ^**. If* $\varphi(M_\varphi(R)) = R$*, then the image of every word from* $M_\varphi(R^*)$ *covers by the image of a set of some words from* $(M_\varphi(R))^*$.

Proof. Let w^* be an arbitrary word from $M_\varphi(R^*)$, i.e. $w^* \in M_\varphi(R^*)$. That means (the regular language) $\varphi(w^*)$ contains in R^* or $\varphi(w^*) \subseteq R^*$, i.e. each word ω^* over Σ^* from $\varphi(w^*)$ is a concatenation $\omega^* = \omega_1^*\omega_2^* \ldots \omega_k^*$ of a finite number k of words $\omega_1^*,\ \omega_2^*,\ \ldots\ \omega_k^*$ from R, $\omega_1^*,\ \omega_2^*,\ \ldots\ \omega_k^* \in R$. All of them belong to the image of some word from the maximal rewriting of R because its image covers R, i.e. for any ω_j^* $(j = \overline{1-k})$ there exists a word w_j^* from $M_\varphi(R)$ $(w_j^* \in M_\varphi(R))$ such that $\omega_j^* \in \varphi\left(w_j^*\right)$. Then $\varphi(w^*) = \{\omega^*\} = \{\omega_1^*\omega_2^* \ldots \omega_k^*\} \subseteq$
$\subseteq \left\{\varphi(w_1^*)\,\varphi(w_2^*)\ \ldots\ \varphi(w_k^*)\right\} = \left\{\varphi(w_1^* w_2^*\ \ldots\ w_k^*) \,/\, w_1^* w_2^*\ \ldots\ w_k^* \in (M_\varphi(R))^*\right\}$, which shows $\varphi(w^*)$ really covers by the image at φ of the set $\{w_1^* w_2^* \ldots w_k^*\}$ of all words $w_1^* w_2^* \ldots w_k^*$ from $(M_\varphi(R))^*$ of the indicated type.

Theorem 3. could be expressed in a more detailed way with a description of the participating in it languages which are regular: *Let the (regular language which is the) image of the maximal rewriting (a regular language) of a regular language at a regular language substitution covers the entire given regular language. Then the (regular language which is the) image of any word from the maximal rewriting (a regular language) of the Kleene closure (a regular language) of the initial regular language covers by the image of the set of some words from the Kleene closure (a regular language) of the maximal rewriting (a regular language) of this given regular language. Everything is at the same given regular language substitution.*

This expression of the statement gives a reason for **a conjecture** the constructed set of some words from the Kleene closure of the maximal rewriting of this given regular language, whose image covers the image of any initial word from the maximal rewriting of the Kleene closure of the initial regular language, is also a regular language. The conjecture still remains unproved and not rejected.

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