DYNAMICS OF A PLANAR MODEL: MELNIKOV'S APPROACH, APPLICATIONS

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Abstract. *In this paper, we focus on the Hamiltonian, which gives rise to a specific dynamical system. We demonstrate some modules for investigating the dynamics of the proposed model. Some investigations in the light of Melnikov's approach is considered. A possible application of the Melnikov functions can find in modeling and synthesis of radiation antenna diagrams is also discussed.*

Key words: Modified Planar Model, Melnikov Function, Antenna Factor.

Mathematics Subject Classification: 65L07, 34A34.

1. The model

A number of authors devote their research to the phase-space flow of a particle in a forced cubic and higher order potentials. This problem has very direct application in mechanics and engineering sciences and can also be considered as a normal form of a more complex Hamiltonian system. The publications on this topic are significant and varied (see [1, 2, 3, 4, 5, 6, 7]). We focus on the Hamiltonian, which gives rise to the following modified dynamical system:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = bx - \displaystyle\sum_{i=0}^{\left[\frac{n}{2}\right]-1} b_i x^{n-2i} - \epsilon \left(bx - \sum_{i=0}^{\left[\frac{n}{2}\right]-1} b_i x^{n-2i} \right) \sum_{j=1}^{N} g_i \sin(j\omega t). \end{cases} \tag{1}$$

1.1. The case $n = 3$, $b = b_0 = 1$

The Melnikov function [8] is of the form

$$M(t_0) = \int_{-\infty}^{\infty} y_0(t)(x_0(t) - x_0^3(t)) \sum_{j=1}^{N} g_j \sin(j\omega(t + t_0))dt \tag{2}$$

with double homoclinic orbit given by: $x_0(t) = \pm\sqrt{2}\,\mathrm{sech}(t)$; $y_0(t) = \mp\sqrt{2}\,\mathrm{sech}(t)\,\tanh(t)$. The following statements are valid

Proposition 1.1. *If $N = 1$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation*

$$\begin{aligned} M(t_0) &= -\frac{1}{6} g_1 \pi \omega^2 (-2+\omega^2)\,\mathrm{csch}\left(\frac{\pi\omega}{2}\right)\cos(t_0\omega) \\ &= F_1(\omega; g_1)\cos(t_0\omega) = 0. \end{aligned} \tag{3}$$

The factor $F_1(\omega; g_1)$ as a function of the parameters ω and g_1 is depicted in Fig. 1 for a) $\omega = 1.3$, $g_1 = 1$ (thick); b) $\omega = 1$, $g_1 = 1.1$ (red); c) $\omega = 0.9$, $g_1 = 1.15$ (green). With a suitable change of variable $t = k\cos\theta + k_1$, the expression $|M^*(\theta)|$ can be used to model a characteristic antenna factor in confidential intervals [9].

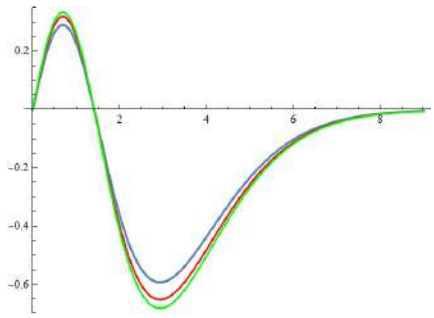
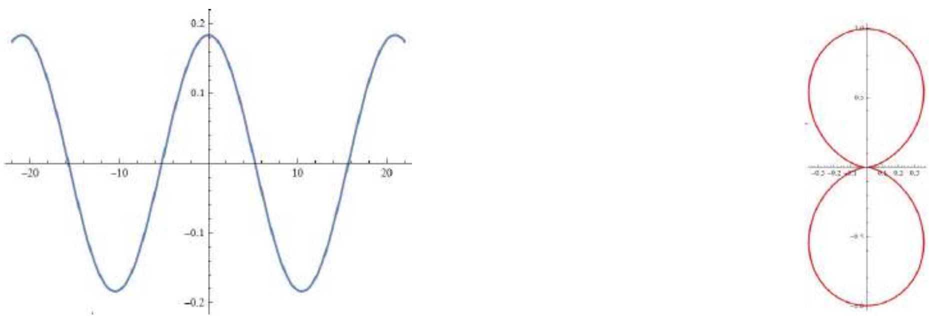


Figure 1. The factor $F_1(\omega; g_1)$

Example 1.1. *For $N = 1$, $\omega = 0.3$, $g_1 = 1$ Melnikov function $M(t_0)$ is depicted in Fig. 2.a. For the fixed values of N, ω and g_1 and $k = 5.2$, $k_1 = 0.001$ the Melnikov antenna factor (dipole) is presented in Fig. 2.b.*



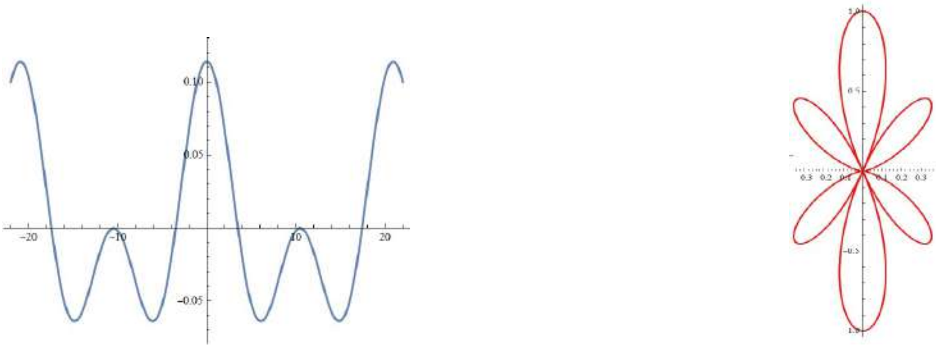
(a) The Melnikov function *(b) The Melnikov antenna factor*

Figure 2. Case $N = 1$ (Example 1)

Proposition 1.2. *If $N = 2$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation*

$$M(t_0) = \frac{1}{12} e^{-2it_0\omega} \pi\omega^2 \Big(- \Big((e^{it_0\omega} + e^{3it_0\omega}) g_1(-2+\omega^2) \operatorname{csch}(\frac{\pi\omega}{2}) \Big) - \\ -8(1+e^{4it_0\omega}) g_2(-1+2\omega^2) \operatorname{csch}(\pi\omega) \Big) = 0. \tag{4}$$

Example 1.2. *For $N = 2$, $\omega = 0.3$, $g_1 = 0.31$, $g_2 = 0.2$ Melnikov function $M(t_0)$ is depicted in Fig. 3.a. For the fixed values of N, ω, g_1, g_2 and $k = 10.1$, $k_1 = 0.001$ the Melnikov antenna factor is presented in Fig. 3.b.*



(a) The Melnikov function *(b) The Melnikov antenna factor*

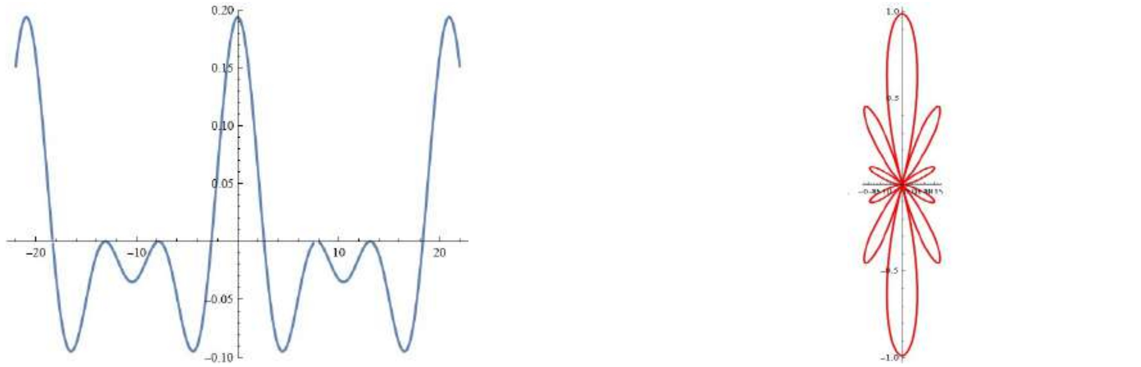
Figure 3. Case $N = 2$ (Example 2)

Proposition 1.3. *If $N = 3$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation*

$$\begin{aligned} M(t_0) = -&\frac{e^{-3it_0\omega}\pi\omega^2}{24(1+2\cosh(\pi\omega))} \Big(\big(2e^{2it_0\omega} g_1(-2+\omega^2) \\ &+ 2e^{4it_0\omega} g_1(-2+\omega^2) + \\ &+ 9g_3(-2+9\omega^2) + 9e^{6it_0\omega} g_3(-2+9\omega^2)\big) \cosh(\frac{\pi\omega}{2}) + \\ &+ e^{it_0\omega} \big(4(1+e^{4it_0\omega}) g_2(-1+2\omega^2) \\ &+ 8(1+e^{4it_0\omega}) g_2(-1+2\omega^2)\cosh(\pi\omega) + \\ &+ e^{it_0\omega}(1+e^{2it_0\omega}) g_1(-2+\omega^2)\cosh(\frac{3\pi\omega}{2})\big)\Big) \times \\ &\times \operatorname{csch}\left(\frac{\pi\omega}{4}\right) \operatorname{sech}(\frac{\pi\omega}{4}) \operatorname{sech}(\frac{\pi\omega}{2}) \end{aligned} \tag{5}$$

Note. Proposition 1.3 holds in the limit $-\frac{2}{3} < Im(\omega) < \frac{2}{3}$.

Example 1.3. *For $N = 3$, $\omega = 0.3$, $g_1 = 0.31$, $g_2 = 0.28$, $g_3 = 0.22$ Melnikov function $M(t_0)$ is depicted in Fig. 4.a. For the fixed values of N, ω, g_1, g_2, g_3 and $k = 12.7$, $k_1 = 0.001$ the Melnikov antenna factor is presented in Fig. 4.b.*



(a) The Melnikov function *(b) The Melnikov antenna factor*

Figure 4. Case $N = 3$ (Example 3)

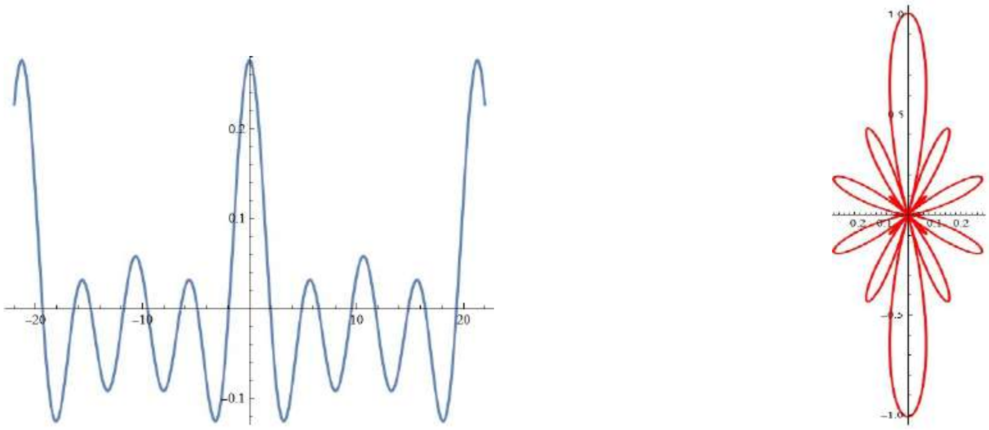
If $N = 4$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation (see Fig. 5)

$$
\begin{aligned}
M(t0) = \frac{1}{24} e^{-4 i t0 \omega} \pi \omega^2 \Big(& -e^{3 i t0 \omega} \left(1 + e^{2 i t0 \omega}\right) g1 \left(-2 + \omega^2\right) \mathrm{Coth}\left[\frac{\pi \omega}{4}\right] - 8 e^{2 i t0 \omega} \left(1 + e^{4 i t0 \omega}\right) g2 \left(-1 + 2 \omega^2\right) \mathrm{Coth}\left[\frac{\pi \omega}{2}\right] + 18 e^{i t0 \omega} g3\, \mathrm{Coth}\left[\frac{3 \pi \omega}{4}\right] + \\
& 18 e^{7 i t0 \omega} g3\, \mathrm{Coth}\left[\frac{3 \pi \omega}{4}\right] - 81 e^{i t0 \omega} g3\, \omega^2 \mathrm{Coth}\left[\frac{3 \pi \omega}{4}\right] - 81 e^{7 i t0 \omega} g3\, \omega^2 \mathrm{Coth}\left[\frac{3 \pi \omega}{4}\right] + 32\, g4\, \mathrm{Coth}[\pi \omega] + 32 e^{8 i t0 \omega} g4\, \mathrm{Coth}[\pi \omega] - \\
& 256\, g4\, \omega^2 \mathrm{Coth}[\pi \omega] - 256 e^{8 i t0 \omega} g4\, \omega^2 \mathrm{Coth}[\pi \omega] - 2 e^{3 i t0 \omega} g1\, \mathrm{Tanh}\left[\frac{\pi \omega}{4}\right] - 2 e^{5 i t0 \omega} g1\, \mathrm{Tanh}\left[\frac{\pi \omega}{4}\right] + e^{3 i t0 \omega} g1\, \omega^2 \mathrm{Tanh}\left[\frac{\pi \omega}{4}\right] + \\
& e^{5 i t0 \omega} g1\, \omega^2 \mathrm{Tanh}\left[\frac{\pi \omega}{4}\right] - 8 e^{2 i t0 \omega} g2\, \mathrm{Tanh}\left[\frac{\pi \omega}{2}\right] - 8 e^{6 i t0 \omega} g2\, \mathrm{Tanh}\left[\frac{\pi \omega}{2}\right] + 16 e^{2 i t0 \omega} g2\, \omega^2 \mathrm{Tanh}\left[\frac{\pi \omega}{2}\right] + 16 e^{6 i t0 \omega} g2\, \omega^2 \mathrm{Tanh}\left[\frac{\pi \omega}{2}\right] - \\
& 18 e^{i t0 \omega} g3\, \mathrm{Tanh}\left[\frac{3 \pi \omega}{4}\right] - 18 e^{7 i t0 \omega} g3\, \mathrm{Tanh}\left[\frac{3 \pi \omega}{4}\right] + 81 e^{i t0 \omega} g3\, \omega^2 \mathrm{Tanh}\left[\frac{3 \pi \omega}{4}\right] + 81 e^{7 i t0 \omega} g3\, \omega^2 \mathrm{Tanh}\left[\frac{3 \pi \omega}{4}\right] - 32\, g4\, \mathrm{Tanh}[\pi \omega] - \\
& 32 e^{8 i t0 \omega} g4\, \mathrm{Tanh}[\pi \omega] + 256\, g4\, \omega^2 \mathrm{Tanh}[\pi \omega] + 256 e^{8 i t0 \omega} g4\, \omega^2 \mathrm{Tanh}[\pi \omega] \Big) = 0
\end{aligned}
$$

Figure 5. The case $N = 4$: Melnikov function $M(t_0)$ using our module implemented in CAS Mathematica.

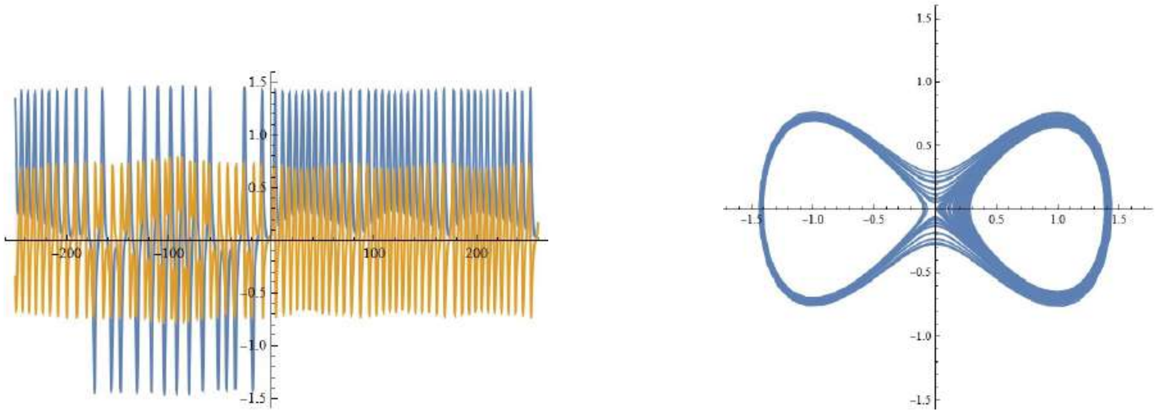
Example 1.4. *For $N = 4$, $\omega = 0.295$, $g_1 = 0.28$, $g_2 = 0.25$, $g_3 = 0.22$, $g_4 = 0.68$ Melnikov function $M(t_0)$ is depicted in Fig. 6.a. For the fixed values of N, ω, g_1, g_2, g_3, $g_4$and $k = 9.7$, $k_1 = 0.001$ the Melnikov antenna factor is presented in Fig. 6.b.*

Example 1.5. *For given $N = 2$, $\omega = 0.9$, $g_1 = 2.9$, $g_2 = 1.1$, $\epsilon = 0.01$ the simulations on the system (1) for $x_0 = 0.1$; $y_0 = 0.1$ are depicted on Fig. 7.*



(a) The Melnikov function *(b) The Melnikov antenna factor*

Figure 6. Case $N = 4$ (Example 4)



(a) The solutions of the system (1) *(b) Phase plot*

Figure 7. Case $N = 2$ (Example 5)

The reader can generate a Melnikov antenna array for a fixed number of emitters. For example, if $N = 5$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation (see Fig. 8).

Example 1.6. *For $N = 5$, $\omega = 0.34$, $g_1 = 0.1$, $g_2 = 0.05$, $g_3 = 0.2$, $g_4 = 0.02$, $g_5 = 0.1$ Melnikov function $M(t_0)$ is depicted in Fig. 9.a. For the fixed values of N, ω, g_1, g_2, g_3, g_4, g_5 and $k = 10.4$, $k_1 = 0.001$ the Melnikov antenna factor is presented in Fig. 9.b.*

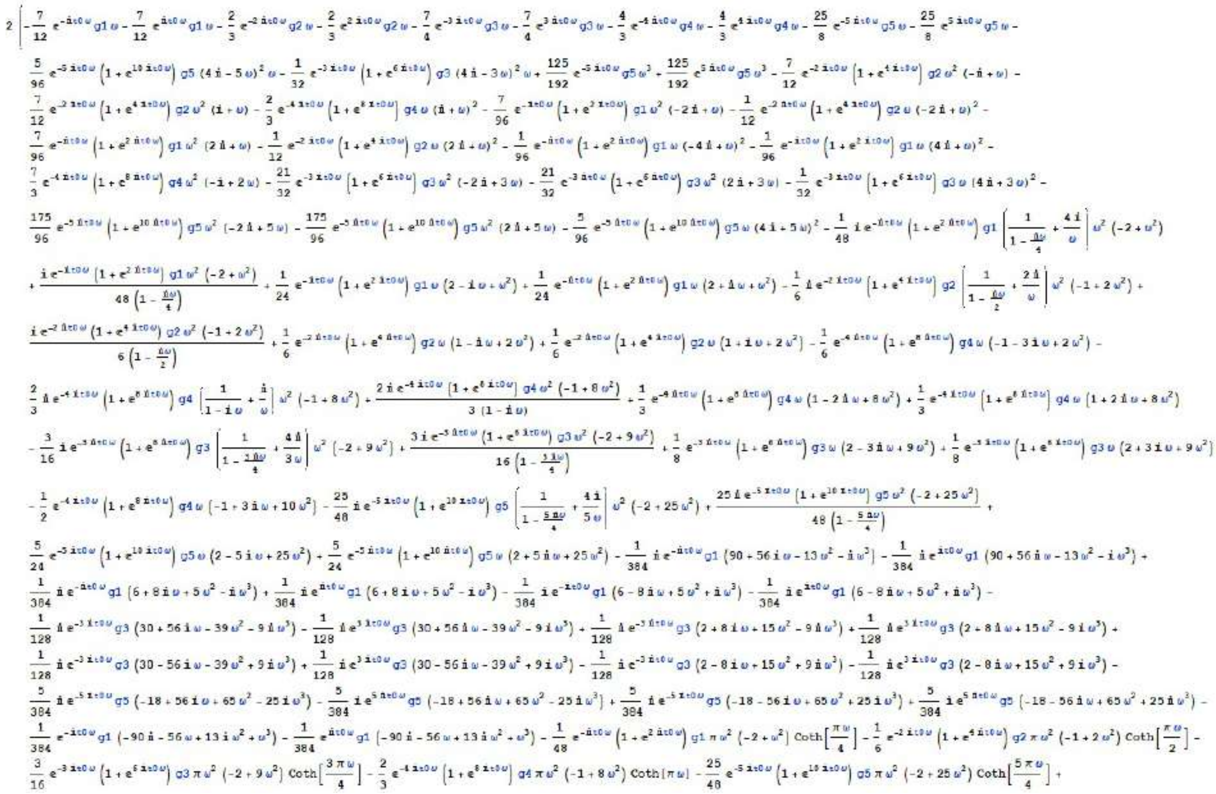
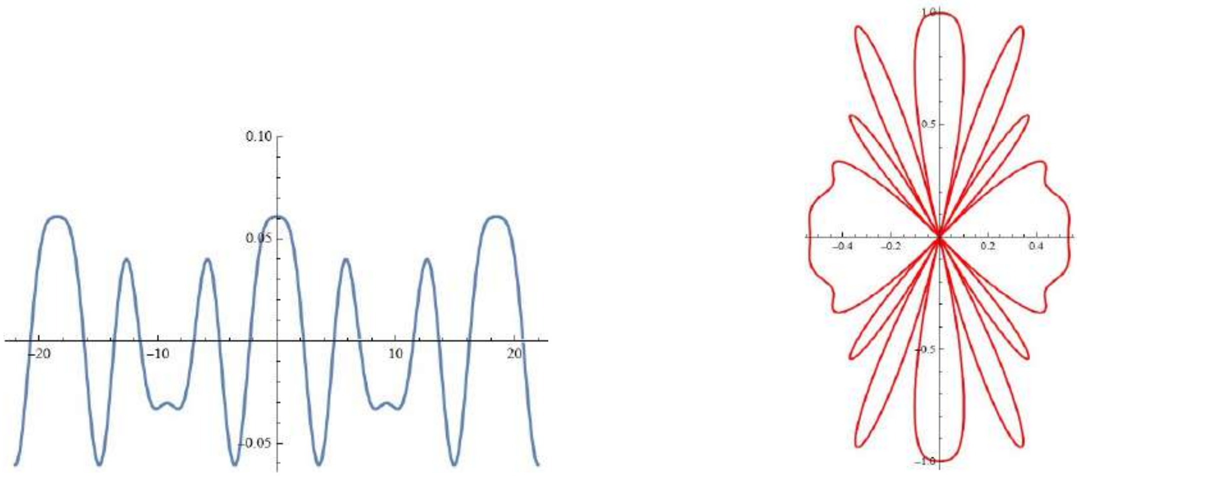


Figure 8. The case $N = 5$: Melnikov function $M(t_0)$ using our module implemented in CAS Mathematica



(a) The Melnikov function *(b) The Melnikov antenna factor*

Figure 9. Case $N = 5$ (Example 6)

Using our module implemented in CAS Mathematica in Fig. 10 we illustrate the generated equation $M(t_0) = 0$ for $N = 6$. For example, for fixed $N = 6$ Melnikov function and Melnikov antenna array are depicted in Fig. 11.

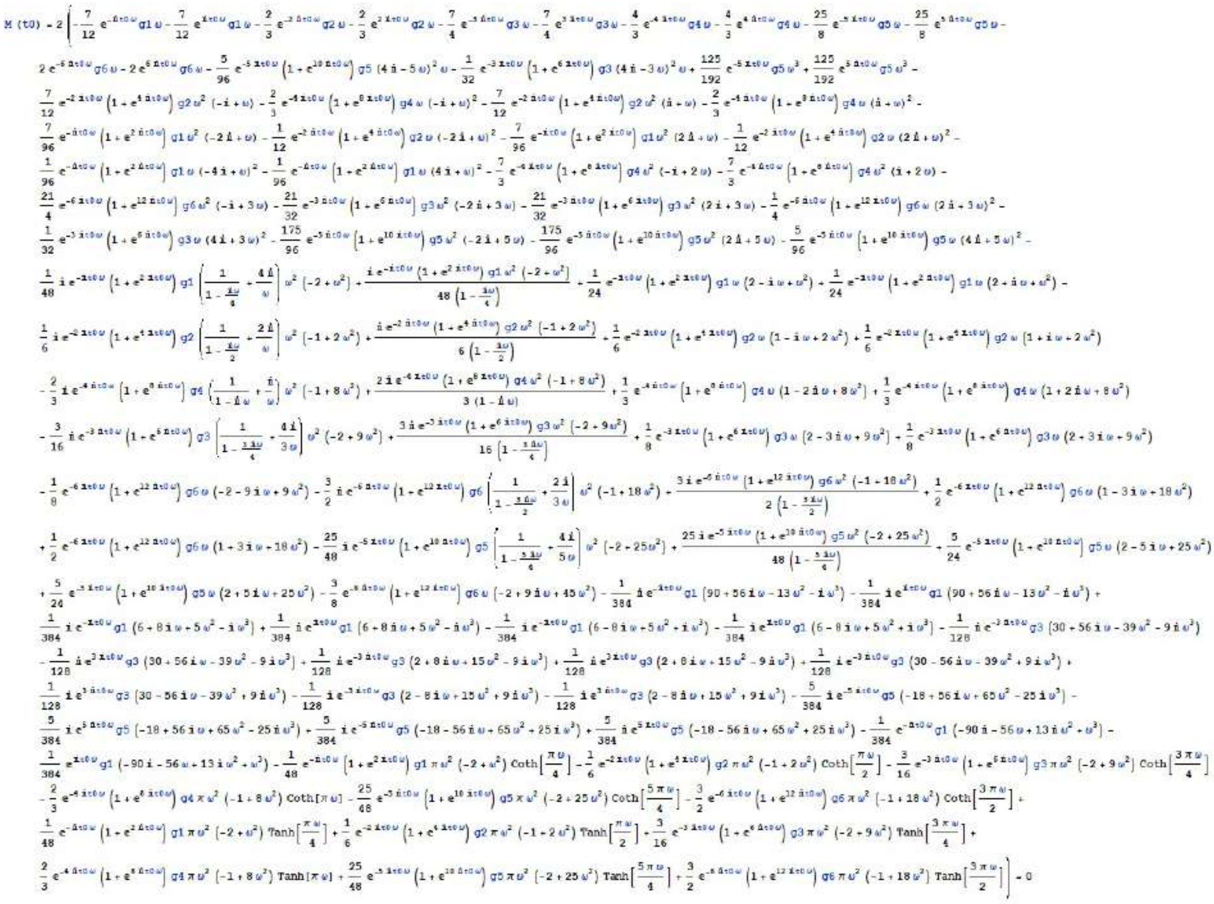
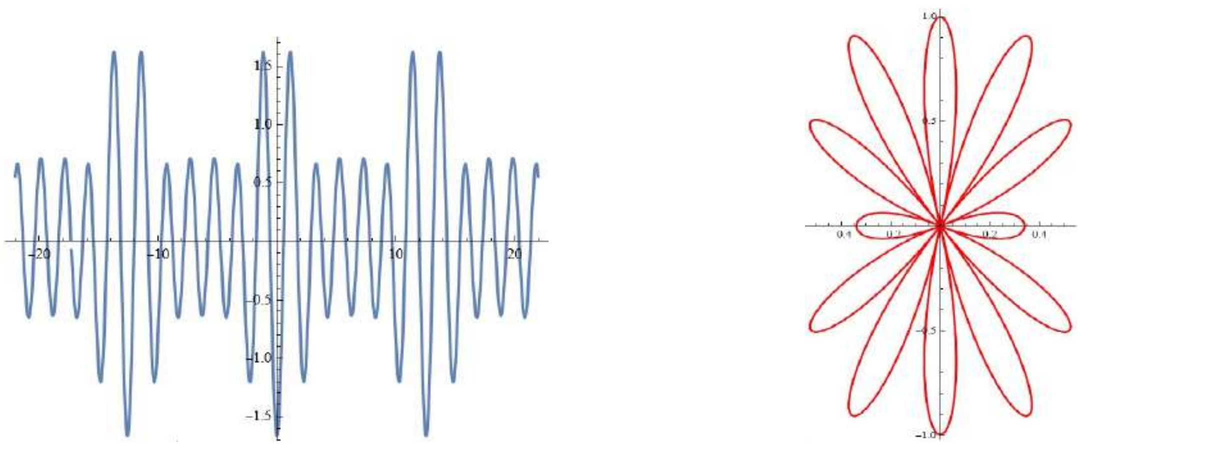


Figure 10. The generated equation $M(t_0) = 0$ for $N = 6$ using our module implemented in CAS Mathematica.



(a) The Melnikov function *(b) The Melnikov antenna factor*

Figure 11. Case $N = 6$

Of course, this relatively new idea of justification and right to exist is subject to serious research by specialists working in this scientific direction. In a number of cases the Melnikov function can be used to approximate electrical stages.

Example 1.7. *Let* $N = 5$; $\omega = 0.31$; $g_1 = 0.09$; $g_2 = 0.09$; $g_3 = 0.001$; $g_4 = 0.001$; $g_5 = 0.001$. *A good approximation of the electrical stage by Melnikov function is depicted on Fig. 12.*

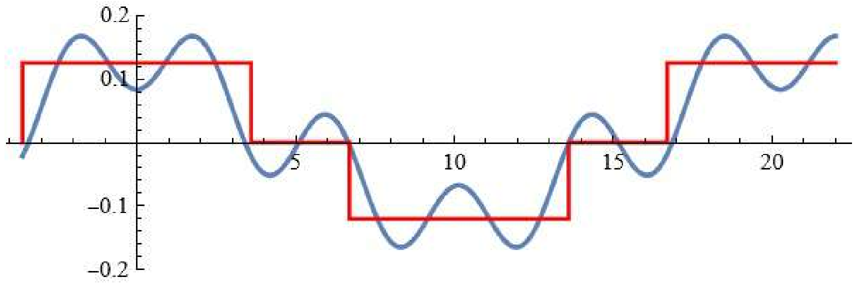


Figure 12. A good approximation of the electrical stage by Melnikov function (Example 7)

1.2. The case $n = 5$

In this case, the reader can continue the studies related to the generation of the Melnikov functions given in the previous section, and we will skip them here. It is sufficient to use the explicit form of homo/heteroclinic orbits. For more details, see [10]. A representation for $b = -0.4$; $b_1 = -0.7$, $b_0 = 0.1$ is given in Fig. 13.

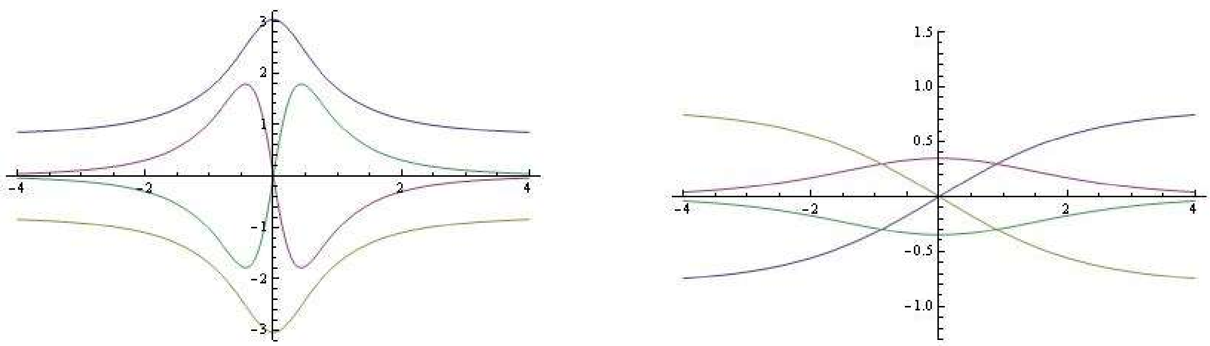
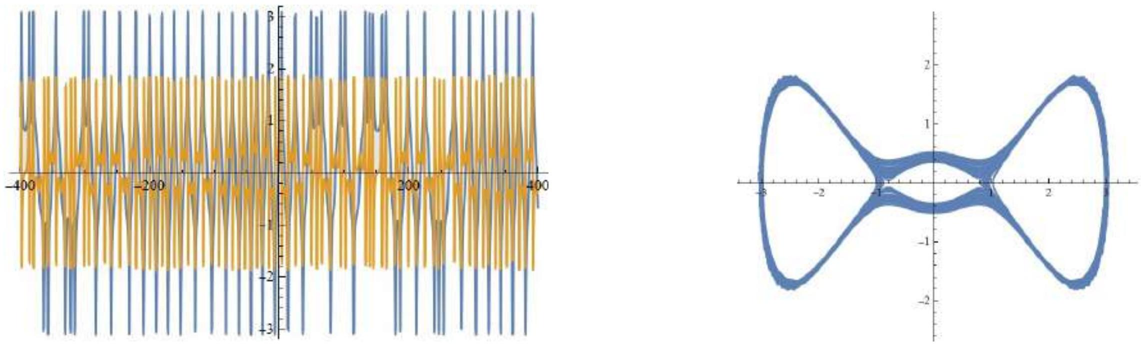


Figure 13. The homo/heteroclinic orbits

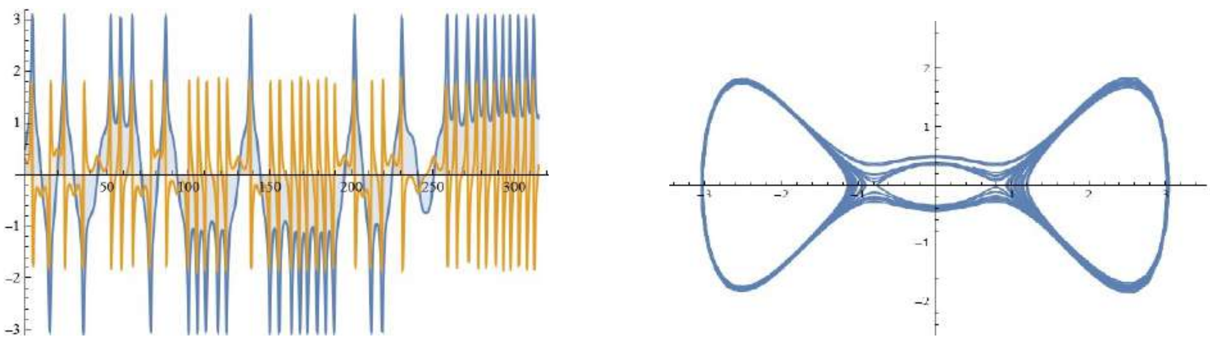
Example 1.8. *For given* $N = 2$, $\omega = 0.3$, $g_1 = 2.9$, $g_2 = 0.8$, $\epsilon = 0.01$ *the simulations on the system for* $x_0 = 0.6$; $y_0 = 0.3$ *are depicted on Fig. 14.*



(a) The solutions of the system *(b) Phase plot*

Figure 14. Case $N = 2$ (Example 8)

Example 1.9. *For given $N = 4$, $\omega = 0.1$, $g_1 = 1.9$, $g_2 = 0.2$, $g_3 = 0.1$, $g_4 = 1.6$, $\epsilon = 0.03$ the simulations on the system for $x_0 = 0.5$; $y_0 = 0.3$ are depicted on Fig. 15.*



(a) The solutions of the system *(b) Phase plot*

Figure 15. Case $N = 4$ (Example 9)

2. Concluding Remarks

If $M(t_0) = 0$ and $\frac{M(t_0)}{dt_0} \neq 0$ for some t_0 and some sets of parameters, then chaos occurs. From the above statements, the reader can formulate the Melnikov condition for chaotic behavior of the proposed dynamic model (1). Nonstandard numerical methods connected to the investigation of the roots of nonlinear equation $M(t_0) = 0$ can be found in [11]. The investigations can be included as an integral part of a planned much more general Web-based application for scientific computing [12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

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