

# DYNAMICS OF A PLANAR MODEL: MELNIKOV'S APPROACH, APPLICATIONS

Maria Vasileva, Nikolay Kyurkchiev, Tsvetelin Zaeovski,  
Anton Iliev, Vesselin Kyurkchiev, Asen Rahnev

**Abstract.** *In this paper, we focus on the Hamiltonian, which gives rise to a specific dynamical system. We demonstrate some modules for investigating the dynamics of the proposed model. Some investigations in the light of Melnikov's approach is considered. A possible application of the Melnikov functions can find in modeling and synthesis of radiation antenna diagrams is also discussed.*

**Key words:** Modified Planar Model, Melnikov Function, Antenna Factor.

**Mathematics Subject Classification:** 65L07, 34A34.

## 1. The model

A number of authors devote their research to the phase-space flow of a particle in a forced cubic and higher order potentials. This problem has very direct application in mechanics and engineering sciences and can also be considered as a normal form of a more complex Hamiltonian system. The publications on this topic are significant and varied (see [1, 2, 3, 4, 5, 6, 7]). We focus on the Hamiltonian, which gives rise to the following modified dynamical system:

$$\left\{ \begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = bx - \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} b_i x^{n-2i} - \epsilon \left( bx - \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} b_i x^{n-2i} \right) \sum_{j=1}^N g_j \sin(j\omega t). \end{array} \right. \quad (1)$$

### 1.1. The case $n = 3$ , $b = b_0 = 1$

The Melnikov function [8] is of the form

$$M(t_0) = \int_{-\infty}^{\infty} y_0(t)(x_0(t) - x_0^3(t)) \sum_{j=1}^N g_j \sin(j\omega(t + t_0)) dt \quad (2)$$

with double homoclinic orbit given by:  $x_0(t) = \pm\sqrt{2} \operatorname{sech}(t)$ ;  $y_0(t) = \mp\sqrt{2} \operatorname{sech}(t) \tanh(t)$ . The following statements are valid

**Proposition 1.1.** *If  $N = 1$ , then the roots of Melnikov function  $M(t_0)$  are given as solutions of the equation*

$$\begin{aligned} M(t_0) &= -\frac{1}{6}g_1\pi\omega^2(-2 + \omega^2) \operatorname{csch}\left(\frac{\pi\omega}{2}\right) \cos(t_0\omega) \\ &= F_1(\omega; g_1) \cos(t_0\omega) = 0. \end{aligned} \quad (3)$$

The factor  $F_1(\omega; g_1)$  as a function of the parameters  $\omega$  and  $g_1$  is depicted in Fig. 1 for a)  $\omega = 1.3$ ,  $g_1 = 1$  (thick); b)  $\omega = 1$ ,  $g_1 = 1.1$  (red); c)  $\omega = 0.9$ ,  $g_1 = 1.15$  (green). With a suitable change of variable  $t = k \cos \theta + k_1$ , the expression  $|M^*(\theta)|$  can be used to model a characteristic antenna factor in confidential intervals [9].

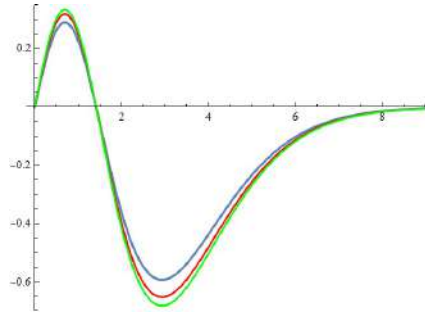
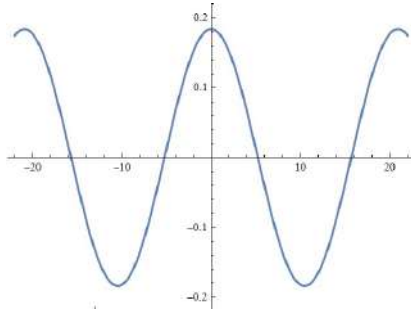
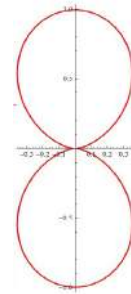


Figure 1. The factor  $F_1(\omega; g_1)$

**Example 1.1.** *For  $N = 1$ ,  $\omega = 0.3$ ,  $g_1 = 1$  Melnikov function  $M(t_0)$  is depicted in Fig. 2.a. For the fixed values of  $N$ ,  $\omega$  and  $g_1$  and  $k = 5.2$ ,  $k_1 = 0.001$  the Melnikov antenna factor (dipole) is presented in Fig. 2.b.*



(a) The Melnikov function



(b) The Melnikov antenna factor

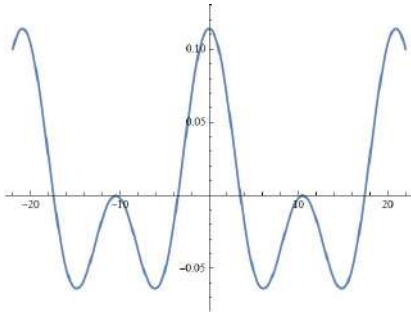
Figure 2. Case  $N = 1$  (Example 1)

**Proposition 1.2.** *If  $N = 2$ , then the roots of Melnikov function  $M(t_0)$  are given as solutions of the equation*

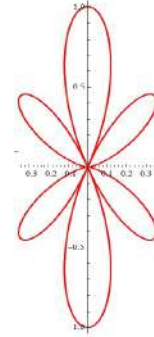
$$M(t_0) = \frac{1}{12}e^{-2it_0\omega}\pi\omega^2 \left( - \left( (e^{it_0\omega} + e^{3it_0\omega})g_1(-2 + \omega^2) \operatorname{csch}\left(\frac{\pi\omega}{2}\right) \right) - \right. \quad (4)$$

$$\left. -8(1 + e^{4it_0\omega})g_2(-1 + 2\omega^2) \operatorname{csch}(\pi\omega) \right) = 0.$$

**Example 1.2.** *For  $N = 2$ ,  $\omega = 0.3$ ,  $g_1 = 0.31$ ,  $g_2 = 0.2$  Melnikov function  $M(t_0)$  is depicted in Fig. 3.a. For the fixed values of  $N$ ,  $\omega$ ,  $g_1$ ,  $g_2$  and  $k = 10.1$ ,  $k_1 = 0.001$  the Melnikov antenna factor is presented in Fig. 3.b.*



(a) The Melnikov function



(b) The Melnikov antenna factor

Figure 3. Case  $N = 2$  (Example 2)

**Proposition 1.3.** *If  $N = 3$ , then the roots of Melnikov function  $M(t_0)$  are given as solutions of the equation*

$$M(t_0) = -\frac{e^{-3it_0\omega}\pi\omega^2}{24(1 + 2\cosh(\pi\omega))} \left( (2e^{2it_0\omega}g_1(-2 + \omega^2) \right. \quad (5)$$

$$+ 2e^{4it_0\omega}g_1(-2 + \omega^2) +$$

$$+ 9g_3(-2 + 9\omega^2) + 9e^{6it_0\omega}g_3(-2 + 9\omega^2)) \cosh\left(\frac{\pi\omega}{2}\right) +$$

$$+ e^{it_0\omega}(4(1 + e^{4it_0\omega})g_2(-1 + 2\omega^2)$$

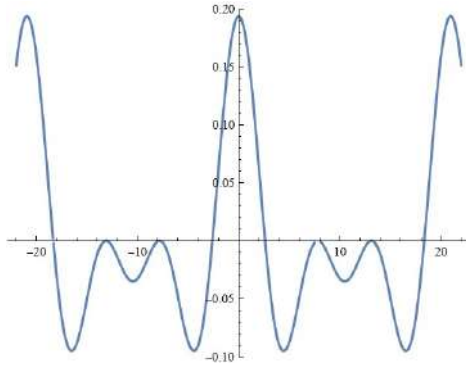
$$+ 8(1 + e^{4it_0\omega})g_2(-1 + 2\omega^2) \cosh(\pi\omega) +$$

$$+ e^{it_0\omega}(1 + e^{2it_0\omega})g_1(-2 + \omega^2) \cosh\left(\frac{3\pi\omega}{2}\right)) \times$$

$$\times \operatorname{csch}\left(\frac{\pi\omega}{4}\right) \operatorname{sech}\left(\frac{\pi\omega}{4}\right) \operatorname{sech}\left(\frac{\pi\omega}{2}\right)$$

**Note.** Proposition 1.3 holds in the limit  $-\frac{2}{3} < \operatorname{Im}(\omega) < \frac{2}{3}$ .

**Example 1.3.** For  $N = 3$ ,  $\omega = 0.3$ ,  $g_1 = 0.31$ ,  $g_2 = 0.28$ ,  $g_3 = 0.22$  Melnikov function  $M(t_0)$  is depicted in Fig. 4.a. For the fixed values of  $N$ ,  $\omega$ ,  $g_1$ ,  $g_2$ ,  $g_3$  and  $k = 12.7$ ,  $k_1 = 0.001$  the Melnikov antenna factor is presented in Fig. 4.b.



(a) The Melnikov function



(b) The Melnikov antenna factor

Figure 4. Case  $N = 3$  (Example 3)

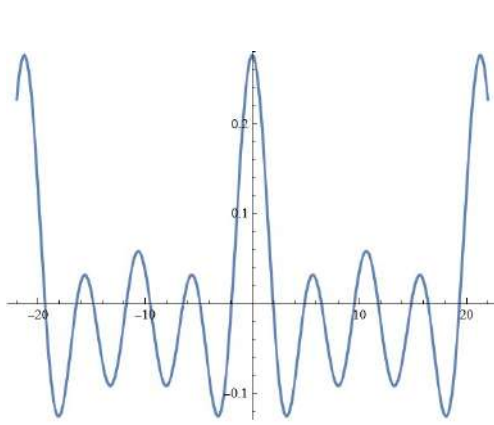
If  $N = 4$ , then the roots of Melnikov function  $M(t_0)$  are given as solutions of the equation (see Fig. 5)

$$\begin{aligned} M(t_0) = & \frac{1}{24} e^{-4\pi\omega} \pi \omega^2 \left( -e^{3\pi\omega} (1 + e^{2\pi\omega}) g_1 (-2 + \omega^2) \coth\left[\frac{\pi\omega}{4}\right] - 8e^{2\pi\omega} (1 + e^{4\pi\omega}) g_2 (-1 + 2\omega^2) \coth\left[\frac{\pi\omega}{2}\right] + 18e^{\pi\omega} g_3 \coth\left[\frac{3\pi\omega}{4}\right] + \right. \\ & 18e^{7\pi\omega} g_3 \coth\left[\frac{3\pi\omega}{4}\right] - 81e^{\pi\omega} g_3 \omega^2 \coth\left[\frac{3\pi\omega}{4}\right] - 81e^{7\pi\omega} g_3 \omega^2 \coth\left[\frac{3\pi\omega}{4}\right] + 32g_4 \coth[\pi\omega] + 32e^{8\pi\omega} g_4 \coth[\pi\omega] - \\ & 256g_4 \omega^2 \coth[\pi\omega] - 256e^{8\pi\omega} g_4 \omega^2 \coth[\pi\omega] - 2e^{3\pi\omega} g_1 \tanh\left[\frac{\pi\omega}{4}\right] - 2e^{5\pi\omega} g_1 \tanh\left[\frac{\pi\omega}{4}\right] + e^{3\pi\omega} g_1 \omega^2 \tanh\left[\frac{\pi\omega}{4}\right] + \\ & e^{5\pi\omega} g_1 \omega^2 \tanh\left[\frac{\pi\omega}{4}\right] - 8e^{2\pi\omega} g_2 \tanh\left[\frac{\pi\omega}{2}\right] - 8e^{6\pi\omega} g_2 \tanh\left[\frac{\pi\omega}{2}\right] + 16e^{2\pi\omega} g_2 \omega^2 \tanh\left[\frac{\pi\omega}{2}\right] + 16e^{6\pi\omega} g_2 \omega^2 \tanh\left[\frac{\pi\omega}{2}\right] - \\ & 18e^{\pi\omega} g_3 \tanh\left[\frac{3\pi\omega}{4}\right] - 18e^{7\pi\omega} g_3 \tanh\left[\frac{3\pi\omega}{4}\right] + 81e^{\pi\omega} g_3 \omega^2 \tanh\left[\frac{3\pi\omega}{4}\right] + 81e^{7\pi\omega} g_3 \omega^2 \tanh\left[\frac{3\pi\omega}{4}\right] - 32g_4 \tanh[\pi\omega] - \\ & \left. 32e^{8\pi\omega} g_4 \tanh[\pi\omega] + 256g_4 \omega^2 \tanh[\pi\omega] + 256e^{8\pi\omega} g_4 \omega^2 \tanh[\pi\omega] \right) = 0 \end{aligned}$$

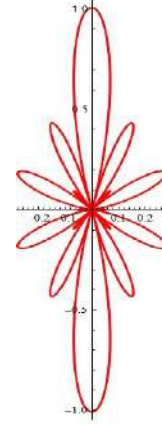
Figure 5. The case  $N = 4$ : Melnikov function  $M(t_0)$  using our module implemented in CAS Mathematica.

**Example 1.4.** For  $N = 4$ ,  $\omega = 0.295$ ,  $g_1 = 0.28$ ,  $g_2 = 0.25$ ,  $g_3 = 0.22$ ,  $g_4 = 0.68$  Melnikov function  $M(t_0)$  is depicted in Fig. 6.a. For the fixed values of  $N$ ,  $\omega$ ,  $g_1$ ,  $g_2$ ,  $g_3$ ,  $g_4$  and  $k = 9.7$ ,  $k_1 = 0.001$  the Melnikov antenna factor is presented in Fig. 6.b.

**Example 1.5.** For given  $N = 2$ ,  $\omega = 0.9$ ,  $g_1 = 2.9$ ,  $g_2 = 1.1$ ,  $\epsilon = 0.01$  the simulations on the system (1) for  $x_0 = 0.1$ ;  $y_0 = 0.1$  are depicted on Fig. 7.

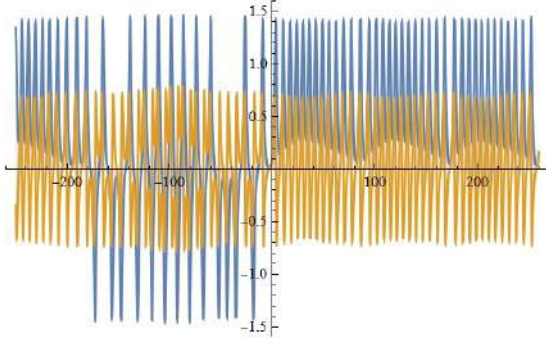


(a) The Melnikov function

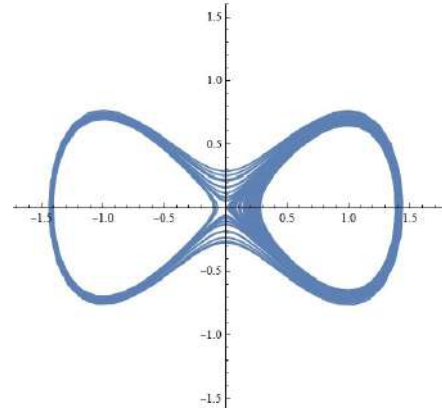


(b) The Melnikov antenna factor

Figure 6. Case  $N = 4$  (Example 4)



(a) The solutions of the system (1)



(b) Phase plot

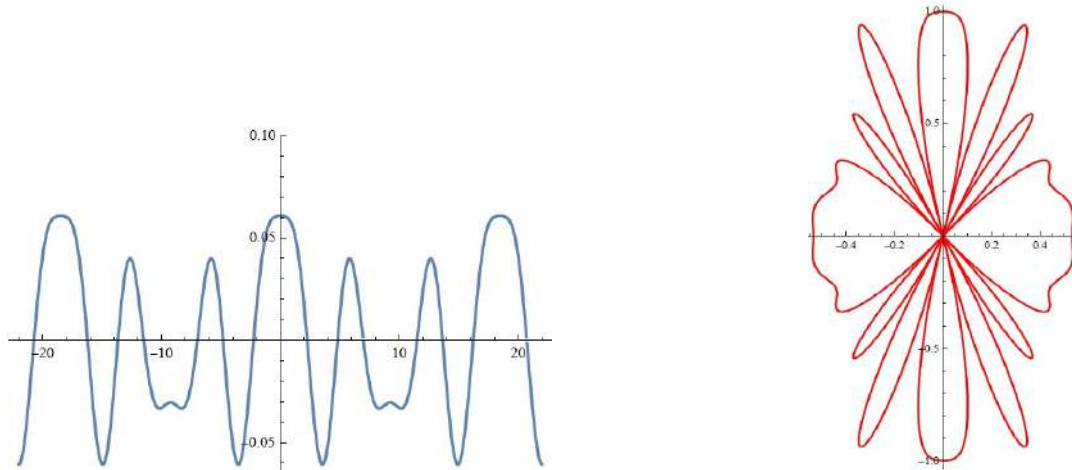
Figure 7. Case  $N = 2$  (Example 5)

The reader can generate a Melnikov antenna array for a fixed number of emitters. For example, if  $N = 5$ , then the roots of Melnikov function  $M(t_0)$  are given as solutions of the equation (see Fig. 8).

**Example 1.6.** For  $N = 5$ ,  $\omega = 0.34$ ,  $g_1 = 0.1$ ,  $g_2 = 0.05$ ,  $g_3 = 0.2$ ,  $g_4 = 0.02$ ,  $g_5 = 0.1$  Melnikov function  $M(t_0)$  is depicted in Fig. 9.a. For the fixed values of  $N$ ,  $\omega$ ,  $g_1$ ,  $g_2$ ,  $g_3$ ,  $g_4$ ,  $g_5$  and  $k = 10.4$ ,  $k_1 = 0.001$  the Melnikov antenna factor is presented in Fig. 9.b.

$$\begin{aligned}
& 2 \left[ -\frac{7}{12} e^{-2i0u} g1 u - \frac{7}{12} e^{2i0u} g1 u - \frac{2}{3} e^{-2i0u} g2 u - \frac{2}{3} e^{2i0u} g2 u - \frac{7}{4} e^{-2i0u} g3 u - \frac{7}{4} e^{2i0u} g3 u - \frac{4}{3} e^{-2i0u} g4 u - \frac{4}{3} e^{2i0u} g4 u - \frac{25}{8} e^{-2i0u} g5 u - \frac{25}{8} e^{2i0u} g5 u - \right. \\
& \frac{5}{96} e^{-5i0u} (1 + e^{10i0u}) g5 (4i - 5u)^2 u - \frac{1}{32} e^{-2i0u} (1 + e^{6i0u}) g3 (4i - 3u)^2 u + \frac{125}{192} e^{-5i0u} g5 u^3 - \frac{125}{192} e^{5i0u} g5 u^3 - \frac{7}{12} e^{-2i0u} (1 + e^{4i0u}) g2 u^2 (-i + u) - \\
& \frac{7}{12} e^{-2i0u} (1 + e^{4i0u}) g2 u^2 (i + u) - \frac{2}{3} e^{-4i0u} (1 + e^{8i0u}) g4 u (i + u)^2 - \frac{7}{96} e^{-10i0u} (1 + e^{20i0u}) g1 u^2 (-2i + u) - \frac{1}{12} e^{-2i0u} (1 + e^{4i0u}) g2 u (-2i + u)^2 - \\
& \frac{7}{96} e^{-5i0u} (1 + e^{10i0u}) g1 u^2 (2i + u) - \frac{1}{12} e^{-2i0u} (1 + e^{6i0u}) g3 u (2i + u)^2 - \frac{1}{96} e^{-5i0u} (1 + e^{10i0u}) g1 u (-4i + u)^2 - \frac{1}{96} e^{-i0u} (1 + e^{2i0u}) g1 u (4i + u)^2 - \\
& \frac{7}{3} e^{-4i0u} (1 + e^{8i0u}) g4 u^2 (-i + 2u) - \frac{21}{32} e^{-3i0u} (1 + e^{6i0u}) g3 u^2 (-2i + 3u) - \frac{21}{32} e^{-3i0u} (1 + e^{6i0u}) g3 u^2 (2i + 3u) - \frac{1}{32} e^{-3i0u} (1 + e^{6i0u}) g3 u (4i + 3u)^2 - \\
& \frac{175}{96} e^{-5i0u} (1 + e^{10i0u}) g5 u^2 (-2i + 5u) - \frac{175}{96} e^{-5i0u} (1 + e^{10i0u}) g5 u^2 (2i + 5u) - \frac{5}{96} e^{-5i0u} (1 + e^{10i0u}) g5 u (4i + 5u)^2 - \frac{1}{48} i e^{-8i0u} (1 + e^{16i0u}) g1 \left( \frac{1}{1 - \frac{i u}{4}} + \frac{4i}{u} \right) u^2 (-2 + u^2) \\
& + \frac{i e^{-10i0u} (1 + e^{20i0u}) g1 u^2 (-2 + u^2)}{48 \left( 1 - \frac{i u}{4} \right)} + \frac{1}{24} e^{-10i0u} (1 + e^{20i0u}) g1 u (2 - i u + u^2) + \frac{1}{24} e^{-8i0u} (1 + e^{16i0u}) g1 u (2 + i u + u^2) - \frac{1}{6} i e^{-7i0u} (1 + e^{14i0u}) g2 \left[ \frac{1}{1 - \frac{i u}{2}} + \frac{2i}{u} \right] u^2 (-1 + 2u^2) + \\
& \frac{i e^{-7i0u} (1 + e^{14i0u}) g2 u^2 (-1 + 2u^2)}{6 \left( 1 - \frac{i u}{2} \right)} + \frac{1}{6} e^{-7i0u} (1 + e^{14i0u}) g2 u (1 - i u + 2u^2) + \frac{1}{6} e^{-5i0u} (1 + e^{10i0u}) g2 u (1 + i u + 2u^2) - \frac{1}{6} e^{-5i0u} (1 + e^{10i0u}) g4 u (-1 + 3i u + 2u^2) - \\
& \frac{2}{3} i e^{-4i0u} (1 + e^{8i0u}) g4 \left[ \frac{1}{1 - i u} + \frac{i}{u} \right] u^2 (-1 + 8u^2) + \frac{2i e^{-4i0u} (1 + e^{8i0u}) g4 u^2 (-1 + 8u^2)}{3 (1 - i u)} + \frac{1}{3} e^{-4i0u} (1 + e^{8i0u}) g4 u (1 - 2i u + 8u^2) + \frac{1}{3} e^{-4i0u} (1 + e^{8i0u}) g4 u (1 + 2i u + 8u^2) \\
& - \frac{3}{16} i e^{-3i0u} (1 + e^{6i0u}) g3 \left[ \frac{1}{1 - \frac{i u}{4}} + \frac{4i}{3u} \right] u^2 (-2 + 9u^2) + \frac{3i e^{-3i0u} (1 + e^{6i0u}) g3 u^2 (-2 + 9u^2)}{16 \left( 1 - \frac{i u}{4} \right)} + \frac{1}{8} e^{-3i0u} (1 + e^{6i0u}) g3 u (2 - 3i u + 9u^2) + \frac{1}{8} e^{-3i0u} (1 + e^{6i0u}) g3 u (2 + 3i u + 9u^2) \\
& - \frac{1}{2} e^{-4i0u} (1 + e^{8i0u}) g4 u (-1 + 3i u + 10u^2) - \frac{25}{48} i e^{-5i0u} (1 + e^{10i0u}) g5 \left[ \frac{1}{1 - \frac{i u}{4}} + \frac{4i}{5u} \right] u^2 (-2 + 25u^2) + \frac{25i e^{-5i0u} (1 + e^{10i0u}) g5 u^2 (-2 + 25u^2)}{48 \left( 1 - \frac{i u}{4} \right)} + \\
& \frac{5}{24} e^{-5i0u} (1 + e^{10i0u}) g5 u (2 - 5i u + 25u^2) + \frac{5}{24} e^{-5i0u} (1 + e^{10i0u}) g5 u (2 + 5i u + 25u^2) - \frac{1}{384} i e^{-8i0u} g1 (90 - 56i u - 13u^2 - i u^3) - \frac{1}{384} i e^{8i0u} g1 (90 + 56i u - 13u^2 - i u^3) + \\
& \frac{1}{384} i e^{-8i0u} g1 (6 + 8i u + 5u^2 - i u^3) + \frac{1}{384} i e^{8i0u} g1 (6 + 8i u + 5u^2 - i u^3) - \frac{1}{384} i e^{-8i0u} g1 (6 - 8i u + 5u^2 + i u^3) - \frac{1}{384} i e^{8i0u} g1 (6 - 8i u + 5u^2 + i u^3) - \\
& \frac{1}{128} i e^{-7i0u} g3 (30 + 56i u - 39u^2 - 9i u^3) - \frac{1}{128} i e^{7i0u} g3 (30 + 56i u - 39u^2 - 9i u^3) + \frac{1}{128} i e^{-7i0u} g3 (2 + 8i u + 15u^2 - 9i u^3) + \frac{1}{128} i e^{7i0u} g3 (2 + 8i u + 15u^2 - 9i u^3) + \\
& \frac{1}{128} i e^{-7i0u} g3 (30 - 56i u - 39u^2 + 9i u^3) + \frac{1}{128} i e^{7i0u} g3 (30 - 56i u - 39u^2 + 9i u^3) - \frac{1}{128} i e^{-7i0u} g3 (2 - 8i u + 15u^2 + 9i u^3) - \frac{1}{128} i e^{7i0u} g3 (2 - 8i u + 15u^2 + 9i u^3) - \\
& \frac{5}{384} i e^{-5i0u} g5 (-18 - 56i u + 65u^2 - 25i u^3) - \frac{5}{384} i e^{5i0u} g5 (-18 - 56i u + 65u^2 - 25i u^3) + \frac{5}{384} i e^{-5i0u} g5 (-18 - 56i u + 65u^2 + 25i u^3) - \frac{5}{384} i e^{5i0u} g5 (-18 - 56i u + 65u^2 + 25i u^3) - \\
& \frac{1}{384} e^{-10i0u} g1 (-90i - 56u + 13i u^2 + u^3) - \frac{1}{384} e^{10i0u} g1 (-90i - 56u + 13i u^2 + u^3) - \frac{1}{48} e^{-8i0u} (1 + e^{16i0u}) g1 \pi u^2 (-2 + u^2) \coth\left[\frac{\pi u}{4}\right] - \frac{1}{48} e^{-8i0u} (1 + e^{16i0u}) g2 \pi u^2 (-1 + 2u^2) \coth\left[\frac{\pi u}{2}\right] - \\
& \frac{3}{16} e^{-3i0u} (1 + e^{6i0u}) g3 \pi u^2 (-2 + 9u^2) \coth\left[\frac{3\pi u}{4}\right] - \frac{2}{3} e^{-4i0u} (1 + e^{8i0u}) g4 \pi u^2 (-1 + 8u^2) \coth[\pi u] - \frac{25}{48} e^{-5i0u} (1 + e^{10i0u}) g5 \pi u^2 (-2 + 25u^2) \coth\left[\frac{5\pi u}{4}\right] +
\end{aligned}$$

Figure 8. The case  $N = 5$ : Melnikov function  $M(t_0)$  using our module implemented in CAS Mathematica



(a) The Melnikov function

(b) The Melnikov antenna factor

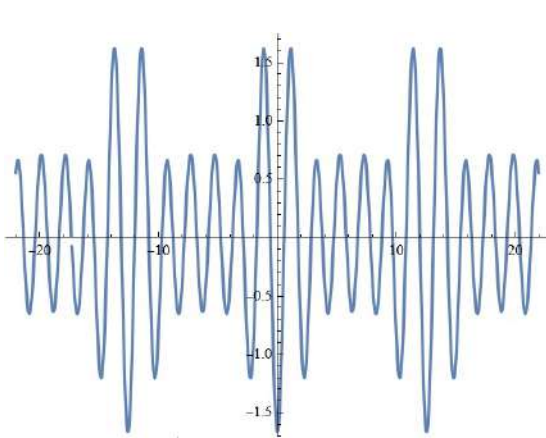
Figure 9. Case  $N = 5$  (Example 6)

Using our module implemented in CAS Mathematica in Fig. 10 we illustrate the generated equation  $M(t_0) = 0$  for  $N = 6$ . For example, for

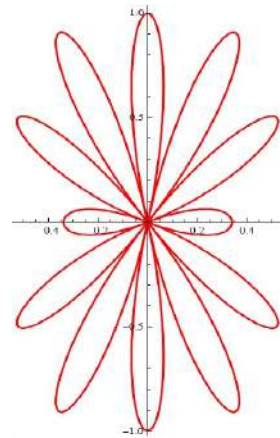
fixed  $N = 6$  Melnikov function and Melnikov antenna array are depicted in Fig. 11.

$$\begin{aligned}
 M(t_0) = & 2 \left[ -\frac{7}{12} e^{-i t_0 u} g_1 u - \frac{7}{12} e^{i t_0 u} g_1 u - \frac{2}{3} e^{-i t_0 u} g_2 u - \frac{2}{3} e^{i t_0 u} g_2 u - \frac{7}{4} e^{-i t_0 u} g_3 u - \frac{7}{4} e^{i t_0 u} g_3 u - \frac{4}{3} e^{-i t_0 u} g_4 u - \frac{4}{3} e^{i t_0 u} g_4 u - \frac{25}{8} e^{-i t_0 u} g_5 u - \frac{25}{8} e^{i t_0 u} g_5 u - \right. \\
 & 2 e^{-i t_0 u} g_6 u - 2 e^{i t_0 u} g_6 u - \frac{5}{96} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 \left( 4 i - 5 u \right)^2 u - \frac{1}{32} e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 \left( 4 i - 3 u \right)^2 u + \frac{125}{192} e^{-i t_0 u} g_5 u^3 + \frac{125}{192} e^{i t_0 u} g_5 u^3 - \\
 & \frac{7}{12} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_2 u^2 \left( -i + u \right) - \frac{2}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 u \left( -i + u \right)^2 - \frac{7}{12} e^{-i t_0 u} \left( 1 + e^{i 4 t_0 u} \right) g_2 u^2 \left( i + u \right) - \frac{2}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 u \left( i + u \right)^2 - \\
 & \frac{7}{96} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 u^2 \left( -2 i + u \right) - \frac{1}{12} e^{-i t_0 u} \left( 1 + e^{i 4 t_0 u} \right) g_2 u \left( -2 i + u \right)^2 - \frac{7}{96} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 u^2 \left( 2 i + u \right) - \frac{1}{12} e^{-i t_0 u} \left( 1 + e^{i 4 t_0 u} \right) g_2 u \left( 2 i + u \right)^2 - \\
 & \frac{1}{96} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 u \left( -4 i + u \right)^2 - \frac{1}{96} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 u \left( 4 i + u \right)^2 - \frac{7}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 u^2 \left( -i + 2 u \right) - \frac{7}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 u^2 \left( i + 2 u \right) - \\
 & \frac{21}{4} e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 u^2 \left( -i + 3 u \right) - \frac{21}{32} e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 u^2 \left( -2 i + 3 u \right) - \frac{21}{32} e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 u^2 \left( 2 i + 3 u \right) - \frac{1}{4} e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 u \left( 2 i + 3 u \right)^2 - \\
 & \frac{1}{32} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_3 u \left( 4 i + 3 u \right)^2 - \frac{175}{96} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 u^2 \left( -2 i + 5 u \right) - \frac{175}{96} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 u^2 \left( 2 i + 5 u \right) - \frac{5}{96} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 u \left( 4 i + 5 u \right)^2 - \\
 & \frac{1}{48} i e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 \left( \frac{1}{1 - \frac{i u}{2}} - \frac{4 i}{u} \right) u^2 \left( -2 + u^2 \right) - \frac{i e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 u^2 \left( -2 + u^2 \right)}{48 \left( 1 - \frac{i u}{2} \right)} + \frac{1}{24} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 u \left( 2 - i u + u^2 \right) - \frac{1}{24} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 u \left( 2 + i u + u^2 \right) - \\
 & \frac{1}{6} i e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_2 \left( \frac{1}{1 - \frac{i u}{2}} - \frac{2 i}{u} \right) u^2 \left( -1 + 2 u^2 \right) + \frac{i e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_2 u^2 \left( -1 + 2 u^2 \right)}{6 \left( 1 - \frac{i u}{2} \right)} + \frac{1}{6} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_2 u \left( 1 - i u + 2 u^2 \right) + \frac{1}{6} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_2 u \left( 1 + i u + 2 u^2 \right) \\
 & - \frac{2}{3} i e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 \left( \frac{1}{1 - \frac{i u}{2}} - \frac{4 i}{u} \right) u^2 \left( -1 + 8 u^2 \right) + \frac{2 i e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 u^2 \left( -1 + 8 u^2 \right)}{3 \left( 1 - \frac{i u}{2} \right)} + \frac{1}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 u \left( 1 - 2 i u + 8 u^2 \right) + \frac{1}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 u \left( 1 + 2 i u + 8 u^2 \right) \\
 & - \frac{3}{16} i e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 \left( \frac{1}{1 - \frac{i u}{2}} - \frac{4 i}{u} \right) u^2 \left( -2 + 9 u^2 \right) + \frac{3 i e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 u^2 \left( -2 + 9 u^2 \right)}{16 \left( 1 - \frac{i u}{2} \right)} + \frac{1}{8} e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 u \left( 2 - 3 i u + 9 u^2 \right) + \frac{1}{8} e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 u \left( 2 + 3 i u + 9 u^2 \right) \\
 & - \frac{1}{8} e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 u \left( -2 - 9 i u + 9 u^2 \right) - \frac{3}{2} i e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 \left( \frac{1}{1 - \frac{i u}{2}} - \frac{2 i}{u} \right) u^2 \left( -1 + 18 u^2 \right) + \frac{3 i e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 u^2 \left( -1 + 18 u^2 \right)}{2 \left( 1 - \frac{i u}{2} \right)} + \frac{1}{2} e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 u \left( 1 - 3 i u + 18 u^2 \right) \\
 & + \frac{1}{2} e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 u \left( 1 + 3 i u + 18 u^2 \right) - \frac{25}{48} i e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 \left( \frac{1}{1 - \frac{i u}{2}} - \frac{4 i}{u} \right) u^2 \left( -2 + 25 u^2 \right) + \frac{25 i e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 u^2 \left( -2 + 25 u^2 \right)}{48 \left( 1 - \frac{i u}{2} \right)} + \frac{5}{24} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 u \left( 2 - 5 i u + 25 u^2 \right) \\
 & + \frac{5}{24} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 u \left( 2 + 5 i u + 25 u^2 \right) - \frac{3}{8} i e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 u \left( -2 + 9 i u + 15 u^2 \right) - \frac{1}{384} i e^{-i t_0 u} g_1 \left( 90 + 56 i u - 13 u^2 - i u^3 \right) - \frac{1}{384} i e^{-i t_0 u} g_1 \left( 90 + 56 i u - 13 u^2 - i u^3 \right) + \\
 & \frac{1}{384} i e^{-i t_0 u} g_1 \left( 6 + 8 i u + 5 u^2 - i u^3 \right) + \frac{1}{384} i e^{-i t_0 u} g_1 \left( 6 - 8 i u + 5 u^2 - i u^3 \right) - \frac{1}{384} i e^{-i t_0 u} g_1 \left( 6 - 8 i u + 5 u^2 - i u^3 \right) - \frac{1}{384} i e^{-i t_0 u} g_1 \left( 6 + 8 i u + 5 u^2 - i u^3 \right) \\
 & - \frac{1}{128} i e^{-i t_0 u} g_3 \left( 30 + 56 i u - 39 u^2 - 9 i u^3 \right) + \frac{1}{128} i e^{-i t_0 u} g_3 \left( 2 + 8 i u + 15 u^2 - 9 i u^3 \right) + \frac{1}{128} i e^{-i t_0 u} g_3 \left( 2 + 8 i u + 15 u^2 - 9 i u^3 \right) - \frac{1}{128} i e^{-i t_0 u} g_3 \left( 30 + 56 i u - 39 u^2 - 9 i u^3 \right) \\
 & - \frac{1}{128} i e^{-i t_0 u} g_5 \left( 30 - 56 i u - 39 u^2 + 9 i u^3 \right) - \frac{1}{128} i e^{-i t_0 u} g_5 \left( 2 - 8 i u + 15 u^2 + 9 i u^3 \right) - \frac{1}{128} i e^{-i t_0 u} g_5 \left( 2 - 8 i u + 15 u^2 + 9 i u^3 \right) - \frac{5}{384} i e^{-i t_0 u} g_5 \left( -18 + 56 i u + 65 u^2 - 25 i u^3 \right) - \\
 & \frac{5}{384} i e^{-i t_0 u} g_5 \left( -18 + 56 i u + 65 u^2 - 25 i u^3 \right) - \frac{5}{384} i e^{-i t_0 u} g_5 \left( -18 - 56 i u + 65 u^2 + 25 i u^3 \right) - \frac{1}{384} e^{-i t_0 u} g_1 \left( -90 i - 56 u + 13 i u^2 + u^3 \right) - \\
 & \frac{1}{384} e^{-i t_0 u} g_1 \left( -90 i - 56 u + 13 i u^2 + u^3 \right) - \frac{1}{48} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 \pi u^2 \left( -2 + u^2 \right) \coth \left[ \frac{\pi u}{4} \right] - \frac{1}{6} e^{-i t_0 u} \left( 1 + e^{i 4 t_0 u} \right) g_2 \pi u^2 \left( -1 + 2 u^2 \right) \coth \left[ \frac{\pi u}{2} \right] - \frac{3}{16} e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 \pi u^2 \left( -2 + 9 u^2 \right) \coth \left[ \frac{3 \pi u}{4} \right] \\
 & - \frac{2}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 \pi u^2 \left( -1 + 8 u^2 \right) \coth \left[ \pi u \right] - \frac{25}{48} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 \pi u^2 \left( -2 + 25 u^2 \right) \coth \left[ \frac{5 \pi u}{4} \right] - \frac{3}{2} e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 \pi u^2 \left( -1 + 18 u^2 \right) \coth \left[ \frac{3 \pi u}{2} \right] + \\
 & \frac{1}{48} e^{-i t_0 u} \left( 1 + e^{i 2 t_0 u} \right) g_1 \pi u^2 \left( -2 + u^2 \right) \tanh \left[ \frac{\pi u}{4} \right] + \frac{1}{6} e^{-i t_0 u} \left( 1 + e^{i 4 t_0 u} \right) g_2 \pi u^2 \left( -1 + 2 u^2 \right) \tanh \left[ \frac{\pi u}{2} \right] + \frac{3}{16} e^{-i t_0 u} \left( 1 + e^{i 6 t_0 u} \right) g_3 \pi u^2 \left( -2 + 9 u^2 \right) \tanh \left[ \frac{3 \pi u}{4} \right] + \\
 & \frac{2}{3} e^{-i t_0 u} \left( 1 + e^{i 8 t_0 u} \right) g_4 \pi u^2 \left( -1 + 8 u^2 \right) \tanh \left[ \pi u \right] + \frac{25}{48} e^{-i t_0 u} \left( 1 + e^{i 10 t_0 u} \right) g_5 \pi u^2 \left( -2 + 25 u^2 \right) \tanh \left[ \frac{5 \pi u}{4} \right] + \frac{3}{2} e^{-i t_0 u} \left( 1 + e^{i 12 t_0 u} \right) g_6 \pi u^2 \left( -1 + 18 u^2 \right) \tanh \left[ \frac{3 \pi u}{2} \right] \Big] - 0
 \end{aligned}$$

Figure 10. The generated equation  $M(t_0) = 0$  for  $N = 6$  using our module implemented in CAS Mathematica.



(a) The Melnikov function



(b) The Melnikov antenna factor

Figure 11. Case  $N = 6$

Of course, this relatively new idea of justification and right to exist is subject to serious research by specialists working in this scientific direction. In a number of cases the Melnikov function can be used to approximate electrical stages.

**Example 1.7.** Let  $N = 5$ ;  $\omega = 0.31$ ;  $g_1 = 0.09$ ;  $g_2 = 0.09$ ;  $g_3 = 0.001$ ;  $g_4 = 0.001$ ;  $g_5 = 0.001$ . A good approximation of the electrical stage by Melnikov function is depicted on Fig. 12.

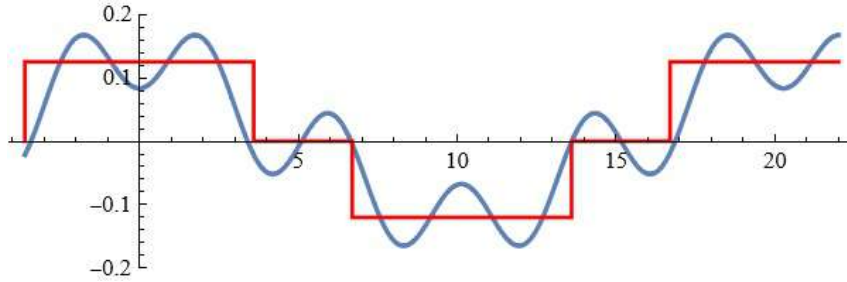


Figure 12. A good approximation of the electrical stage by Melnikov function (Example 7)

## 1.2. The case $n = 5$

In this case, the reader can continue the studies related to the generation of the Melnikov functions given in the previous section, and we will skip them here. It is sufficient to use the explicit form of homo/heteroclinic orbits. For more details, see [10]. A representation for  $b = -0.4$ ;  $b_1 = -0.7$ ,  $b_0 = 0.1$  is given in Fig. 13.

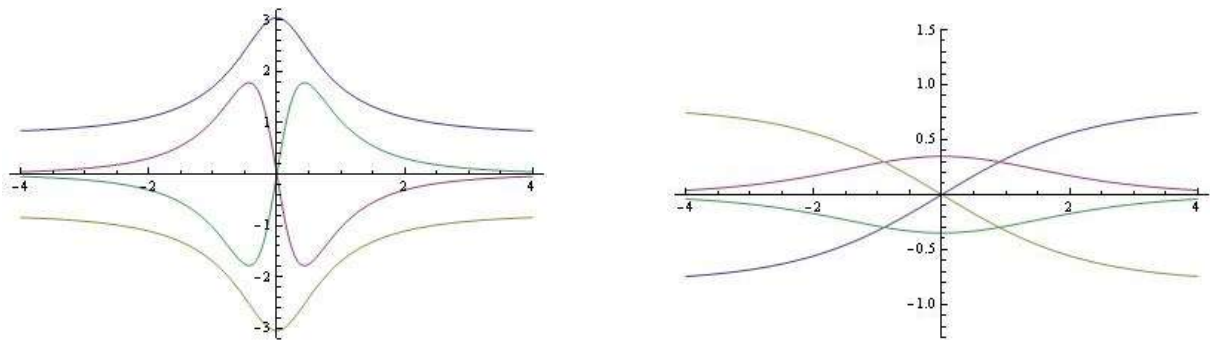
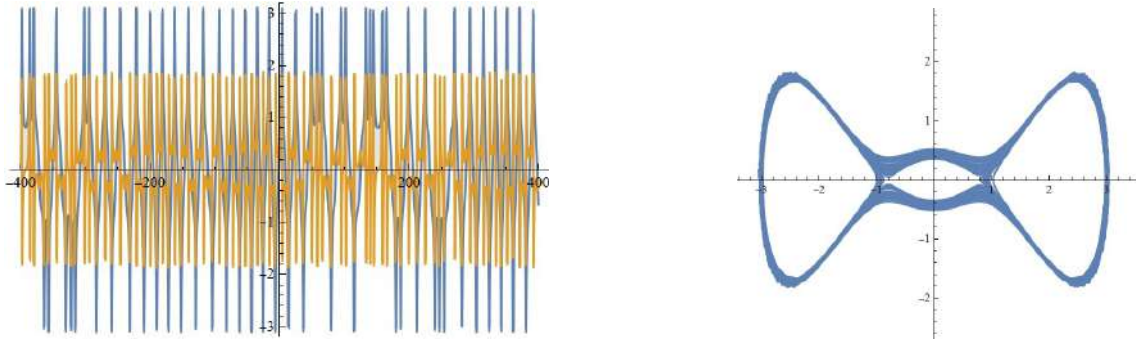


Figure 13. The homo/heteroclinic orbits

**Example 1.8.** For given  $N = 2$ ,  $\omega = 0.3$ ,  $g_1 = 2.9$ ,  $g_2 = 0.8$ ,  $\epsilon = 0.01$  the simulations on the system for  $x_0 = 0.6$ ;  $y_0 = 0.3$  are depicted on Fig. 14.

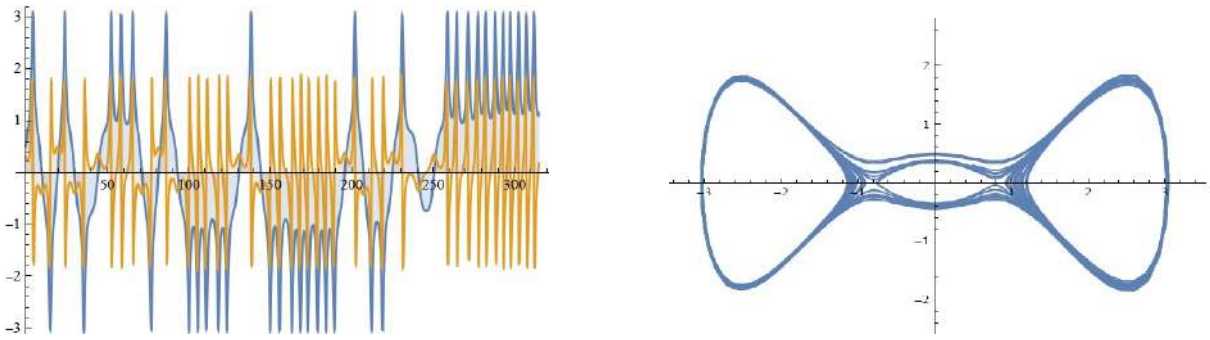


(a) The solutions of the system

(b) Phase plot

 Figure 14. Case  $N = 2$  (Example 8)

**Example 1.9.** For given  $N = 4$ ,  $\omega = 0.1$ ,  $g_1 = 1.9$ ,  $g_2 = 0.2$ ,  $g_3 = 0.1$ ,  $g_4 = 1.6$ ,  $\epsilon = 0.03$  the simulations on the system for  $x_0 = 0.5$ ;  $y_0 = 0.3$  are depicted on Fig. 15.



(a) The solutions of the system

(b) Phase plot

 Figure 15. Case  $N = 4$  (Example 9)

## 2. Concluding Remarks

If  $M(t_0) = 0$  and  $\frac{M(t_0)}{dt_0} \neq 0$  for some  $t_0$  and some sets of parameters, then chaos occurs. From the above statements, the reader can formulate the Melnikov condition for chaotic behavior of the proposed dynamic model (1). Nonstandard numerical methods connected to the investigation of the roots of nonlinear equation  $M(t_0) = 0$  can be found in [11]. The investigations can be included as an integral part of a planned much more general Web-based application for scientific computing [12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

## Acknowledgments

This study is financed by the project No FP23-FMI-002 “Intelligent software tools and applications in research in Mathematics, Informatics,

and Pedagogy of Education” of the Scientific Fund of the Paisii Hilendarski University of Plovdiv, Bulgaria.

### References

- [1] V. Rom-Kedar, A. Poje, Universal properties of chaotic transport in the presence of diffusion, *Physics of Fluids*, 11 (8), 1999, 2044–2057, DOI:10.1063/1.870067.
- [2] V. Rom-Kedar, Transport rates of a class of two-dimensional maps and flows, *Physica D*, 43, 1990, 229–268, DOI: 10.1016/0167-2789(90)90135-C.
- [3] V. Rom-Kedar, The topological approximation method, *Transport, Chaos and Plasma Physics*, World Scientific Press, River Edge, NJ, 1993, 39–57, ISBN: 981-02-1619-X.
- [4] V. Rom-Kedar, Homoclinic tangles – classification and applications, *Nonlinearity*, 7, 1994, 441–473, DOI 10.1088/0951-7715/7/2/008.
- [5] V. Rom-Kedar, Secondary homoclinic bifurcation theorems, *Chaos*, 5 (2), 1995, 385–401, DOI:/10.1063/1.166109.
- [6] V. Rom-Kedar, S. Wiggins, Transport in two-dimensional maps, *Arch. Rat. Mech. Annals*, 109, 1990, 239–298.
- [7] A. Golev, V. Arnaudova, Notes on a modified planar model: intrinsic properties, simulations, *International Journal of Differential Equations and Applications*, 23 (1), 2024, 71–81, DOI: 10.12732/i-jdea.v23i1.6.
- [8] V. Melnikov, On the stability of a center for time-periodic perturbation, *Transactions of the Moscow Mathematical Society*, 12, 1963, 1–57.
- [9] N. Kyurkchiev, A. Andreev, *Approximation and Antenna and Filters synthesis. Some Moduli in Programming Environment MATHEMATICA*, LAP LAMBERT Academic Publishing, Saarbrücken, 2014, 150 pp., ISBN: 978-3-659-53322-8.
- [10] N. Kyurkchiev, Tsv. Zaeviski, A. Iliev, V. Kyurkchiev, A. Rahnev, Modeling of Some Classes of Extended Oscillators: Simulations, Algorithms, Generating Chaos, Open Problems, *Algorithms*, 17 (3), 2024, 121, DOI: 10.3390/a17030121.
- [11] A. Iliev, N. Kyurkchiev, *Nontrivial Methods in Numerical Analysis: Selected Topics in Numerical Analysis*, LAP LAMBERT Academic Publishing, Saarbrücken, 2010, ISBN: 978-3-8433-6793-6.

- [12] A. Golev, T. Terzieva, A. Iliev, A. Rahnev, N. Kyurkchiev, Simulation on a generalized oscillator model: WEB-based application, *C. R. Acad. Bulg. Sci.*, 77 (2), 2024, 230–237, DOI: 10.7546/CRABS.2024.02.08.
- [13] N. Kyurkchiev, Tsv. Zaeovski, A. Iliev, V. Kyurkchiev, A. Rahnev, Nonlinear dynamics of a new class of micro-electromechanical oscillators—open problems, *Symmetry*, 16 (2), 2024, 253, DOI:10.3390/sym16020253.
- [14] N. Kyurkchiev, Tsv. Zaeovski, A. Iliev, V. Kyurkchiev, A. Rahnev, Generating Chaos in Dynamical Systems: Applications, Symmetry Results, and Stimulating Examples, *Symmetry*, 16, 2024, 938, DOI:10.3390/sym16080938.
- [15] N. Kyurkchiev, Tsv. Zaeovski, A. Iliev, V. Kyurkchiev, A. Rahnev, Dynamics of a new class of oscillators: Melnikov's approach, possible application to antenna array theory, *Math. and Inf.*, 2024, 67, 1–15, DOI:10.53656/math2024-4-1-dyn.
- [16] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On some extended oscillator models: a technique for simulating and studying their dynamics*, Plovdiv University Press, Plovdiv, 2022, 138 pp., ISBN: 978-619-7663-13-6.
- [17] N. Kyurkchiev, A. Iliev, V. Kyurkchiev, M. Vasileva, A. Rahnev, Some investigations and simulations on the planar Rayleigh-Lienard differential system (WEB platform upgrade), *International Journal of Differential Equations and Applications*, 22 (1), 2023, 81–102, DOI: 10.12732/ijdea.v22i1.7.
- [18] N. Kyurkchiev, A. Iliev, On the hypothetical oscillator model with second kind Chebyshev's polynomial-correction: type of limit cycles, simulations and possible applications, *Algorithms*, 15 (12), 2022, 462, DOI:10.3390/a15120462.
- [19] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, On a class of orthogonal polynomials as corrections in Lienard differential system. Applications, *Algorithms*, 16 (6), 2023, 297, DOI:10.3390/a16060297.
- [20] N. Kyurkchiev, Tsv. Zaeovski, On a hypothetical oscillator: investigations in the light of the Melnikov's approach, some simulations, *International Journal of Differential Equations and Applications*, 22(1), 2023, 67–79, DOI: 10.12732/ijdea.v22i1.6.
- [21] M. Vasileva, V. Kyurkchiev, A. Iliev, A. Rahnev, Tsv. Zaeovski, N.

Kyurkchiev, *Some investigations and simulations on the generalized Rayleigh systems, Duffing systems with periodic parametric excitation, Mathieu and Hopf oscillators*, Plovdiv, Plovdiv University Press, 2024, ISBN: 978-619-7768-03-9.

Maria Vasileva<sup>1</sup>, Nikolay Kyurkchiev<sup>2</sup>, Tsvetelin Zaevski<sup>3</sup>,  
Anton Iliev<sup>4</sup>, Vesselin Kyurkchiev<sup>5</sup> and Asen Rahnev<sup>6</sup>

<sup>1,2,4,5,6</sup> Paisii Hilendarski University of Plovdiv,

Faculty of Mathematics and Informatics,

236 Bulgaria Blvd., 4027 Plovdiv, Bulgaria

<sup>2,3</sup> Institute of Mathematics and Informatics,

Bulgarian Academy of Sciences,

Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, Bulgaria

<sup>3</sup> Sofia University St. Kliment Ohridski,

Faculty of Mathematics and Informatics,

5, James Bouchier Blvd., Sofia 1164, Bulgaria

Corresponding author: [mariavasileva@uni-plovdiv.bg](mailto:mariavasileva@uni-plovdiv.bg)