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BEST PROXIMITY PAIRS OF NONCYCLIC MAPPINGS WITH A CONTRACTIVE ITERATE ON $\mathbb{P}$ SETS

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Abstract. The well-known Banach contraction principle has a plethora of generalizations. In this paper we have focused our attention on three of them: best proximity pairs of a noncyclic map in a complete metric space with the UC property, the contraction condition holding for some elements in a space with a relation introduced in it, formalized by a $\mathbb{P}$ set, and the mapping being with a contractive iterate at a point. We prove a theorem combining these three generalizations, providing sufficient conditions for the existence and uniqueness of best proximity pairs of noncyclic mappings with a contractive iterate on $\mathbb{P}$ sets. The paper concludes with an illustrative example.

1. Introduction

Ever since the introduction of the Banach contraction principle, the theory of fixed points has been ubiquitous in both theoretical and applied mathematics. It has been used in the solution of a plethora of problems even in its unmodified form. Such results have warranted the generalization of the Banach contraction principle in a multitude of ways.

One way to weaken the principle is to look for a closest element in some sense, instead of a fixed point [3]. If we have a mapping $T : A \cup B \to A \cup B$

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such that $T : A \to B$ and $T : B \to A$ and $A \cap B = \emptyset$, then clearly there would not exist an element $x \in A \cup B$ such that $Tx = x$. Then we can seek an element that is a best proximity point. Seeing as such objects are usually discussed in Banach spaces, a best proximity point would be a point that satisfies the optimization problem $\min_{x\in A\cup B} \|Tx - x\|$. This concept has not only flourished in a theoretical environment but also has found applications, some being in game theory [15] and differential equations [16]. An offspring of this idea is the concept of a best proximity pair when the mapping is such that $T : A \to A$ and $T : B \to B$ [6].

A generalization that has been initiated by [25], but having received recognition after the publication of [20], is the introduction of a partial ordering to the metric space X and the restriction of the contractive condition to only a subset of comparable elements. There are a multitude of results in partially ordered metric spaces [1, 20]. Later such notions have been generalized by replacing the partial ordering with a general relation between the elements in X, represented by a $\mathbb{P}$ set [17].

A way to relax the contractive condition is introduced in [21], where the methodology is to allow for the contractive condition to hold for some iteration of the mapping $T^{n(x)}$ that depends on the specific element $x \in X$, instead of for every iteration.

Recently, a result combining all three of these results has been proven [8], looking at the case where $n(x)$ is an odd natural number. The goal of this paper is to continue research into unifying these ideas, focusing on mappings that can lead to best proximity pairs.

2. Preliminaries

In this paper, we use the following notation: $\mathbb{R}$ for the set of real numbers, $\mathbb{N}$ for the set of the naturals ($\mathbb{N} = \{1, 2, 3, \dots\}$), $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$, $\mathbb{C}$ for the set of complex numbers, $\mathbb{Z}$ for the set of the integers, $\lfloor\cdot\rfloor$ for the floor function ($\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$, as $\lfloor r\rfloor = \max\{n \in \mathbb{Z} : n \le r\}$), (X, ρ) for the metric space X, with metric ρ, $B[x, r]$ for the closed ball in the metric space (X, ρ), with center $x \in X$, and radius $r \ge 0$ (that is, $B[x, r] = \{u \in X : \rho(u, x) \le r\}$), and $B(x, r)$ for the open ball in the metric space (X, ρ), with center $x \in X$, and radius $r \ge 0$ ($B(x, r) = \{u \in X : \rho(u, x) < r\}$).

The study of best proximity points obtained via a mapping of the form $T : A\cup B \to A\cup B$, $T : A \to B$, $T : B \to A$ was pioneered in [3]. In that paper, the authors use the notion of a cyclic contraction map.

Definition 2.1 ([3])**.** Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is a cyclic map if

$$T(A) \subset B \text{ and } T(B) \subset A.$$

Definition 2.2 ([3])**.** Let A and B be nonempty subsets of a metric space X, such that $A \cap B = \emptyset$. We say that $x \in A \cup B$ is a best proximity point if

$$d(x, Tx) = \operatorname{dist}(A, B).$$

Definition 2.3 ([3])**.** Let (X, ρ) be a metric space, A and B be subsets of X. We say that the map $T : A \cup B \to A \cup B$ is a cyclic contraction map if it is a cyclic map and satisfies the inequality

$$\rho(Tx, Ty) \leq \alpha\rho(x, y) + (1 - \alpha)\operatorname{dist}(A, B)$$

for some $\alpha \in (0, 1)$ and every $x \in A$, $y \in B$.

For the purposes of getting an existence and uniqueness result, the uniform convexity of the underlying Banach space is of paramount importance.

Definition 2.4 ([2, 5])**.** Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0, 2]$ we define the modulus of convexity of $\|\cdot\|$ by

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\}.$$

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The space $(X, \|\cdot\|)$ is then called a uniformly convex Banach space.

Whenever the space under consideration is a Banach space $(X, \|\cdot\|)$, we will use the metric induced by the norm, i.e., $\rho(x, y) = \|x - y\|$.

Theorem 2.5 ([3])**.** *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map. Then there exists a unique best proximity point x of T in A.*

However, requiring the Banach space to be uniformly convex is far too restricting. A replacement for the uniform convexity in metric spaces has been introduced in [24].

Definition 2.6 ([24])**.** Let (X, ρ) be a metric space and $A, B \subseteq X$. Let for any three sequences $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \subseteq A$, $\{y_n\}_{n=1}^{\infty} \subseteq B$, so that

- $\lim\limits_{n\to\infty} \rho(x_n, y_n) = \operatorname{dist}(A, B)$,
- $\lim\limits_{n\to\infty} \rho(z_n, y_n) = \operatorname{dist}(A, B)$.

there hold $\lim_{n\to\infty}\rho(x_n,z_n)=0$. Then, we say that the ordered pair (A,B) satisfies the property UC.

Theorem 2.7 ([24]). *Let A and B be nonempty closed subsets of a complete metric space (X,ρ), such that the ordered pairs (A,B) satisfy the property UC. Let $T:A\cup B\to A\cup B$ be a cyclic map and there exist $k\in[0,1)$, so that the inequality*

$$\rho(Tx,Ty)\le k\max\{\rho(x,y),\rho(x,Tx),\rho(y,Ty)\}+(1-k)\mathrm{dist}(A,B)$$

holds for all $x\in A$ and $y\in B$.

Then there is a unique best proximity point x of T in A, the sequence of successive iterations $\{T^{2n}x_0\}_{n=1}^{\infty}$ converges to x for any initial guess $x_0\in A$. There is at least one best proximity point $y\in B$ of T in B. Moreover, the best proximity point $y\in B$ of T in B is unique, provided that the ordered pair (B,A) has the UC property.

Since [24], different concepts have been considered in order to substitute the uniform convexity [10, 22, 23, 24]. In particular, observations made in [12, 13, 27] lead to the obtaining of results for fixed and best proximity points in $(\mathbb{R},|\cdot|)$.

Corollary 2.8 ([12, 13, 27]). *Let A and B be real intervals. Then the ordered pair (A,B) satisfies the property UC.*

Lemma 2.9 ([4]). *Let $A,B\subseteq X$, where (X,ρ) is a metric space, and the ordered pair (A,B) satisfies the property UC. Then, for every $\varepsilon>0$ there is $\delta>0$ so that $\mathrm{diam}(A\cap B[y,\mathrm{dist}(A,B)+\delta])\le\varepsilon$ for any $y\in B$.*

Further connections between these concepts have been established in [26].

One of the first results about fixed points in partially ordered metric spaces, presented in a more sophisticated context, can be found in [25]. However, only after the publication of [20] was interest by the scientific community piqued. In that paper, the contractive condition $\rho(Tx,Ty)\le k\rho(x,y)$ is modified in such a way as to be valid only for $x\preccurlyeq y$.

Theorem 2.10 ([20]). *Let $(X,d,\preccurlyeq)$ be a partially ordered complete metric space and $f:X\to X$ be a continuous, monotone (i.e., either order preserving or order reversing) map, such that there is $k\in[0,1)$ so that the inequality*

$$d(Tx,Ty)\le kd(x,y)$$

holds true for arbitrary $x,y\in X$, satisfying $x\succcurlyeq y$. A fixed point $\xi\in X$ of T exists if there is $x_0\in X$ such that either $x_0\preccurlyeq fx_0$ or $x_0\succcurlyeq fx_0$.

The fixed point ξ will be unique if each pair of elements $x,y\in X$ possesses a lower bound or an upper bound.

Introducing a partial order and restricting the contractive condition to comparable elements in some manner has been generalized by the introduction of $\mathbb{P}$ sets ($\mathbb{P} \subset X \times X$) in [17]. In a short amount of time, a series of articles focuses on coupled fixed points, where $\mathbb{P}$ sets are subsets of X^4 [17, 18, 19]. Further research into these ideas has been conducted in [7, 9].

An especially important concept for proving such results is stated in the following definition.

Definition 2.11 ([19])**.** Let (X, d) be a metric space, $\mathbb{P} \subset X \times X$ and $F : X \to X$ be a mapping. $\mathbb{P}$ is called F-closed if

$$(x, y) \in \mathbb{P} \Rightarrow (F(x), F(y)) \in \mathbb{P}.$$

We will present some well-known examples [17].

Example 2.12. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. Let $F : X \to X$ be an increasing function, i.e., $F(x) \preccurlyeq F(y)$, provided that $x \preccurlyeq y$. Then the set $\mathbb{P} = \{(x, y) \in X \times X : x \preccurlyeq y\}$ is F-closed.

Example 2.13. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. For $F : X \to X$ let $F(x)$ be comparable with $F(y)$, i.e., $F(x) \asymp F(y)$. Then the set $\mathbb{P} = \{(x, y) \in X \times X : x \asymp y\}$ is F-closed.

In [21] we observe a different relaxation of the Banach contraction principle.

Theorem 2.14 ([21])**.** *Let X be a Banach space, and $T : X \to X$ a continuous mapping satisfying the condition: there exists a constant $\alpha \in (0, 1)$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$*

$$\rho(T^{n(x)}y, T^{n(x)}x) \leq \alpha\rho(y, x).$$

Then T has a unique fixed point z and $\lim_{s\to\infty} T^s x = z$ for each $x \in X$.

Later, the maps introduced in [21] have been named maps iterated at a point. Further developments of this idea have been presented in [8, 9, 11, 14].

All of these ideas have been unified in [8].

Definition 2.15 ([17])**.** Let (X, d) be a metric space. We say that two sequences $\{x_n\}$, $\{y_n\} \subset X$ are Cauchy equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 2.16 ([8])**.** Let (X, d) be a metric space, $A, B \subset X$ and $\mathbb{P} \subset A \times B$, $A \cap B = \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence such that $x_{2n} \in A$ and $x_{2n+1} \in B$. The triple $(A \cup B, d, \mathbb{P})$ is said to be:

- cyclically e-$\mathbb{P}$-regular if for any sequence $\{x_{2n}\}$, convergent to x^*, such that $(x_{2n}, x_{2n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$, there holds $(x^*, x_{2n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N} \cup \{0\}$

- cyclically o-$\mathbb{P}$-regular if for any sequence $\{x_{2n+1}\}$, convergent to x^*, such that $(x_{2n}, x_{2n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$, there holds $(x_{2n}, x^*) \in \mathbb{P}$, for all $n \in \mathbb{N} \cup \{0\}$.

Definition 2.17 ([8])**.** Let X be a non-empty set, $A, B \subset X$, $\mathbb{P} \subset A \times B$ and let $T : A \cup B \to A \cup B$ be a cyclic map. We say that $\mathbb{P}$ is cyclically T-closed if

$$(x, y) \in \mathbb{P} \Rightarrow (Ty, Tx) \in \mathbb{P}.$$

Definition 2.18 ([8])**.** We say that $\mathbb{P}$ has the cyclically transitive property if from $(x, y), (z, y), (z, u) \in \mathbb{P}$ it follows that $(x, u) \in \mathbb{P}$.

Theorem 2.19 ([8])**.** *Let (X, d) be a complete metric space, $A, B \subset X$ be nonempty such that $A \cap B = \emptyset$, the pair (A, B) have the UC property, $\mathbb{P} \subset A \times B$, $T : A \cup B \to A \cup B$ be a cyclic map and there hold*

(i) *$\mathbb{P}$ is cyclically T-closed and has the cyclically transitive property;*
(ii) *the triple $(A \cup B, d, \mathbb{P})$ is cyclically* e-$\mathbb{P}$*-regular;*
(iii) *there exists $x_0 \in A$ such that $(x_0, Tx_0) \in \mathbb{P}$;*
(iv) *there exists $k \in [0, 1)$ such that for all $x \in A \cup B$ there is $n(x) \equiv 1 \pmod 2 \in \mathbb{N}$, such that for all $y \in A \cup B$, where (x, y) or $(y, x) \in \mathbb{P}$, we have*

$$d(T^{n(x)}(x), T^{n(x)}(y)) \leq kd(x, y) + (1 - k)\text{dist}(A, B). \tag{2.1}$$

Then there exists a best proximity point x^* in A and for any arbitrarily chosen $x_0 \in A$, such that $(x_0, Tx_0) \in \mathbb{P}$ the iterated sequence $x_{2n} = T^{2n}x_0$ converges to a best proximity point. Furthermore, x^* is a fixed point of T^2. Moreover, there hold

(a) *for any $x \in A$ such that $(x_0, Tx) \in \mathbb{P}$ or $(x, Tx_0) \in \mathbb{P}$, the sequences $x_{2n} = T^{2n}x_0$ and $u_{2n} = T^{2n}x$ are Cauchy equivalent and hence u_{2n} converges to the same point x^*;*
(b) *if $y^* \in A$ is a best proximity point and either*

$$(x_0, Ty^*) \in \mathbb{P} \text{ or } (y^*, Tx_0) \in \mathbb{P},$$

or there exists $z \in A$ so that

$$(x_0, Tz), (y^*, Tz) \in \mathbb{P} \text{ or } (z, Tx_0), (z, Ty^*) \in \mathbb{P},$$

then $y^* = x^*$;
(c) *if additionally we suppose that for every $x, y \in A$ such that neither $(x, Ty) \in \mathbb{P}$ or $(y, Tx) \in \mathbb{P}$ there is $z \in A$ so that either*

$$(x, Tz), (y, Tz) \in \mathbb{P} \text{ or } (z, Tx), (z, Ty) \in \mathbb{P},$$

then x^* is the unique proximity point.

A very similar concept to best proximity points is a best proximity pair.

Definition 2.20 ([6]). Let A and B be nonempty subsets of a metric space X, such that $A \cap B = \emptyset$. A mapping $T : A \cup B \to A \cup B$ is called noncylcic if

$$T(A) \subseteq A, T(B) \subseteq B.$$

Definition 2.21 ([6]). Let A and B be nonempty subsets of a metric space X, such that $A \cap B = \emptyset$. We say that $(x, y) \in (A, B)$ is a best proximity pair for the noncyclic mapping $T : A \cup B \to A \cup B$ if $Tx = x$, $Ty = y$, $d(x, y) = \text{dist}(A, B)$.

It is the aim of this paper to extend the results for mappings with a contractive iterate at a point on $\mathbb{P}$ sets by considering such maps in the context of best proximity pairs.

3. Main Results

Definition 3.1. Let A and B be sets, $\mathbb{P} \subseteq A \times B$ and $T : A \cup B \to A \cup B$. We say that $\mathbb{P}$ is T-expansive if for every $(x, y) \in \mathbb{P}$, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, there holds $(T^n x, T^m y) \in \mathbb{P}$.

Definition 3.2. Let (X, ρ) be a metric space, A and B be subsets of X, $\mathbb{P} \subseteq A \times B$, $T : A \cup B \to A \cup B$ be a noncyclic map. Let there exist $\lambda \in [0; 1)$, such that for every $(a, b) \in \mathbb{P}$, there is $n_A(a) \in \mathbb{N}$, and $n_B(b) \in \mathbb{N}$, so that

$$\rho(T^{n_A(a)} a, T^{n_A(a)} b) \leq \lambda \rho(a, b) + (1 - \lambda)\text{dist}(A, B)$$

and

$$\rho(T^{n_B(b)} a, T^{n_B(b)} b) \leq \lambda \rho(a, b) + (1 - \lambda)\text{dist}(A, B).$$

Then, we say that T is a noncyclic map with a contractive iterate on $\mathbb{P}$.

Let us first state a result that addresses the behavior of T on the subset A.

Theorem 3.3. *Let (X, ρ) be a complete metric space, A and B be subsets of X. Let T be a noncyclic map with a contractive iterate on $\mathbb{P}$, $\mathbb{P}$ be T-expansive, and the ordered pair (A, B) satisfy the property UC. Let there exist $a \in A$ and $b \in B$ such that $(a, b) \in \mathbb{P}$. Then:*

- *(i) The sequence $T^n a$ is Cauchy and $\lim_{n\to\infty} \rho(T^n a, T^n b) = \text{dist}(A, B)$. If in addition $\lim\limits_{n\to\infty} T^n a = z$ and $(z, b) \in \mathbb{P}$, then $Tz = z$.*
- *(ii) Additionally, if $(c, b) \in \mathbb{P}$, then the sequences $T^n a$ and $T^n c$ are Cauchy equivalent.*
- *(iii) If $\alpha \in A$ is such that $T\alpha = \alpha$ there exists $\zeta \in B$ such that $(a, \zeta), (\alpha, \zeta) \in \mathbb{P}$, then $z = \alpha$.*

(iv) *If additionally we suppose that for every* $a_1, a_2 \in A$ *there is* $\zeta \in B$ *such that* $(a_1, \zeta), (a_2, \zeta) \in \mathbb{P}$*, then there exists a unique fixed point* z *of* T *in* A *such that* $\lim_{n\to\infty} \rho(z, T^n\beta) = \operatorname{dist}(A, B)$ *for* $(\cdot, \beta) \in \mathbb{P}$*.*

As is shown below, (A, B) having the UC property is paramount for establishing that $T^n a$ is Cauchy. However, the sequence $T^n b$ may not even be convergent. If we introduce one more condition (mainly, the pair (B, A) having the UC property), we can prove the following result for best proximity pairs.

Theorem 3.4. *Let* (X, ρ) *be a complete metric space,* A *and* B *be subsets of* X*. Let* T *be a noncyclic map with a contractive iterate on* $\mathbb{P}$*,* $\mathbb{P}$ *be* T*-expansive, and the ordered pairs* (A, B) *and* (B, A) *satisfy the property* UC*. Let there exist* $a \in A$ *and* $b \in B$ *such that* $(a, b) \in \mathbb{P}$*. Then:*

(I) *The sequences* $T^n a$*,* $T^n b$ *are Cauchy. If in addition* $\lim\limits_{n\to\infty} T^n a = z$ *and* $(z, b) \in \mathbb{P}$*, then* $Tz = z$*. Similarly, if in addition* $\lim\limits_{n\to\infty} T^n b = w$ *and* $(a, w) \in \mathbb{P}$*, then* $Tw = w$*. If both hold, then* (z, w) *is a best proximity pair for* T*.*

(II) *Additionally, if* $(c, b) \in \mathbb{P}$*, then the sequences* $T^n a$ *and* $T^n c$ *are Cauchy equivalent. Similarly, if* $(a, d) \in \mathbb{P}$*, then the sequences* $T^n b$ *and* $T^n d$ *are Cauchy equivalent.*

(III) *If we also have another best proximity pair* (α, β) *and either* (z, β) *or* (α, w) *is in* $\mathbb{P}$*, or there exists* $\zeta \in A \cup B$ *such that* $(a, \zeta), (\alpha, \zeta) \in \mathbb{P}$ *or* $(\zeta, b), (\zeta, \beta) \in \mathbb{P}$*, then* $(z, w) = (\alpha, \beta)$*.*

(IV) *If additionally we suppose that for every* $a_1, a_2 \in A, b_1, b_2 \in B$ *there is* $\omega \in A, \zeta \in B$ *so that* $(a_1, \zeta), (a_2, \zeta) \in \mathbb{P}$ *and* $(\omega, b_1), (\omega, b_2) \in \mathbb{P}$*, then the best proximity pair is unique.*

4. Auxiliary Results

Lemma 4.1. *Let* (X, ρ) *be a metric space,* A *and* B *be subsets of* X*,* $T : A\cup B \to A\cup B$ *be a noncyclic map with a contractive iterate on* $\mathbb{P} \subseteq A\times B$*, and* $\mathbb{P}$ *be* T*-expansive. Then, the sequences* $\{T^n a\}_{n=0}^{\infty}$ *and* $\{T^n b\}_{n=0}^{\infty}$ *are bounded for every* $(a, b) \in \mathbb{P}$*.*

Proof. By assumption, T is a noncyclic map with a contractive iterate on $\mathbb{P}$, and $\mathbb{P}$ is T-expansive. Thus, there is $n_B(b) \in \mathbb{N}$, such that for every $n \geq n_B(b)$ there holds

$$\begin{aligned}
\rho(T^n a, T^{n_B(b)} b) &\leq \lambda\rho(T^{n-n_B(b)} a, b) + (1-\lambda)\operatorname{dist}(A, B) \\
&\leq \lambda\rho(T^{n-n_B(b)} a, T^{n_B(b)} b) + (1-\lambda)\operatorname{dist}(A, B) \\
&\quad +\lambda\rho(T^{n_B(b)} b, b).
\end{aligned}$$

Applying the last inequality, $p = \left\lfloor \dfrac{n}{n_B(b)} \right\rfloor$ times consecutively, we get

$$\begin{aligned}
& \rho(T^n a, T^{n_B(b)} b) \\
\leq\ & \lambda \rho(T^{n-n_B(b)} a, T^{n_B(b)} b) + (1-\lambda)\mathrm{dist}(A,B) + \lambda \rho(T^{n_B(b)} b, b) \\
\leq\ & \lambda^2 \rho(T^{n-2n_B(b)} a, T^{n_B(b)} b) + (1-\lambda^2)\mathrm{dist}(A,B) + \lambda \rho(T^{n_B(b)} b, b) \\
& + \lambda^2 \rho(T^{n_B(b)} b, b) \\
\leq\ & \lambda^3 \rho(T^{n-3n_B(b)} a, T^{n_B(b)} b) + (1-\lambda^3)\mathrm{dist}(A,B) + \lambda \rho(T^{n_B(b)} b, b) \\
& + \lambda^2 \rho(T^{n_B(b)} b, b) + \lambda^3 \rho(T^{n_B(b)} b, b) \\
& \cdots \\
\leq\ & \lambda^p \rho(T^{n-pn_B(b)} a, T^{n_B(b)} b) + (1-\lambda^p)\mathrm{dist}(A,B) + \sum_{k=1}^{p} \lambda^k \rho(T^{n_B(b)} b, b).
\end{aligned}$$

Therefore, for every $n \in \mathbb{N}$, the inequality

$$\rho(T^n a, T^{n_B(b)} b) \leq \max_{0 \leq k < n_B(b)} \rho(T^k a, T^{n_B(b)} b) + \mathrm{dist}(A,B) + \frac{\lambda}{1-\lambda} \rho(T^{n_B(b)} b, b)$$

is true. Thus, $\{T^n a\}_{n=0}^{\infty}$ is bounded. By a similar argument, we obtain that $\{T^n b\}_{n=0}^{\infty}$ is also bounded. □

Lemma 4.2. *Let (X,ρ) be a metric space, $A, B \subseteq X$, $T : A \cup B \to A \cup B$ be a noncyclic map with a contractive iterate on $\mathbb{P} \subseteq A \times B$, and $\mathbb{P}$ be T-expansive. Let $(a,b) \in \mathbb{P}$ and the sequences $\{q_n^a\}_{n=0}^{\infty}, \{q_n^b\}_{n=0}^{\infty} \subseteq \mathbb{N}_0$ be defined as $q_0^a = q_0^b = 0$, $q_{n+1}^a = q_n^a + n_A(T^{q_n^a} a)$, $q_{n+1}^b = q_n^b + n_B(T^{q_n^b} b)$. Then,*

- *(i)* $\lim_{n\to\infty} \sup_{i \geq q_n^b} \rho(T^i a, T^{q_b^n} b) = \mathrm{dist}(A,B)$.
- *(ii)* $\lim_{n\to\infty} \sup_{i \geq q_n^a} \rho(T^{q_a^n} a, T^i b) = \mathrm{dist}(A,B)$.
- *(iii) If the ordered pair (A,B) satisfies the property UC, then the sequence $\{T^n a\}_{n=0}^{\infty}$ is Cauchy.*

Proof. Using Lemma 4.1, it follows that $\sup_{n \in \mathbb{N}_0} \rho(T^n a, b) = M < \infty$.

By assumption, T is a noncyclic map with a contractive iterate on $\mathbb{P}$ and $\mathbb{P}$ is T-expansive. Therefore, for every $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$, the inequality

$$\rho(T^{q_n^b+k} a, T^{q_n^b} b) \leq \lambda \rho(T^{q_{n-1}^b+k} a, T^{q_{n-1}^b} b) + (1-\lambda)\mathrm{dist}(A,B)$$

is true. Thus, for any $n \in \mathbb{N}_0$, there holds

$$\begin{aligned}
\sup_{i\geq q_n^b} \rho(T^i a, T^{q_n^b} b) &\leq \lambda \sup_{i\geq q_{n-1}^b} \rho(T^i a, T^{q_{n-1}^b} b) + (1-\lambda)\mathrm{dist}(A,B) \\
&\leq \lambda^2 \sup_{i\geq q_{n-2}^b} \rho(T^i a, T^{q_{n-2}^b} b) + (1-\lambda^2)\mathrm{dist}(A,B) \\
&\quad \dots \\
&\leq \lambda^{n-1} \sup_{i\geq q_1^b} \rho(T^i a, T^{q_1^b} b) + (1-\lambda^{n-1})\mathrm{dist}(A,B) \\
&\leq \lambda^{n} \sup_{i\leq \mathbb{N}_0} \rho(T^i a, b) + (1-\lambda^{n})\mathrm{dist}(A,B) \\
&\leq \lambda^n M + (1-\lambda^n)\mathrm{dist}(A,B).
\end{aligned} \tag{4.1}$$

By assumption, $T(A) \subseteq A$, $T(B) \subseteq B$, $a \in A$, and $b \in B$. Consequently, $\sup_{i\geq q_n^b} \rho(T^i a, T^{q_n^b} b) \geq \mathrm{dist}(A,B)$, for each $n \in \mathbb{N}$. Using the last inequality and (4.1), we can observe that

$$\lim_{n\to\infty} \sup_{i\geq q_n^b} \rho(T^i a, T^{q_n^b} b) = \mathrm{dist}(A,B). \tag{4.2}$$

By similar arguments, we can see that $\lim_{n\to\infty} \sup_{i\geq q_n^a} \rho(T^{q_n^a} a, T^i b) = \mathrm{dist}(A,B)$.

If the ordered pair (A,B) satisfies the property UC, from the limit (4.2) and $\{T^n a\}_{n=0}^\infty \subseteq A$, it follows that for every $\delta > 0$ there is $N \in \mathbb{N}$, so that $\{T^n a\}_{n=q_N^b}^\infty \subseteq A \cap B[T^{q_N^b} b, \mathrm{dist}(A,B)+\delta]$, i.e.,

$$\mathrm{diam}(\{T^n a\}_{n=q_N^b}^\infty) \leq \mathrm{diam}(A \cap B[T^{q_N^b} b, \mathrm{dist}(A,B)+\delta]).$$

Using the last inequality and Lemma 2.9, we obtain that for every ε there is $N \in \mathbb{N}$, such that $\mathrm{diam}(\{T^n a\}_{n=q_N^b}^\infty) \leq \varepsilon$. Hence, the sequence $\{T^n a\}_{n=0}^\infty$ is Cauchy. □

Lemma 4.3. *Let (X,ρ) be a metric space, $A, B \subseteq X$, $T : A\cup B \to A \cup B$ be a noncyclic map with a contractive iterate on $\mathbb{P} \subseteq A \times B$ and $\mathbb{P}$ be T-expansive. Let $(a,b) \in \mathbb{P}$. Then, $\lim_{n\to\infty} \rho(T^n a, T^n b) = \mathrm{dist}(A,B)$.*

Proof. Let the sequence $\{q_n\}_{n=0}^\infty \subseteq \mathbb{N}_0$ be defined recursively as $q_0 = 0$, $q_{n+1} = q_n + n_A(T^{q_n} a)$.

By Lemma 4.2 (ii) and (iii), it follows that

- For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that if $m \geq N$ and $i \geq q_m$, then $\mathrm{dist}(A,B) - \varepsilon \leq \rho(T^{q_m} a, T^i b) \leq \mathrm{dist}(A,B) + \varepsilon$.
- For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that if $m \geq N$ and $i \geq q_m$, then $0 \leq \rho(T^{q_m} a, T^i a) \leq \varepsilon$.

Therefore, for every ε, there is $N \in \mathbb{N}$, such that if $m \geq N$ and $i \geq q_m$, it holds that

$$\begin{aligned}\operatorname{dist}(A,B) - 2\varepsilon &\leq \rho(T^{q_m}a, T^i b) - \rho(T^{q_m}a, T^i a)\\ &\leq \rho(T^i a, T^i b) \leq \rho(T^{q_m}a, T^i b) + \rho(T^{q_m}a, T^i a)\\ &\leq \operatorname{dist}(A,B) + 2\varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \rho(T^n a, T^n b) = \operatorname{dist}(A,B)$. □

Lemma 4.4. *Let (X,ρ) be a complete metric space, $A, B \subseteq X$, $T : A \cup B \to A \cup B$ be a noncyclic map with a contractive iterate on $\mathbb{P} \subseteq A \times B$, and $\mathbb{P}$ be T-expansive. Let $(a,b), (c,b) \in \mathbb{P}$. Then, the sequences $T^n a$ and $T^n c$ are Cauchy equivalent and $\lim_{n\to\infty} T^n a = \lim_{n\to\infty} T^n c = z$.*

Proof. From Lemma 4.2 (iii) and the assumption that (X,ρ) is complete, we get

$$\lim_{n\to\infty} T^n a = a' \in X \text{ and } \lim_{n\to\infty} T^n c = c' \in X. \tag{4.3}$$

Let the sequence $\{q_n\}_{n=0}^{\infty} \subseteq \mathbb{N}_0$ be defined as $q_0 = 0$, $q_{n+1} = q_n + n_B(T^{q_n}b)$. Then, from Lemma 4.2 (i), we obtain that $\lim_{n\to\infty} \rho(T^{q_n}a, T^{q_n}b) = \operatorname{dist}(A,B)$ and $\lim_{n\to\infty} \rho(T^{q_n}c, T^{q_n}b) = \operatorname{dist}(A,B)$. Using the last two limits and the assumption that the ordered pair satisfies the property UC, it follows that

$$\lim_{n\to\infty} \rho(T^{q_n}a, T^{q_n}c) = 0,$$

that is, the sequences $T^n a$ and $T^n c$ are Cauchy equivalent. From (4.3) and the continuity of the metric, we can conclude that

$$\rho(a', c') = \lim_{n\to\infty} \rho(T^n a, T^n c) = \lim_{n\to\infty} \rho(T^{q_n}a, T^{q_n}c) = 0,$$

i.e., $a' = c'$. □

Lemma 4.5. *Let (X,ρ) be a complete metric space, $A, B \subseteq X$, $T : A \cup B \to A \cup B$ be a noncyclic map with a contractive iterate on $\mathbb{P} \subseteq A \times B$, and $\mathbb{P}$ be T-expansive. Let $(a,b) \in \mathbb{P}$, $\lim_{n\to\infty} T^n a = z$, and $(z,b) \in \mathbb{P}$. Then $Tz = z$.*

Proof. By Lemma 4.4, the assumption that $\lim_{n\to\infty} T^n a = z$ and the continuity of the metric, we obtain

$$\lim_{n\to\infty} \rho(z, T^n b) = \operatorname{dist}(A,B). \tag{4.4}$$

By assumption $(z,b) \in \mathbb{P}$, therefore $z \in A$. T is a noncyclic map with a contractive iterate on $\mathbb{P}$ and $\mathbb{P}$ is T-expansive. Consequently,

$$\operatorname{dist}(A,B) \leq \rho(T^{n_A(z)}z, T^{n_A(z)+k}b) \leq \lambda(z, T^k b) + (1-\lambda)\operatorname{dist}(A,B)$$

for any $k \in \mathbb{N}$. From the last inequality and (4.4), we can observe that $\lim_{n\to\infty} \rho(T^{n_A(z)}z, T^n b) = \operatorname{dist}(A,B)$. By the last limit, (4.4) and the ordered pair satisfying the property UC, it follows that $\lim_{n\to\infty} \rho(z, T^{n_A(z)}z) = 0$. Thus, $z = T^{n_A(z)}z$.

Let us suppose that $Tz = z' \neq z$. Then $T^{kn_A(z)+1}z = z'$ for every $k \in \mathbb{N}_0$. Then, the equality

$$\lim_{n\to\infty} T^n z = z \tag{4.5}$$

does not hold. Using $(a,b), (z,b) \in \mathbb{P}$, $\lim_{n\to\infty} T^n a = z$ and Lemma 4.3, it follows that $\lim_{n\to\infty} T^n z = \lim_{n\to\infty} T^n a = z$. The last limit contradicts (4.5). Therefore, $Tz = z$. □

5. Proof of the Main Result

5.1. **Proof of Theorem 3.3.** (i) All of the conditions of Lemma 4.2 are fulfilled. Therefore, the sequence $T^n a$ is Cauchy. From Lemma 4.3 we get that $\lim_{n\to\infty} \rho(T^n a, T^n b) = \operatorname{dist}(A,B)$. Additionally, due to (X,ρ) being a complete metric space, if $\lim_{n\to\infty} T^n a = z$ and $(z,b) \in \mathbb{P}$, then from Lemma 4.5 we get that $Tz = z$.

(ii) This immediately follows from Lemma 4.4.

(iii) Let α be such that $T\alpha = \alpha$ and there exists $\zeta \in B$ such that the inclusion $(a,\zeta), (\alpha,\zeta) \in \mathbb{P}$ holds. Then from (II) we can conclude that $T^n z$ and $T^n \alpha$ are Cauchy equivalent, that is,

$$\lim_{n\to\infty} \rho(T^n z, T^n \alpha) = \lim_{n\to\infty} \rho(z,\alpha) = \rho(z,\alpha) = 0,$$

or $z = \alpha$.

(iv) Let $a_1, a_2 \in A$ and $\zeta \in B$ be such that $(a_1,\zeta), (a_2,\zeta) \in \mathbb{P}$. From (i) we get that $T^n a_1 \to z_1$ and $T^n a_2 \to z_2$, where $z_1, z_2 \in A$ such that $\rho(T^n a_i, T^n \zeta) = \operatorname{dist}(A,B), Tz_i = z_i, i = 1,2$. From (ii) we can conclude that $z_1 = z_2$. □

5.2. **Proof of Theorem 3.4.** Since the subsets $A, B \subseteq X$ have symmetric roles, we can use Theorem 3.3 to make conclusions for elements from both subsets.

(I) From (i) the sequences $T^n a$, $T^n b$ are Cauchy. If $\lim_{n\to\infty} T^n a = z$ and $(z,b) \in \mathbb{P}$, then $Tz = z$. Similarly, if $\lim_{n\to\infty} T^n b = w$ and $(a,w) \in \mathbb{P}$, then $Tw = w$. Having both of them hold leads to $Tz = z$, $Tw = w$ and

$$\rho(z,w) = \lim_{n\to\infty} \rho(T^n a, T^n b) = \operatorname{dist}(A,B).$$

Thus, the pair (z,w) is a best proximity pair for T.

(II) This immediately follows from (ii).

(III) Let (α,β) be another best proximity pair. Without loss of generality, let there exist $\zeta\in B$ such that $(z,\zeta),(\alpha,\zeta)\in\mathbb{P}$. Then from (iii) $z=\alpha$ and $w=\beta$. Consequently, $(z,w)=(\alpha,\beta)$.

(IV) This follows immediately from (iv). □

6. Illustrative Example

Example 6.1. Let us consider the metric space $(\mathbb{R},|\cdot|)$. Let $A=[0,\infty)$, $B=(-\infty,-1]$ and $\mathbb{P}=A\times B$. Let $\beta(x)=\left\lceil\frac{1}{x}\right\rceil \pmod 2$ and the mapping $T:A\cup B\to A\cup B$ be defined as

$$Tx=\begin{cases}\dfrac{1}{\left\lceil\frac{1}{x}\right\rceil-1} & : x\in A\setminus\{0\} \text{ and } \beta(x)=0\\[2ex] \dfrac{x}{4} & : x=0 \text{ or } x\in A \text{ and } \beta(x)=1\\[2ex] \dfrac{-1}{\left\lceil\frac{-1}{1+x}\right\rceil-1}-1 & : x\in B\setminus\{-1\} \text{ and } \beta(-x-1)=0\\[2ex] \dfrac{x+1}{4}-1 & : x=-1 \text{ or } x\in B \text{ and } \beta(-x-1)=1.\end{cases}$$

Let us first note that $\operatorname{dist}(A,B)=1$.

By Corollary 2.8, we get that both ordered pairs (A,B) and (B,C) satisfy the property UC.

It can be shown that $T(A)\subseteq A$ and $T(B)\subseteq B$. Additionally, $\mathbb{P}=A\times B$. Thus, $\mathbb{P}$ is T-expansive.

It is the case that for each $(x,y)\in\mathbb{P}$ there holds

$$\left|\begin{array}{l} T^2x\le\dfrac{x}{2}\\[3ex] T^2y\ge\dfrac{y+1}{2}-1.\end{array}\right.$$

From the last system, $T^2x\in A$, $T^2y\in B$, $x\in A$ and $y\in B$ it follows that for every $(x,y)\in\mathbb{P}$

$$|T^2x-T^2y|\le\left|\frac{x}{2}-\left(\frac{y+1}{2}-1\right)\right|\le\frac{1}{2}|x-y|+\frac{1}{2},$$

i.e.,

$$|T^2x-T^2y|\le\frac{1}{2}|x-y|+\frac{1}{2}\operatorname{dist}(A,B).$$

Thus, T is a noncyclic map with a contractive iterate on $\mathbb{P}$ with $\lambda = \frac{1}{2}$, $n_A(x) = 2$ for each $x \in A$ and $n_B(x) = 2$ for each $x \in B$. Thus, using the facts that the ordered pairs (A, B) and (B, C) satisfy the property UC, $\mathbb{P}$ being T-expansive and $(\mathbb{R}, |\cdot - \cdot|)$ being complete, we can apply Theorem 3.4.

Clearly, from $\mathbb{P} = A \times B$, the conditions of (I), (II), (III) and (IV) are fulfilled. Therefore, there exists a unique best proximity pair. One can easily confirm that $T0 = 0, T(-1) = -1$ and $|0 - (-1)| = \operatorname{dist}(A, B)$. Thus, $(0, -1)$ is the unique best proximity pair.

As a final remark, from (II) it follows that 0 is the unique fixed point of T in A and -1 is the only fixed point of T in B.

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ДВОЙКИ ТОЧКИ НА НАЙ-ДОБРО ПРИБЛИЖЕНИЕ ЗА НЕЦИКЛИЧНИ ИЗОБРАЖЕНИЯ СЪС СВИВАЩИ ИТЕРАЦИИ ВЪРХУ $\mathbb{P}$ МНОЖЕСТВО

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Резюме. Добре познатият принцип на Банах за свиващите изображения има множество обобщения. В тази статия сме фокусирали вниманието си върху три от тях: двойки точки на най-добро приближение за нециклично изображение в пълно метрично пространство със свойството UC, условие за свиване, валидно за някои елементи в пространството с въведена в него релация, формализирана от $\mathbb{P}$ множество, и изображението да има свиващи итерации в точка. Доказваме теорема, комбинираща тези три обобщения, предоставяйки достатъчни условия за съществуване и единственост на двойки точки на най-добро приближение за нециклични изображения със свиващи итерации върху $\mathbb{P}$ множества. Статията завършва с илюстративен пример.