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BOUNDARY VALUE PROBLEMS FOR DIFFERENCE EQUATIONS AND ULAM–HYERS STABILITY

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Abstract. Ulam–Hyers stability for boundary value problem for difference equations is discussed and some misunderstandings in the literature are pointed out. A new approach is suggested to avoid the commented main misunderstanding. The classical difference inequality is combined with an inequality for the boundary condition. Note that this approach does not keep the main idea of Ulam type stability because the solution of the studied boundary value problem is fixed and it has to be close to any solution of the defined inequalities. In connection with this we define a modified Ulam–Hyers stability and sufficient conditions are obtained. The results are illustrated on several examples.

1. Introduction

Ulam-type stability results are useful in many applications in numerical analysis, optimization, etc., where finding the exact solution is quite difficult. The main idea in the application of Ulam type stability to initial value problems is that the initial value of the studied initial value problem depends on the particular chosen solution of the corresponding inequality (see, for example, [1], [3], [5], [6], [7], [10]). Unfortunately, this idea is not used in the application to some

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initial value problems for delay equations as well as for boundary value problems (BVP) and there are some misunderstandings in some published papers, such as [9] for delays, [2], [4], [8] for BVP for fractional delay differential equations. Inspired by above, we consider a scalar nonlinear difference equation with a boundary conditions. We suggest a correct way to study Ulam–Hyers stability for the given BVP by adding to the corresponding inequality an inequality concerning the boundary conditions. We define a modified Ulam–Hyers stability which main point is the application of the fixed unique solution of the studied BVP and its closeness to any solution of the defined different inequalities. We provide several examples to discuss the motivation of the studied problem as well as to illustrate the application of the suggested approach to a particular BVP. The examples are solved by the help of Wolfram Mathematica.

2. Statement of the problem and definition of solution

Let $\mathbb{Z}_+$ be the set of all nonegative integers; $a, b \in \mathbb{Z}_+ : a < b < \infty$; $\mathbb{Z}[a,b] = \{z \in \mathbb{Z}_+ : a \leq z \leq b\}$.

Consider the nonlinear difference equation

$$x(n) = f(n, x(n-1), x(n)) \text{ for } n \in \mathbb{Z}[a+1, b], \tag{1}$$

with a boundary condition

$$\alpha x(a) + \beta x(b) = \gamma, \tag{2}$$

where $f : \mathbb{Z}[a+1,b] \times \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, \gamma \in \mathbb{R}, \alpha \neq 0$ are given constants.

Remark 2.1. Note that partial cases of the boundary condition (2) are the initial value problem ($\alpha = 1$, $\beta = 0$), the periodic boundary condition ($\alpha = 1$, $\beta = -1$, $\gamma = 0$), the anti-periodic boundary condition ($\alpha = 1$, $\beta = 1$, $\gamma = 0$).

Usually, the difference equation describes the development of a certain phenomenon by recursively defining a sequence, each of whose terms is defined as a function of the preceding terms, once one or more initial terms are known (see, for example, [3]). In our studied problem, the present state is also involved nonlinearly in the right side part. It makes the answer of the question about the existence of the solution more complicated.

Remark 2.2. Note that the nonlinear equation $z = f(n, x, z)$ could not have a solution w.r.t. $z \in \mathbb{R}$ for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

In connection with the written above we introduce the following assumption:

(A) For any $n \in \mathbb{Z}[a+1,b]$ and $x \in \mathbb{R}$ the algebraic equation $z = f(n, x, z)$ has a unique solution $z = z(n, x) \in \mathbb{R}^N$.

Remark 2.3. The assumption (A) guarantees the existence of a solution of (1) with an initial condition $x(a) = x_0$ with an arbitrary initial value $x_0 \in \mathbb{R}$.

Remark 2.4. If condition (A) is satisfied we denote $F(n, x) = f(n, x, z(n, x))$.

Example 2.1. Let $f(n, x, z) = nx + 0.5z$. Then the equation mentioned in Remark 2.2 is reduced to $z = nx + 0.5z$ with a solution $z = 2nx$ and Eq. (1) is reduced to $x(n) = 2nx(n-1), n \in \mathbb{Z}[a, b]$ with a solution $x(n) = A2^k \prod_{i=1}^{k}(a + i),\ n = a + k,\ k = 1, 2, \dots, b - a$, where $\alpha A + \beta A 2^{b-a} \prod_{i=1}^{b-a}(a + i) = \gamma$ or $A = \frac{\gamma}{\alpha + \beta 2^{b-a} \prod_{i=1}^{b-a}(a+i)}$ iff $\alpha \neq -\beta 2^{b-a} \prod_{i=1}^{b-a}(a + i)$.

In our further work we will assume condition (A) is satisfied and according to Remark 2.4 we consider the difference equation

$$x(n) = F(n, x(n-1)) \quad \text{for} \quad n \in \mathbb{Z}[a + 1, b]. \tag{3}$$

3. Ulam–Hyers stability

Let $\varepsilon > 0$. We consider the following inequality

$$|y(n) - F(n, y(n-1))| \leq \varepsilon \quad \text{for} \quad n \in \mathbb{Z}[a + 1, b], \tag{4}$$

where $y(a) \in \mathbb{R}$ is an arbitrary number.

Remark 3.1. If the function $y(n),\ n \in \mathbb{Z}[a + 1, b]$ satisfies the inequality (4) then there exists a function $h : \mathbb{Z}[a + 1, b] \to \mathbb{R} : \ |h(n)| \leq \varepsilon,\ n \in \mathbb{Z}[a + 1, b]$ such the the equality

$$y(n) - F(n, y(n-1)) + h(n) = 0 \quad \text{for} \quad n \in \mathbb{Z}[a + 1, b] \tag{5}$$

holds.

The inequality (4) plays the main role in the Ulam–Hyers stability. When Ulam–Hyers stability is applied to the difference equations with an initial condition then an arbitrary solution $y(n),\ n \in \mathbb{Z}[a, b]$ of the inequality (4) is chosen, the solution $x(n),\ n \in \mathbb{Z}[a + 1, b]$ of the equation (3) is taken with $z(a) = y(a)$ and the closeness between both functions $y(n)$ and $x(n)$ is studied.

The application of Ulam–Hyers stability for boundary value problems is not so simple because for any chosen solution $y(n),\ n \in \mathbb{Z}[a, b]$ of the inequality (4), if we consider a solution $x(n),\ n \in \mathbb{Z}[a, b]$ of BVP (3), (2), both functions $y(n)$ and $x(n)$, differently than the case of initial condition, have no connection and they could be not close enough. This misunderstanding could be seen in several published papers (see, for example, [4], [8]).

Note that for a fixed ε the inequality (4) has many solutions. Some of them could satisfy the boundary condition (2) but some of them could not. This does not allow us to claim the closeness between the solutions of inequality (4) and the solutions of BVP (3), (2). We will illustrate it in an example.

Example 3.1. To be more clear and to emphasize on the application of Ulam–Hyers stability we consider $a = 0, b = 2$ and consider the following BVP

$$x(n) = 0.5\frac{x(n-1)}{1+nx^2(n-1)} \quad \text{for } n \in \mathbb{Z}[1,4], \qquad x(0) = 2x(4)+1. \tag{6}$$

Denote $x(0) = A$ then the solution of (6) is

$$\begin{aligned} x(0) &= A \\ x(1) &= \frac{0.5A}{1+A^2} \\ x(2) &= \frac{0.25A(1+A^2)|}{1+2.5A^2+A^4} \\ x(3) &= \frac{0.125A(1+A^2)(1+2.5A^2+A^4)}{1+5.1875A^2+8.625A^4+5.1875A^6+A^8} \\ x(4) &= \frac{A_1}{A_2}, \end{aligned}$$

where

$$A_1 = 0.0625A(1+A^2)(1+2.5A^2+A^4)(1+5.1875A^2+8.625A^4+5.1875A^6+A^8)$$

and

$$\begin{aligned} A_2 =& 1+10.4375A^2+44.5977A^4+101.063A^6+131.867A^8 \\ &+101.063A^{10}+44.5977A^{12}+10.4375A^{14}+A^{16}. \end{aligned}$$

From the boundary condition we get $A - 2x(4) = 1$ which has one real solution $A \approx 1.05291$.

For particular value of A the solution is given in Table 1.

TABLE 1. Values of the solution of (6).

Solution	**A=1.05291**
$x(0)$	1.05291
$x(1)$	0.249668
$x(2)$	0.110996
$x(3)$	0.05352
$x(4)$	0.0264569

Now, let $\varepsilon > 0$ be an arbitrary number and consider the inequality

$$\left| y(n) - 0.5\frac{y(n-1)}{1+ny^2(n-1)} \right| \leq \varepsilon \quad \text{for } n = 1, 2. \tag{7}$$

We will obtain one of the possible solutions of the inequality (7) to illustrate the suggested way to apply the ideas of Ulam–Hyers stability.

Let $y(0) = 1$. Then two of the possible solutions of (7) are given in Table 2.

TABLE 2. Values of the solution of inequality (7).

Solution	$\varepsilon = 0.1$	$\varepsilon = 0.5$
$y(1)$	0.3	0.5
$y(2)$	0.177119	0.416667
$y(3)$	0.130942	0.386986
$y(4)$	0.128686	0.371006

It is obvious that the solutions of (7) do not satisfy the boundary condition $y(0) - 2y(4) = 1$.

Now we will study Ulam–Hyers stability of BVP(3), (2). Example 3.1 shows that we could not proceed directly by the application only of inequality (4) in study Ulam–Hyers stability.

We will consider the difference inequality (4) with an inequality for the boundary condition, i.e., we will consider the following inequalities

$$\begin{aligned} &|y(n) - F(n, y(n-1))| \leq \varepsilon \quad \text{for } n \in \mathbb{Z}[a+1, b], \\ &|\alpha y(a) - \beta y(b) - \gamma| \leq \varepsilon. \end{aligned} \tag{8}$$

Note that for any ε inequalities (8) have many solutions.

Remark 3.2. If the function $y(n),\ n \in \mathbb{Z}[a+1, b]$ satisfies the inequalities (8) then there exist a function $h:\ \mathbb{Z}[a+1, b] \to \mathbb{R}:\ |h(n)| \leq \varepsilon,\ n \in \mathbb{Z}[a+1, b]$ and constants $K_n : |K_n| \leq \varepsilon, n \in \mathbb{Z}[a+1, b]$, such the the equalities

$$\begin{aligned} &y(n) - F(n, y(n-1)) - h(n) = 0 \quad \text{for } n \in \mathbb{Z}[a+1, b], \\ &\alpha y(a) - \beta y(b) - \gamma + K_n = 0 \end{aligned} \tag{9}$$

hold.

Definition 3.1. The BVP (3), (2) is *modified Ulam–Hyers stable* if there exists a sequence of positive real numbers $C_n > 0,\ n \in \mathbb{Z}[a, b]$ such that for each $\varepsilon > 0$ and for each solution $y(n)$ of the inequalities (8) the inequality

$$|y(n) - x(n)| \leq C_n \varepsilon, \quad n \in \mathbb{Z}[a, b] \tag{10}$$

holds where $x(n)$ is the solution of BVP (3), (2).

We will introduce the following assumption:

(H) The function F is Lipshitz, i.e., $|F(n,x)-F(n,y)| \leq L|x-y|$, $x, y \in \mathbb{R}$, $n \in \mathbb{Z}[a,b]$ with $|\alpha| > L^{b-a}|\beta|$.

Theorem 3.1. *Let the condition (H) be satisfied. Then the BVP* (3), (2) *is modified Ulam–Hyers stable.*

Proof. Let $\varepsilon > 0$ be an arbitrary given number. Let $y(n)$, $n \in \mathbb{Z}[a,b]$ be a solution of the inequalities (8) and $z(n)$, $n \in \mathbb{Z}[a,b]$ be the unique solution of BVP (3), (2). According to Remark 3.2 the equalities (9) hold.

We denote $|z(a)-y(a)| = A$, apply induction, condition (H) and equalities (9) and obtain

$$\begin{aligned}
|z(a+1)-y(a+1)| &\leq |F(a+1,z(a))-F(a+1,y(a))| + |h(a+1)| \\
&\leq LA + \varepsilon, \\
|z(a+2)-y(a+2)| &\leq L|z(a+1)-y(a+1)| + \varepsilon \leq L^2A + (1+L)\varepsilon, \\
|z(a+3)-y(a+3)| &\leq L|z(a+2)-y(a+2)| + \varepsilon \\
&\leq L^3A + (1+L+L^2)\varepsilon, \\
&\dots\dots\dots \\
|z(k)-y(k)| &\leq L^{k-a}A + \varepsilon \sum_{i=0}^{k-a-1} L^i = L^{k-a}A + \varepsilon\frac{1-L^{k-a}}{1-L}, \qquad (11) \\
&k \in \mathbb{Z}[a+1,b], \\
&\dots\dots\dots \\
|z(b)-y(b)| &\leq L|z(b-1)-y(b-1)| + \varepsilon \\
&\leq L^{b-a}A + \varepsilon \sum_{i=0}^{b-a-1} L^i = L^{b-a}A + \varepsilon\frac{1-L^{b-a}}{1-L}.
\end{aligned}$$

Then from the boundary condition we get

$$\begin{aligned}
A = |z(a)-y(a)| &= \left|\frac{\beta}{\alpha}(z(b)-y(b)) + \frac{K_n}{\alpha}\right| \leq \left|\frac{\beta}{\alpha}\right| |z(b)-y(b)| + \frac{|K_n|}{|\alpha|} \\
&\leq \left|\frac{\beta}{\alpha}\right| |z(b)-y(b)| + \frac{\varepsilon}{|\alpha|},
\end{aligned}$$

and by (11)

$$|z(b)-y(b)| \leq L^{b-a}\left(\left|\frac{\beta}{\alpha}\right| |z(b)-y(b)| + \frac{\varepsilon}{|\alpha|}\right) + \varepsilon\frac{1-L^{b-a}}{1-L}$$

or

$$|z(b)-y(b)|\left(1-L^{b-a}\Big|\frac{\beta}{\alpha}\Big|\right)\leq\varepsilon\left(\frac{1-L^{b-a}}{1-L}+\frac{L^{b-a}}{|\alpha|}\right).$$

Therefore,

$$|z(b)-y(b)|\leq\varepsilon\frac{|\alpha|\frac{1-L^{b-a}}{1-L}+L^{b-a}}{|\alpha|-|\beta|L^{b-a}}$$

and

$$\begin{aligned} A &\leq \Big|\frac{\beta}{\alpha}\Big|\left(\varepsilon\frac{|\alpha|\frac{1-L^{b-a}}{1-L}+L^{b-a}}{|\alpha|-|\beta|L^{b-a}}\right)+\frac{\varepsilon}{|\alpha|} \\ &= \frac{1}{|\alpha|}\left(1+|\beta|\frac{|\alpha|\frac{1-L^{b-a}}{1-L}+L^{b-a}}{|\alpha|-|\beta|L^{b-a}}\right)\varepsilon. \end{aligned} \tag{12}$$

From inequalities (11) and (12) we get

$$|z(k)-y(k)|\leq\left(L^{k-a}\frac{1}{|\alpha|}\left(1+|\beta|\frac{|\alpha|\frac{1-L^{b-a}}{1-L}+L^{b-a}}{|\alpha|-|\beta|L^{b-a}}\right)+\frac{1-L^{k-a}}{1-L}\right)\varepsilon,$$
$$k\in\mathbb{Z}[a+1,b],$$

which proves the modified Ulam–Hyers stability with the constants

$$C_k=L^{k-a}\frac{1}{|\alpha|}\left(1+|\beta|\frac{|\alpha|\frac{1-L^{b-a}}{1-L}+L^{b-a}}{|\alpha|-|\beta|L^{b-a}}\right)+\frac{1-L^{k-a}}{1-L},\quad k\in\mathbb{Z}[a,b]. \tag{13}$$

□

Remark 3.3. In the case $L=1$ we have $C_k=k-a+1+|\beta|\frac{b-a+1}{|\alpha|-|\beta|}$ in the proof of Theorem 3.1.

In the case $L\neq 1$ we have $C_k=L^{k-a}\left(1+|\beta|\frac{1-L^{b-a+1}}{(1-L)|\alpha|-L^{b-a}|\beta|}\right)+\frac{1-L^{k-a}}{1-L}$ in the proof of Theorem 3.1.

Example 3.2. Consider BVP (6) with a unique solution given in Table 1.

The function $F(n,x)=0.5\frac{x}{1+nx}$ is Lipschitz with a constant $L=0.5$ and the inequality $|\alpha|-L^{b-a}|\beta|=1-2*0.5^2=0.5<1$ holds, i.e., condition (H) is satisfied.

According to Eq. (13) we have $C_0=5.5,\ C_1=3.75,\ C_2=2.875,\ C_3=2.4375,\ C_4=2.21875.$

Now, let $\varepsilon>0$ be an arbitrary number and consider the inequalities

$$\left|y(n)-0.5\frac{y(n-1)}{1+ny^2(n-1)}\right|\leq\varepsilon\ \text{ for } n=1,2,\qquad |y(0)-2y(2)-1|\leq\varepsilon. \tag{14}$$

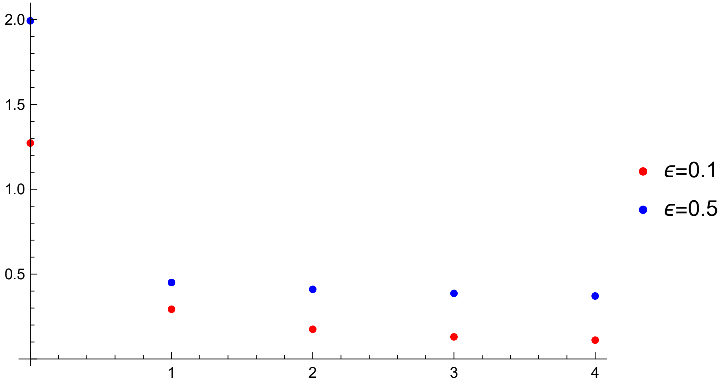

FIGURE 1. Graph of the solutions of (14) for $\varepsilon = 0.1,\ \varepsilon = 0.5$.

Let $\varepsilon = 0.1$ and we will obtain one possible solution of the inequalities (14). We could not proceed as in Example 3.1, and so we denote $y(0) = B$ and obtain $y(1),\ y(2),\ y(3),\ y(4)$ as functions of B. Then from the boundary condition $B - 2y(4) + 1 - 0.5 * 0.1$ we obtain the only real value $B \approx 1.27189$ and one solution of (14) (see Table 3).

Let $\varepsilon = 0.5$ and we will obtain one possible solution of the inequalities (14). We could not proceed as in Example 3.1, and therefore we denote $y(0) = B$ and obtain $y(1),\ y(2),\ y(3),\ y(4)$ as functions of B. Then from the boundary condition $B - 2y(4) + 1 - 0.5 * 0.5$ we obtain the only real value $B \approx 1.9919$ and one solution of (14) (see Table 3 and Figure 1).

TABLE 3. Values of the solution of inequalities (14).

Solution	$\varepsilon = 0.1$	$\varepsilon = 0.5$
$y(0)$	1.27189	1. 9919
$y(1)$	0.29294	0.450486
$y(2)$	0.175014	0.410216
$y(3)$	0.130143	0.3863
$y(4)$	0.110943	0.370952

Compare the differences $|x(n) - y(n)|$ for $n = 0, 1, 2, 3, 4$ (see Table 4, Figure 2, Figure 3).

TABLE 4. Solution of BVP (6), inequalities (14) for various ε and the bounds εC_n.

n	**BVP** (6)	(14) with $\varepsilon = 0.1$	$0.1C_n$	(14) with $\varepsilon = 0.5$	$0.5C_n$
0	1.05291	1.27189	0.55	1. 9919	2.75
1	0.249668	0.29294	0.375	0.450486	1.875
2	0.110996	0.175014	0.2875	0.410216	1.4375
3	0.110996	0.130143	0.24375	0.3863	1.21875
4	0.0264569	0.110943	0.221875	0.370952	1.10938

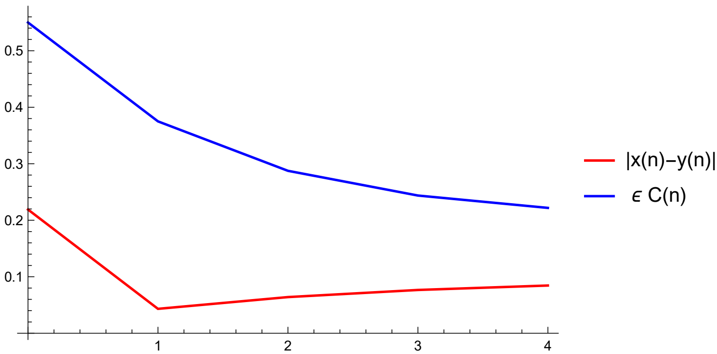

FIGURE 2. Graphs of the difference between the solution of BVP (6) and the particular solution of of (14) for $\varepsilon = 0.1$ and the bound $\varepsilon C(n)$.

From Table 4, Figure 2 and Figure 3 it could be seen that the given particular solution of inequalities (14) satisfies inequalities (10), i.e., they illustrate the modified Ulam–Hyers stability for the BVP (6).

4. CONCLUSION

In this paper we suggest a way to study correctly Ulam–Hyers stability for boundary value problems for difference equations. We consider not only the corresponding difference inequality but also we combine it with an appropriate inequality for the boundary condition. We define modified Ulam–Hyers stability and obtain sufficient conditions. The theoretical study is illustrated with examples.

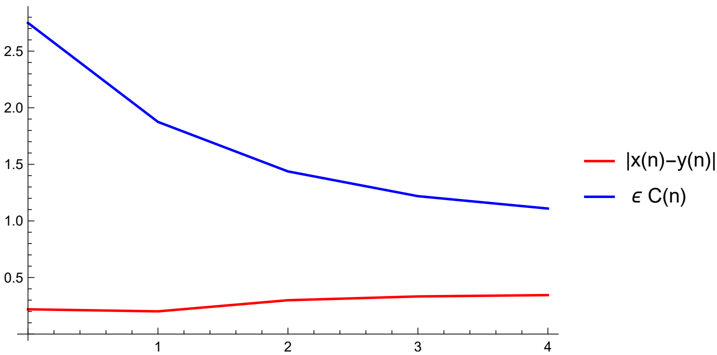

FIGURE 3. Graph of the difference between the solution of BVP (6) and the particular solution of (14) for $\varepsilon = 0.5$ and the bound $\varepsilon C(n)$.

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REFERENCES

[1] C. Buse, D. O'Regan, and O. Saierli, Hayers-Ulam stability for linear differences with time dependent and periodic coefficients, *Symmetry* **11** (2019), 339. doi:10.3390/sym11030339

[2] F. Chen and Y. Zhou, Existence and Ulam Stability of solutions for discrete fractional boundary value problem, *Discr. Dynam. Nature Soc.* **2013**, Art. ID 459161, 7 pp. doi:10.1155/2013/459161

[3] S. N. Elaydi, *An Introduction to Difference Equations*, 3rd ed., Springer, New York, NY, USA, 2005.

[4] S. S. Haider and M. U. Rehman, Ulam-Hyers-Rassias stability and existence of solutions to nonlinear fractional difference equations with multipoint summation boundary condition, *Acta Math. Sci.* **40** (2020), 589–602. doi:10.1007/s10473-020-0219-1

[5] S. Hristova and K. Stefanova, Ulam type stability for scalar nonlinear non-instantaneous impulsive difference equations with computer realization, *AIP Conf. Proc.* **2333**, 110001 (2021). doi:10.1063/5.0041629

[6] S. M. Jung and Y. W. Nam, On the Hyers-Ulam stability of the first-order difference equation, *J. Function Spaces* **2016**, 6078298 (2016). doi:10.1155/2016/6078298

[7] Y. W. Nam, Hyers-Ulam stability of elliptic Mobius difference equation, *Cogent Math. Stat.* **5**, 1492338 (2018). doi:10.1080/25742558.2018.1492338

[8] A. G. M. Selvam and R. Dhineshbabu, Ulam stability of solutions of a discrete boundary value problem with fractional order, *J. Phys.: Conf. Ser.* **1597**, 012023 (2020). doi:10.1088/1742-6596/1597/1/012023

[9] A. K. Tripathy and B. Dihudi, Hyers-Ulam stability of first order nonlinear delay difference equation, *Intern. J. Theoret. Appl. Math.* **8** (2022), 58–64. doi:10.11648/j.ijtam.20220803.12

[10] I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26** (2010), 103–107. http://www.jstor.org/stable/43999438

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Гранична задача за диференчни уравнения и устойчивост на Улам-Хайърс

Снежана Христова

Резюме. В статията е дискутирана устойчивост на Улам-Хайърс за гранична задача за диференчни уравнения и са посочени някои неточности в литературата. Предложен е нов подход за приложение на Улам-Хайърс устойчивостта за гранични задачи, който да избягва тези неточности. Класическото диференчно неравенство е комбинирано с неравенство за граничните условия. Този подход не запазва основната идея на устойчивостта на Улам, защото решението на дадената задача е фиксирано и то трябва да е достатъчно близко до всяко решение на съответното неравенство. Във връзка с това, се дефинира по подходящ начин модифицирана устойчивост на Улам и са получени достатъчни условия за нея. Теоретичните резултати са илюстрирани с няколко примера.