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# ON SOME EXAMPLES OF COMPLEX NORDEN MANIFOLDS

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Abstract. Examples of complex manifolds with Norden metric are presented. Two-dimensional Kähler Norden manifolds are obtained as real interpretation of smooth curves in the complex plane. Examples of Norden manifolds with abelian complex structure are constructed on Lie groups and Lie algebras.

## 1. Introduction

Smooth manifolds with additional tensor structures of type (1,1) and compatible pseudo-Riemannian metric are subject of intensive scientific research. It is well known that on an almost complex manifold, there exists a Riemannian metric compatible with the almost complex structure in such a way that in the tangent space at each point of the manifold an isometry arises. Such manifolds are called almost Hermitian. On the other hand, the Russian geometer A. P. Norden [9, 10] defined another kind of metric on an almost complex manifold, termed by him B-metric, which is compatible with the almost complex structure in such a manner that an anti-isometry is induced. Norden introduced and studied manifolds endowed with B-metric and complex structure parallel with respect to the Levi-Civita connection called by him B-manifolds (also known as

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Kähler manifolds with B-metric). Later, this metric is termed Norden metric and such manifolds are studied by many authors under different names: almost complex manifolds with Norden metric, almost complex Norden manifolds or almost anti-Hermitian manifolds.

The systematic research on almost complex manifolds with Norden metric was initiated by K. Gribachev, D. Mekerov and G. Djelepov in [6] where these manifolds are called Generalized B-manifolds. In the same work, the first incomplete classification of these manifolds is presented. The first complete classification with respect to the covariant derivative of the almost complex structure is introduced by G. Ganchev and A. Borisov in [4]. This classification consists of three basic classes whose intersection is the class of the Kähler manifolds.

Many examples of almost complex Norden manifolds have emerged in the scientific literature. Among the first ones presented are the examples by G. Ganchev, K. Gribachev and V. Mihova in [5], and by R. Castro, L. M. Hervella, E. García-Río in [1].

In this article, we consider almost complex manifolds with Norden metric with zero Nijenhuis tensor of the almost complex structure. In this case the almost complex structure is said to be integrable, i.e. it is a complex structure. The aim of this work is to construct examples of such manifolds and to study their geometric properties. The article is organized as follows. In section 2, we give basic information about almost complex Norden manifolds. In section 3, we present examples of flat 2-dimensional Kähler manifolds obtained as real interpretation of smooth curves in the complex plane. In section 4, we present examples of manifolds with Norden metric and abelian complex structure constructed on Lie groups and Lie algebras. The manifolds are in the classes $\mathcal{W}_1$ and $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the Ganchev-Borisov classification.

## 2. Preliminaries

Let $M$ be a differentiable manifold with real dimension $\dim M = 2n$. Smooth vector fields in $\mathfrak{X}(M)$ and vectors in the tangent space $T_pM$, $p \in M$, will be denoted by $X, Y, Z, W$.

The manifold $(M, J, g)$ is called an *almost complex Norden manifold* (almost complex manifold with Norden metric) if $J$ is an almost complex structure, and $g$ is a pseudo-Riemannian metric, called Norden metric, such that

$$J^2 = -\mathrm{id}, \qquad g(JX, JY) = -g(X, Y) \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M)$.

The tensor field

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \tag{2.2}$$

is called the Nijenhuis tensor of $J$. The almost complex structure $J$ is said to be integrable if $N = 0$. According to [8], an almost complex structure is complex if and only if $N = 0$.

Since (2.1) yield $g(X,JY) = g(JX,Y)$, the tensor field $\widetilde{g}$ defined by

$$\widetilde{g}(X,Y) = g(X,JY), \tag{2.3}$$

also possesses the Norden metric property. The metric $\widetilde{g}$ is called the associated metric of $g$ (twin metric). Both metrics are necessarily of neutral signature $(n,n)$.

The fundamental tensor field $F$ of type (0,3) is defined by

$$F(X,Y,Z) = g\left((\nabla_X J)Y, Z\right), \tag{2.4}$$

where $\nabla$ is the Levi-Civita connection of $g$, and has the following properties:

$$F(X,Y,Z) = F(X,Z,Y), \qquad F(X,JY,JZ) = F(X,Y,Z). \tag{2.5}$$

Let $\{e_i\}$ $(i = 1,2,\dots,2n)$ be an arbitrary basis in $T_pM$, $p \in M$, and $(g^{ij})$ be the inverse matrix of $(g_{ij})$. Then, the 1-forms $\theta$ and $\theta^*$ associated with $F$ are defined by

$$\theta(Z) = g^{ij}F(e_i,e_j,Z), \qquad \theta^* = \theta \circ J. \tag{2.6}$$

A classification of the almost complex Norden manifolds with respect to the covariant derivative of $J$, i.e. to $F$, is introduced by G. Ganchev and A. Borisov in [4]. This classification consists of three basic classes $\mathcal{W}_i$ $(i = 1,2,3)$ and their direct sums $\mathcal{W}_i \oplus \mathcal{W}_j$ $(i \neq j)$. The special class of the Kähler Norden manifolds $\mathcal{W}_0$ is contained in all other classes. In our work, we consider only the classes of complex Norden manifolds which are:

1. The class $\mathcal{W}_0$ of the Kähler Norden manifolds

$$F(X,Y,Z) = 0 \quad \Leftrightarrow \quad \nabla J = 0; \tag{2.7}$$

2. The class $\mathcal{W}_1$

$$\begin{aligned} F(X,Y,Z) = \tfrac{1}{2n}\big\{&g(X,Y)\theta(Z) + g(X,Z)\theta(Y) + g(X,JY)\theta(JZ) \\ &+g(X,JZ)\theta(JY)\big\}; \end{aligned} \tag{2.8}$$

3. The class $\mathcal{W}_2$ of the *special complex Norden manifolds*

$$F(X,Y,JZ)+F(Y,Z,JX)+F(Z,X,JY) = 0, \quad \theta = 0 \quad \Leftrightarrow \quad N = 0, \quad \theta = 0; \tag{2.9}$$

4. The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the complex Norden manifolds

$$F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0 \quad \Leftrightarrow \quad N = 0. \tag{2.10}$$

The curvature tensor $R$ of type (1,3) of the Levi-Civita connection $\nabla$ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.11}$$

and the corresponding (0,4)-tensor is given by

$$R(X,Y,Z,W) = g\big(R(X,Y)Z, W\big). \tag{2.12}$$

The Ricci tensor $\rho$ and the scalar curvature of the manifold are computed by

$$\rho(X,Y) = g^{ij} R(e_i, X, Y, e_j), \qquad \tau = g^{ij} \rho(e_i, e_j). \tag{2.13}$$

## 3. Curves in 2-dimensional complex space and their real interpretation as flat Kähler Norden manifolds

Let $z\ (z_1, z_2, ..., z_n)$, $z_k = x_k + i y_k$, $x_k, y_k \in \mathbb{R}$ $(k = 1, 2, .., n)$, be a vector in the $n$-dimensional complex space $\mathbb{C}^n$. Here, by $i$ we denote the imaginary unit ($i^2 = -1$). The standard mapping (identification) between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ is defined by [9, 10]

$$\varphi: \ z\ (z_1, z_2, ..., z_n) \in \mathbb{C}^n \ \longrightarrow \ \varphi(z) = Z(x_1, x_2, ..., x_n; y_1, y_2, ..., y_n) \in \mathbb{R}^{2n}.$$

By this mapping the transformation $J_0$ in $\mathbb{C}^n$ defined by $z \to iz$ induces a complex structure $J$ in $\mathbb{R}^{2n}$ given by $J = \varphi\ J_0\ \varphi^{-1}$ which is called canonical. The canonical complex structure is determined by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where $I_n$ is the square unit matrix.

Consider the Euclidean scalar product $G_0$ in $\mathbb{C}^n$ which is the complex linearization of the natural scalar product in $\mathbb{R}^n$, i.e. $G_0(z,z) = z_1^2 + z_2^2 + ... + z_n^2$. Then, $G_0$ induces two metrics on $\mathbb{R}^{2n}$ [5]: $g = \mathrm{Re}\ G_0$ and $\widetilde{g} = -\mathrm{Im}\ G_0$. If $Z(x_1, ..., x_n; y_1, ..., y_n) \in \mathbb{R}^{2n}$ then

$$\begin{aligned} g(Z,Z) &= x_1^2 + ... + x_n^2 - y_1^2 - ... - y_n^2, \\ \widetilde{g}(Z,Z) &= -2(x_1 y_1 + ... + x_n y_n). \end{aligned} \tag{3.1}$$

Both metrics $g$ and $\widetilde{g}$ satisfy the second equation in (2.1) and hence they are Norden metrics.

The images by $\varphi$ of the vectors in the standard basis of $\mathbb{C}^n$ and their $J$-associated vectors form the so called adapted basis ($J$-basis) of $\mathbb{R}^{2n}$, i.e. the basis $\{e_1, ..., e_n; Je_1, ..., Je_n\}$ such that

$$g(e_j, e_k) = -g(Je_j, Je_k) = \delta_{jk}, \quad g(e_j, Je_k) = 0, \quad j, k = 1, 2, ..., n.$$

The vector space $\mathbb{R}^{2n}$ endowed with the scalar product rule $g$ defined by (3.1) is called a pseudo-Euclidean $2n$-dimensional vector space with signature $(n, n)$ and is denoted by $\mathbb{R}^{2n}_{(n,n)}$.

3.1. **Real interpretation of curves in $\mathbb{C}^2$ as 2-dimensional flat Kähler Norden manifolds.** In this section we consider smooth curves in $\mathbb{C}^2$ and their real interpretation as 2-dimensional surfaces in $\mathbb{R}^4_{(2,2)}$.

Let $OE_1E_2$ be a coordinate system in $\mathbb{C}^2$, where $\{E_1, E_2\}$ is the standard basis of $\mathbb{C}^2$, and let $c : r(z_1, z_2)$ be a smooth curve in $\mathbb{C}^2$ with $z_k = x_k + iy_k$ $(k = 1, 2)$. The functions $z_1 = z_1(z)$ and $z_2 = z_2(z)$ are analytic functions of one complex variable $z = u + iv$, where $u, v \in \mathbb{R}$. Hence, the pairs of functions $(x_1, y_1)$ and $(x_2, y_2)$ satisfy the Cauchy–Riemann equations, i.e.

$$\frac{\partial x_k}{\partial u} = \frac{\partial y_k}{\partial v}, \quad \frac{\partial x_k}{\partial v} = -\frac{\partial y_k}{\partial u}, \quad k = 1, 2. \tag{3.2}$$

The standard mapping $\varphi$ between $\mathbb{C}^2$ and $\mathbb{R}^4$ maps the vector $r(z_1, z_2) \in \mathbb{C}^2$ in the vector $r(x_1, x_2; y_1, y_2) \in \mathbb{R}^4_{(2,2)}$ with respect to the adapted basis, i.e. the basis $\{e_1, e_2, Je_1, Je_2\}$. The equation

$$M^2 : r = r(x_1(u, v), x_2(u, v); y_1(u, v), y_2(u, v)) \tag{3.3}$$

defines a 2-dimensional smooth surface $M^2$ in $\mathbb{R}^4_{(2,2)}$. The basis of the tangent space $T_pM^2$, $p \in M^2$, is defined by the vectors $r_1 = \frac{\partial r}{\partial u}$ and $r_2 = \frac{\partial r}{\partial v}$.

Let $J$ be the canonical complex structure in $\mathbb{R}^4$. Due to the Cauchy-Riemann equations (3.2), we obtain $Jr_1 = r_2$ and $Jr_2 = -r_1$. Hence, the restriction of $J$ on $T_pM^2$ is defined by

$$J_1^1 = J_2^2 = 0, \quad J_1^2 = -J_2^1 = 1. \tag{3.4}$$

Also, due to (3.2) and (3.1), we have $g_{11} = -g_{22}$, i.e. $g$ satisfies the Norden metric property. Thus, the manifold $(M^2, J, g)$ is a 2-dimensional complex Norden manifold.

We prove the following

**Theorem 3.1.** *The 2-dimensional manifold $(M^2, J, g)$ defined by (3.3) which is the real interpretation in $\mathbb{R}^4_{(2,2)}$ of a smooth curve in $\mathbb{C}^2$ is a Kähler Norden manifold.*

*Proof.* From the Cauchy-Riemann equations (3.2) it follows that

$$\frac{\partial g_{11}}{\partial u} = -\frac{\partial g_{12}}{\partial v}, \quad \frac{\partial g_{11}}{\partial v} = \frac{\partial g_{12}}{\partial u}. \tag{3.5}$$

Then, having in mind the formula for the Christoffel symbols of the Levi-Civita connection of $g$, i.e.

$$\Gamma_{jk}^{s}=\frac{1}{2}g^{sl}\left(\partial_{j}g_{kl}+\partial_{k}g_{jl}-\partial_{l}g_{jk}\right),$$

where $\partial_j g_{kl}=\frac{\partial g_{kl}}{\partial u^j}$, and $u^1=u$, $u^2=v$, we prove that

$$\Gamma_{11}^{1}=-\Gamma_{22}^{1}=\Gamma_{12}^{2},\quad \Gamma_{12}^{1}=-\Gamma_{11}^{2}=\Gamma_{22}^{2}. \tag{3.6}$$

To calculate the components of $\nabla J$ we use the formula $\nabla_j J_k^s=\partial_j J_k^s-\Gamma_{jk}^{m}J_m^s+\Gamma_{jm}^{s}J_k^m$. Then, by (3.4) and (3.6), we obtain $\nabla_j J_k^s=0$ for all $j,k,s=1,2$. □

We remark that every 2-dimensional Kähler Norden manifold is flat, i.e. with zero curvature tensor. The proof goes as follows. Let $\{e_1,e_2=Je_1\}$ be the adapted basis of the manifold. Then, the unique essential component of the curvature tensor of type (0,4) is $R_{1221}=R(e_1,e_2,e_2,e_1)$. As it is known [5], the curvature tensor of a Kähler Norden manifold satisfies the Kählerian property which is $R(X,Y,JZ,JW)=-R(X,Y,Z,W)$. Due to this property, it follows that $R(e_1,e_2,e_2,e_1)=0$.

3.2. **Explicit examples of curves in $\mathbb{C}^2$ as 2-dimensional flat Kähler Norden manifolds.** We present three concrete examples of second degree curves to illustrate the results in the previous subsection.

**Example 3.1.** Consider a circle in $\mathbb{C}^2$ with equation

$$z_1^2+z_2^2=r^2, \tag{3.7}$$

where $r=a+ib$, $a,b\in\mathbb{R}$. From the standard parametrization $z_1=r\cos z$, $z_2=r\sin z$, having in mind that $z_k=x_k+iy_k$, and $z=u+iv$, we obtain that the real interpretation of the complex circle (3.7) is the surface in $\mathbb{R}^4_{(2,2)}$ given by

$$S^2:\left|\begin{array}{l} x_1=a\cos u\cosh v+b\sin u\sinh v\\ x_2=a\sin u\cosh v-b\cos u\sinh v\\ y_1=-a\sin u\sinh v+b\cos u\cosh v\\ y_2=a\cos u\sinh v+b\sin u\cosh v.\end{array}\right. \tag{3.8}$$

To study the geometric properties of $S^2$, we compute the basis of the tangent space $\{r_1,r_2\}$ and the components of the metric $g_{jk}$, having in mind the scalar product rule (3.1). Thus, we get

$$g_{11}=-g_{22}=a^2-b^2,\quad g_{12}=g_{21}=-2ab. \tag{3.9}$$

Since all components $g_{jk}$ are constant, we obtain $\Gamma_{jk}^{l}=0$ for all $j,k,l=1,2$. The real interpretation of the complex circle in $\mathbb{R}^4_{(2,2)}$ is the surface $S^2$ with a flat Levi-Civita connection.

**Example 3.2.** Consider a rectangular hyperbola in $\mathbb{C}^2$ given by

$$z_1^2 - z_2^2 = r^2, \tag{3.10}$$

where $r = a + ib$, $a, b \in \mathbb{R}$. The standard parametrization is $z_1 = r\cosh z$, $z_2 = r\sinh z$, where $\cosh z$ and $\sinh z$ are the hyperbolic functions of $z = u+iv$. The 2-dimensional surface which is the real interpretation of the hyperbola is given by

$$H^2 : \left| \begin{array}{l} x_1 = a\cosh u\cos v - b\sinh u\sin v \\ x_2 = a\sinh u\cos v - b\cosh u\sin v \\ y_1 = a\sinh u\sin v + b\cosh u\cos v \\ y_2 = a\cosh u\sin v + b\sinh u\cos v. \end{array} \right. \tag{3.11}$$

We compute the components $g_{jk}$ as follows:

$$\begin{aligned} g_{11} = -g_{22} &= (a^2 - b^2)\cosh 2u\cos 2v - 2ab\sinh 2u\sin 2v, \\ g_{12} = g_{21} &= -(a^2 - b^2)\sinh 2u\sin 2v - 2ab\cosh 2u\cos 2v. \end{aligned} \tag{3.12}$$

Then, the Christoffel symbols are given by

$$\begin{aligned} \Gamma_{11}^1 = -\Gamma_{22}^1 = \Gamma_{12}^2 &= \frac{\sinh 4u}{\cosh 4u + \cos 4v}, \\ \Gamma_{12}^1 = -\Gamma_{11}^2 = \Gamma_{22}^2 &= -\frac{\sin 4v}{\cosh 4u + \cos 4v}. \end{aligned} \tag{3.13}$$

**Example 3.3.** Consider the parabola in $\mathbb{C}^2$ given by

$$z_2^2 = z_1. \tag{3.14}$$

Its real interpretation is the surface

$$P^2 : r(u^2 - v^2,\ u,\ 2uv,\ v). \tag{3.15}$$

Analogously, we compute the components $g_{jk}$

$$\begin{aligned} g_{11} = -g_{22} &= 4u^2 + 1 - 4v^2, \\ g_{12} = g_{21} &= -8uv, \end{aligned} \tag{3.16}$$

and the Christoffel symbols

$$\begin{aligned} \Gamma_{22}^1 = -\Gamma_{11}^1 = -\Gamma_{12}^2 &= \tfrac{4u}{g}\left(1 + 4u^2 + 4v^2\right), \\ \Gamma_{11}^2 = -\Gamma_{12}^1 = -\Gamma_{22}^2 &= \tfrac{4v}{g}\left(4u^2 + 4v^2 - 1\right). \end{aligned} \tag{3.17}$$

Considering the surfaces in examples 3.2 and 3.3, by (3.13) and (3.17), one can verify that $\nabla J = 0$ and $R_{1221} = 0$ for both manifolds $(H^2, J, g)$ and $(P^2, J, g)$.

## 4. Lie groups as Complex Norden manifolds

In this section, following ideas from [1, 7], we construct and study four examples of complex Norden manifolds on Lie groups.

Let $G$ be a real connected $2n$-dimensional Lie group, and let $\mathfrak{g}$ be its associated Lie algebra. If $\{X_1, X_2, ..., X_{2n}\}$ is a basis of left-invariant vector fields on $G$, we endow $G$ with an almost complex structure $J$ and a Norden metric $g$ in the following way

$$JX_i = X_{n+i}, \quad JX_{n+i} = -X_i, \quad i = 1, 2, .., n; \tag{4.1}$$

$$\begin{aligned} &g(X_i, X_i) = -g(X_{n+i}, X_{n+i}) = 1, \qquad i = 1, 2, .., n, \\ &g(X_i, X_j) = 0, \quad i \neq j, \quad i, j = 1, 2, ..., 2n. \end{aligned} \tag{4.2}$$

Then, $(G, J, g)$ is an almost complex Norden manifold.

The almost complex structure $J$ on $G$ is called *abelian* if

$$[JX, JY] = [X, Y] \quad \text{for all } X, Y \in \mathfrak{g}. \tag{4.3}$$

If the structure is abelian, then the Nijenhuis tensor $N = 0$ and thus $(G, J, g)$ is a complex Norden manifold. In [3], it is proved that if a real Lie algebra $\mathfrak{g}$ admits an abelian complex structure then $\mathfrak{g}$ is solvable (more precisely, 2-step solvable).

**Example 4.1.** Consider the real 2-dimensional Lie group $G_2$ defined by

$$G_2 = \left\{ \begin{pmatrix} e^x & y \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}. \tag{4.4}$$

The left-invariant vector fields on $G_2$ are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = e^x \frac{\partial}{\partial y}. \tag{4.5}$$

Then, the associated Lie algebra $\mathfrak{g}_2$ of $G_2$ is defined by

$$\mathfrak{g}_2 : \ [X_1, X_2] = X_2. \tag{4.6}$$

Hence, the structure $J$ on $G_2$ defined by (4.1) is abelian which yields that $(G_2, J, g)$ is a complex Norden manifold.

By help of the Koszul formula

$$\begin{aligned} &2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X), \end{aligned} \tag{4.7}$$

we compute the non-zero components $\nabla_{X_i} X_j$ $(i, j = 1, 2)$ of the Levi-Civita connection

$$\nabla_{X_2} X_2 = -X_1, \quad \nabla_{X_2} X_1 = -X_2. \tag{4.8}$$

Denote the components of the fundamental tensor $F$ by $F_{ijk} = F(X_i, X_j, X_k)$. Then, having in mind (2.4) and (4.8), we obtain the following non-zero components

$$F_{211} = F_{222} = -2. \tag{4.9}$$

By (2.6), the components $\theta_i = \theta(X_i)$ of the 1-form $\theta$ are

$$\theta_1 = 0, \quad \theta_2 = 2. \tag{4.10}$$

Then, having in mind the characteristic condition (2.8) of the class $\mathcal{W}_1$ we prove the following

**Proposition 4.1.** *The 2-dimensional almost complex Norden manifold* $(G_2, J, g)$ *defined by (4.4) with corresponding Lie algebra* $\mathfrak{g}_2$ *given by (4.6) belongs to the class* $\mathcal{W}_1$.

*Proof.* Taking into account the components $F_{ijk}$ and $\theta_i$ from (4.9) and (4.10), we check that for all essential components of $F$ the characteristic condition (2.8) holds. Thus, $(G, J, g)$ is in the class $\mathcal{W}_1$. □

Let us remark that in [1, 2] examples of 2-dimensional manifolds conformally equivalent to Kähler manifolds are studied. Such manifolds are in the class $\mathcal{W}_1$ with closed 1-forms $\theta$ and $\theta^*$. On the manifold which we consider only $\theta^*$ is closed.

**Example 4.2.** Let us generalize the idea of the previous example by considering the $2n$-dimensional Lie group

$$G_{2n} = \left\{ \begin{pmatrix} e^{x_1} & 0 & ... & 0 & x_{n+1} \\ 0 & e^{x_2} & ... & 0 & x_{n+2} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & e^{x_n} & x_{2n} \\ 0 & 0 & ... & 0 & 1 \end{pmatrix} \mid x_1, x_2, ..., x_{2n} \in \mathbb{R} \right\}. \tag{4.11}$$

The left-invariant vector fields on $G_{2n}$ are

$$X_i = \frac{\partial}{\partial x_i}, \quad X_{i+n} = e^{x_i} \frac{\partial}{\partial x_{i+n}}, \quad i = 1, 2, ..., n. \tag{4.12}$$

The associated Lie algebra $\mathfrak{g}_{2n}$ is defined by the following commutator relations

$$\mathfrak{g}_{2n} : \ [X_i, X_{i+n}] = X_{i+n}, \qquad i = 1, 2, ..., n. \tag{4.13}$$

The non-zero components of the Levi-Civita connection are given by

$$\nabla_{X_{i+n}} X_i = -X_{i+n}, \quad \nabla_{X_{i+n}} X_{i+n} = -X_i. \tag{4.14}$$

The non-zero essential components of the tensor $F$ and its associated 1-forms are given by

$$F(X_{i+n}, X_i, X_i) = -2, \quad \theta_{i+n} = \theta^*_i = 2. \tag{4.15}$$

We check that the manifold does not meet the characteristic criteria for the classes $\mathcal{W}_1$ or $\mathcal{W}_2$. Hence, we proved

**Proposition 4.2.** *The manifold $(G_{2n}, J, g)$ defined by (4.11) with corresponding Lie algebra $\mathfrak{g}_{2n}$ given by (4.13) belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$.*

Further, we study the curvature properties of $(G_{2n}, J, g)$ by computing the non-zero essential components of the curvature tensor $R_{ijkl} = R(X_i, X_j, X_k, X_l)$ and the Ricci tensor $\rho_{ij} = \rho(X_i, X_j)$, and the scalar curvature:

$$\begin{aligned} &R(X_i, X_{i+n}, X_{i+n}, X_i) = 1, \\ &\rho(X_i, X_i) = -\rho(X_{i+n}, X_{i+n}) = -1, \quad \tau = -2n. \end{aligned} \tag{4.16}$$

By (4.16) and (4.2) it follows that $\rho = -\frac{\tau}{2n} g$. Hence, we prove

**Proposition 4.3.** *The complex Norden manifold $(G_{2n}, J, g)$ defined by (4.11) and (4.13) is an Einsteinian manifold.*

Let us remark that the Lie algebras (4.6) and (4.13) are particular cases of families of $2n$-dimensional Lie algebras depending on $2n$ real parameters studied in [11].

**Example 4.3.** Consider the 4-dimensional Lie group defined by

$$G_4 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & e^t \end{pmatrix} \mid x, y, z, t \in \mathbb{R} \right\}. \tag{4.17}$$

Is we set $t = 0$ for all $x, y, z \in \mathbb{R}$ then the Heisenberg group is obtained.

The left-invariant vector fields on $G_4$ are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{\partial}{\partial t}. \tag{4.18}$$

Then, the associated Lie algebra $\mathfrak{g}_4$ of $G_4$ is defined by the following non-zero commutator relations

$$\mathfrak{g}_4 : \ [X_1, X_2] = [X_3, X_4] = X_3, \quad [X_2, X_4] = X_2. \tag{4.19}$$

From (4.19) it follows that the almost complex structure $J$ on $G_4$ defined by (4.1) is abelian and thus, $(G_4, J, g)$ is a complex Norden manifold.

By (4.7) and (4.19) the non-zero essential components of the Levi-Civita connection are given by:

$$\begin{gathered}\nabla_{X_2}X_2=-\nabla_{X_3}X_3=X_4,\quad \nabla_{X_1}X_2=\tfrac{1}{2}X_3,\quad \nabla_{X_1}X_3=\tfrac{1}{2}X_2,\\ \nabla_{X_2}X_3=-\tfrac{1}{2}X_1,\quad \nabla_{X_2}X_4=X_2,\quad \nabla_{X_3}X_4=X_3.\end{gathered}\tag{4.20}$$

The non-zero essential components of the fundamental tensor $F$ and its associated Lie 1-forms are:

$$\begin{gathered}F_{112}=\tfrac{1}{2},\quad F_{211}=-1,\quad F_{222}=2,\quad F_{314}=\tfrac{3}{2},\\ \theta_2=-\theta_4^*=4.\end{gathered}\tag{4.21}$$

From the last equations, we check that the manifold does not satisfy the characteristic conditions of the classes $\mathcal{W}_1$ or $\mathcal{W}_2$. Thus, we prove

**Proposition 4.4.** *The manifold $(G_4, J, g)$ defined by (4.17) is in the class $\mathcal{W}_1\oplus\mathcal{W}_2$.*

Having in mind (4.20), the essential non-zero components of the curvature tensor, the Ricci tensor and the scalar curvature are

$$\begin{gathered}R_{1221}=-R_{2332}=\tfrac{3}{4},\quad R_{1331}=\tfrac{1}{4},\quad R_{2442}=-R_{3443}=-1,\quad R_{1234}=-\tfrac{1}{2}\\ \rho_{11}=\tfrac{1}{2},\quad \rho_{22}=\tfrac{5}{2},\quad \rho_{33}=-\tfrac{3}{2},\quad \rho_{44}=-2,\quad \tau=\tfrac{13}{2}.\end{gathered}$$

**Example 4.4.** Consider the 4-dimensional Lie group defined by

$$G'_4=\left\{\begin{pmatrix} e^y\cos z & e^y\sin z & x & 0\\ -e^y\sin z & e^y\cos z & t & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & e^z\end{pmatrix}\ \middle|\ x,y,z,t\in\mathbb{R}\right\}.\tag{4.22}$$

The left-invariant vector fields on $G'_4$ are

$$\begin{gathered}X_1=e^y\cos z\tfrac{\partial}{\partial x}-e^y\sin z\tfrac{\partial}{\partial t},\quad X_2=\tfrac{\partial}{\partial z},\\ X_3=\tfrac{\partial}{\partial y},\quad X_4=e^y\sin z\tfrac{\partial}{\partial x}+e^y\cos z\tfrac{\partial}{\partial t}.\end{gathered}\tag{4.23}$$

Then, the associated Lie algebra $\mathfrak{g}'_4$ of $G'_4$ is defined by the following non-zero commutator relations

$$\mathfrak{g}'_4:\ [X_1,X_2]=[X_3,X_4]=X_4,\quad [X_1,X_3]=-[X_2,X_4]=-X_1.\tag{4.24}$$

The almost complex structure $J$ on $G'_4$ defined by (4.1) is abelian and hence $(G'_4, J, g)$ is a complex Norden manifold.

By (4.7) and (4.24) the non-zero essential components of the Levi-Civita connection are given by:

$$\begin{gathered}\nabla_{X_1}X_1=-\nabla_{X_4}X_4=-X_3,\quad \nabla_{X_1}X_2=-\nabla_{X_4}X_3=X_4,\\ \nabla_{X_1}X_3=\nabla_{X_4}X_2=-X_1,\quad \nabla_{X_1}X_4=\nabla_{X_4}X_1=X_2.\end{gathered}\tag{4.25}$$

Then, by (4.1) and (4.25) we compute the non-zero components of the covariant derivative of $\nabla J$:

$$\begin{aligned}(\nabla_{X_1}J)X_1 &= -(\nabla_{X_4}J)X_4 = -2X_1,\\ (\nabla_{X_1}J)X_2 &= -(\nabla_{X_4}J)X_3 = 2X_2,\\ (\nabla_{X_1}J)X_3 &= (\nabla_{X_4}J)X_2 = 2X_3,\\ (\nabla_{X_1}J)X_4 &= (\nabla_{X_4}J)X_1 = -2X_4.\end{aligned} \tag{4.26}$$

The rest of the components $(\nabla_{X_i}J)X_j$ are zero. Then, the non-zero essential components of the fundamental tensor $F$ and its associated Lie 1-forms are:

$$F_{122} = -F_{111} = F_{414} = 2, \quad \theta_1 = -\theta_3^* = -4. \tag{4.27}$$

From the last equations, we check that the manifold does not satisfy the characteristic conditions of the classes $\mathcal{W}_1$ or $\mathcal{W}_2$. Thus, we prove

**Proposition 4.5.** *The manifold $(G'_4, J, g)$ defined by (4.22) is in the class $\mathcal{W}_1 \oplus \mathcal{W}_2$.*

We start studying the curvature properties of $(G'_4, J, g)$ by computing the non-zero essential components of the curvature tensor $R_{ijkl}$:

$$R_{1221} = -R_{1331} = -R_{2442} = R_{3443} = -R_{1234} = 1. \tag{4.28}$$

Then, we check that the curvature tensor of the considered manifold has the property $R(X, Y, JZ, JW) = R(X, Y, Z, W)$. In [11], such tensors are termed anti-Kähler.

Let us consider the (0,4)-type tensor $K$ defined by

$$K(X, Y, Z, W) = -\frac{1}{4}g\big((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z\big). \tag{4.29}$$

In [11], is proved that on a complex manifold with Norden metric the curvature tensor $R$ satisfies the following identity

$$\begin{aligned}&\underset{X,Y,Z}{\mathfrak{S}}\big\{R(JX, JY, Z, W) + R(X, Y, JZ, JW)\big\}\\ &= -\underset{X,Y,Z}{\mathfrak{S}} g\big((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z\big),\end{aligned}$$

where $\mathfrak{S}$ denotes the cyclic sum over three arguments. Thus, having in mind that the tensor $K$ is anti-symmetric by its first and second pair of arguments, it follows that in the case when $R$ is anti-Kählerian, the tensor $K$ satisfies the first Bianchi identity, i.e. is a curvature-like tensor.

**Theorem 4.6.** *The curvature tensor $R$ of the complex manifold $(G'_4, J, g)$ defined by (4.22) has the form*

$$R(X, Y, Z, W) = -\frac{1}{4}g\big((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z\big). \tag{4.30}$$

*Proof.* Let $X = x^iX_i$, $Y = y^iX_i$, $Z = z^iX_i$ and $W = w^iX_i$, $x^i, y^i, z^i, w^i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$, be arbitrary vectors in $\mathfrak{g}'_4$. By (4.26) and (4.29) we obtain

$$\begin{aligned} K(X,Y,Z,W) = &\left(x^1y^3 - x^3y^1 - x^2y^4 + x^4y^2\right)\left(z^1w^3 - z^3w^1 - z^2w^4 + z^4w^2\right) \\ &- \left(x^1y^2 - x^2y^1 + x^3y^4 - x^4y^3\right)\left(z^1w^2 - z^2w^1 + z^3w^4 - z^4w^3\right). \end{aligned}$$

Taking into account the components of the curvature tensor (4.28), we get the same expression for $R$. Hence, formula (4.30) is valid. □

Having in mind the commutator relations (4.24), in a similar way to the previous theorem we prove the following

**Proposition 4.7.** *The curvature tensor $R$ of $(G'_4, J, g)$ defined by (4.22) has the form*

$$R(X,Y,Z,W) = g\left([X,Y],[Z,W]\right). \tag{4.31}$$

Let us remark that in [11], an example of a family of $2n$-dimensional Lie algebras with curvature tensor of the form (4.30) and (4.31) is studied.

Further, we compute the non-zero essential components of the Ricci tensor and the scalar curvature as follows

$$\rho_{11} = \rho_{22} = -\rho_{33} = -\rho_{44} = 2, \qquad \tau = 8. \tag{4.32}$$

By (4.32) and (4.2) it follows that $\rho = 2g$ and thus we prove

**Proposition 4.8.** *The complex Norden manifold $(G'_4, J, g)$ defined by (4.22) is an Einstein manifold.*

## 5. Conclusion

In this paper, we have presented and studied examples of 2-, 4- and $2n$-dimensional complex manifolds with Norden metric belonging to three classes in the classification of Ganchev and Borisov. The examples of Kähler manifolds are obtained as real interpretations of non-degenerate second degree curves in the complex plane, and the manifolds in the classes $\mathcal{W}_1$ and $\mathcal{W}_1 \oplus \mathcal{W}_2$ are constructed on Lie groups and Lie algebras.

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## НЯКОИ ПРИМЕРИ НА КОМПЛЕКСНИ НОРДЕНОВИ МНОГООБРАЗИЯ

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**Резюме.** Представени са примери на комплексни многообразия с норденова метрика. Получени са двумерни келерови многообразия като реална интерпретация на гладки криви в комплексната равнина. Конструирани са примери на норденови многообразия с абелева комплексна структура върху групи на Ли и алгебри на Ли.