OPERATIVE APPREHENSION OF GEOMETRICAL FIGURES BY PRIMARY AND SECONDARY SCHOOL STUDENTS[1]

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ABSTRACT

The investigation of this study concerned the role of various aspects of figure modification proposed by Duval (1995), i.e., mereologic, optic and place ways, on the operative apprehension of geometrical figures. Data were collected from 225 primary and secondary school students (grades 6, 7, 8). Findings revealed mostly similarities in the way that primary and secondary school students behaved during the application of the different types of modifying the figures, although some differences were found, as performances are regarded. Students displayed greater consistency in applying the mereologic and the optic figure modification rather than the place figure modification. The mereologic modification tasks appeared to be the most difficult for primary school students to solve. In contrast, secondary school students performed better while applying a mereologic modification, rather than the other two types of modifying a figure.

INTRODUCTION AND THEORETICAL FRAMEWORK

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one because in a figure various relations of an object with other objects are depicted (Mesquita,

[1] This study draws on the research project MED19 funded by the University of Cyprus.

1996). However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998). Geometrical figures are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Therefore, the double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (e.g. Mesquita, 1998).

Duval (1995) distinguishes four apprehensions for a "geometrical figure": perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, perceptual apprehension refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. Sequential apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. Discursive apprehension is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through operative apprehension that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one makes the figure larger or narrower, while the place way refers to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. In a problem of geometry, one or more of these operations can highlight a figural modification that gives an insight to the solution of a problem.

Even though previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Mesquita, 1996; Kurina, 2003), the cognitive processes underlying the four apprehensions for a "geometrical figure"

proposed by Duval (1995) have not been empirically verified yet. Recently, Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2009, in press) have confirmed a three level hierarchy about the role of perceptual, operative and discursive apprehension in geometrical figure understanding. A next research by Gagatsis, Deliyianni, Elia, Monoyiou and Michael (2009, submitted for conference) have also confirmed a three level hierarchy about the role the mereologic, the optic and the place modifications exert on operative figure understanding of primary school students. In this paper, we present a further study that focused on analyzing the cognitive processes underlying the various kinds of geometrical figure modifications, including secondary school students. This knowledge may be useful in understanding students' operative apprehension processes of geometrical figures and in providing teaching implications for the improvement of students' geometrical understanding.

Specifically, drawing on Duval's (1995) theoretical model, the study sought answers to the following three research questions:

(1) Does students' performance differ for each type of geometrical figure modification?

(2) Are there any differences in primary and secondary students' performance on using each of the three types of geometrical figure modification?

(3) How consistently do the primary and secondary school students apply each of the three types of geometrical figure modification?

(4) Are there any differences in primary and secondary students' consistency when applying the three types of geometrical figure modification?

METHOD

The study was conducted among 125 students, aged 11 to 12, from primary schools (grade 6) and 120 students, aged 12-14, from grades 7 and 8 from secondary schools in Cyprus. The a priori analysis of the test that was constructed in order to examine the research questions of this study is the following:

1. The first group of tasks includes task 1 (M1), 2 (M2) and 3 (M3) concerning students' mereologic way of modifying a given figure.
2. The second group of tasks includes task 4 (O4), 5 (O5) and 6 (O6). These tasks examine students' optic way of modifying a given figure.
3. The third group of tasks includes task 7 (P7), 8 (P8), 9 (P9) and 10 (P10) that correspond to the place way of modifying a given figure.

Representative samples of the tasks used in the test appear in the Appendix. Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students' answers to the tasks were codified with M, O and P corresponding to mereologic, optic and place way, respectively, followed by the number indicating the exercise number.

The hierarchical clustering of variables (Lerman, 1981) was conducted using the statistical software C.H.I.C. (Bodin, Coutourier, & Gras, 2000). Thus, a hierarchical similarity diagram of the primary and secondary students' responses to the tasks of the test was constructed. The similarity diagram allows for the arrangement of the tasks into clusters according to the homogeneity by which they were handled by the students.

RESULTS

In order to answer the first and second research questions, students' performance on each type of figural modification was calculated, by examining the means and standard deviations of students' performance on each modification type of geometrical figures. As it can be seen in Table 1, primary students' performance on the place modification tasks ($\overline{X}$ =0.66, SD=0.24) was higher than their performance on the optic modification tasks ($\overline{X}$ =0.61, SD= 0.26), but the use of the t-criterion for paired samples revealed that this difference was not statistically significant (p>0.01). In contrast, students' performance was significantly lower on the mereologic modification tasks ($\overline{X}$ =0.28, SD= 0.29) than their performance on the other two types of modification tasks (p<0.01). As secondary school students are concerned, although their performance on the mereologic modification tasks ($\overline{X}$ =0.57, SD=0.26) was higher than their performance on the optic modification tasks ($\overline{X}$ =0.55, SD= 0.23), this difference was not statistically significant (p>0.01), as it was shown by the use of the t-criterion for paired samples. On the other hand, their performance on the place modification tasks was significantly lower ($\overline{X}$ =0.35, SD= 0.33) than their performance on the other two types of modification tasks (p<0.01).

Table 1: *Mean scores and standard deviations on each type of figure modification*

Types of geometrical figure modification	Primary school students		Secondary school students	
	$\overline{X}$ *	SD	$\overline{X}$ *	SD
Mereologic	0.28	0.29	0.57	0.26
Optic	0.61	0.26	0.55	0.23
Place	0.66	0.24	0.35	0.33

*maximum score=1

Figures 1 and 2 present the similarity diagrams of the primary and secondary school students' responses to the tasks of the test respectively. Particularly, two

similarity clusters can be identified in each diagram. Looking at Figure 1, cluster 1 involves students' responses to all the mereologic modification tasks (M1, M2, M3) and two of the place modification tasks (P9, P10). Cluster 2 is comprised of students' responses to all the optic modification tasks (O4, O5, O6) and the other two place modification tasks (P7, P8). In Figure 2 the first cluster includes students' responses to all the mereologic modification tasks (M1, M2, M3), two of the place modification tasks (P9, P10) and one of the optic modification task (O6). In cluster 2 the variables corresponding to the optic modification tasks (O4, O5) and the other two place modification tasks (P7, P8) are involved.

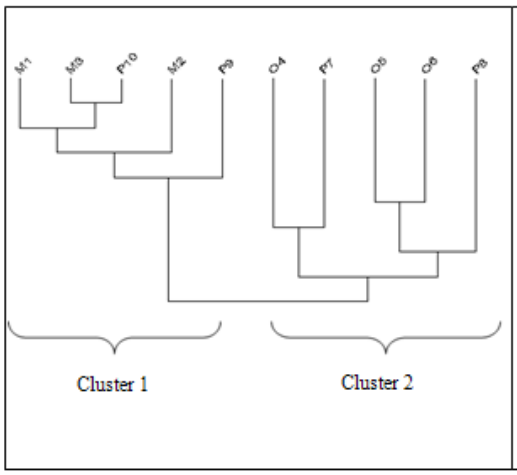

Figure 1: *Similarity diagram of primary school students' responses to the tasks of the test*

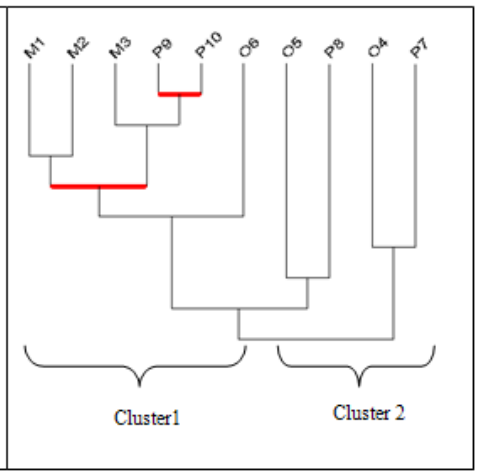

Figure2: *Similarity diagram of secondary school students' responses to the task*

Concerning the third research question, the comparison of the two diagrams indicates that similarities can be found between primary and secondary school students' behaviours, regarding the modifications of the geometrical figure. More specifically, consistency was displayed by primary and secondary school students when applying respectively the mereologic and optic modification of the geometrical figure. This was not the case for the application of the place modification of the figures, since the variables of these types of modification are separated in the two similarity clusters. Primary and secondary school students' behaviour during the place modification of the geometrical figure is similar to the mereologic modification, while similarity with optic modification applications is also found.

Despite the invariance of the relations mentioned above, some differences also arise. In particular, comparing the first cluster of each diagram, we notice that a greater consistency can be found among secondary school students, as regards their responses to the mereologic modification tasks. In addition, a greater consistency is

displayed by the same students regarding the two place modification tasks that are located in cluster 1 of the similarity diagram (P9, P10). Furthermore, as the optic modifications are concerned, primary school students' solutions are more coherent than secondary school students' ones. The variables of this type of modification are gathered in the same cluster for the primary school, while this is not the case for the secondary school, since one of the optic modification variables is placed in a different cluster (O6).

The following two diagrams present the implications between the variables, according to students' behaviour to the tasks of the test. Concerning figure 3, two implicative chains are discriminated. In particular, the first chain indicates that in order students to solve task P9, task P10 must be solved. Furthermore, the solution of the task P10 leads to the accomplishment of the solution for task M3. When succeeding in task M3, students can continue with task O5. Task O4 is situated in the bottom of the second implicative chain. Noticing at figure 4, two implicative chains can also be distinguished. At the top of the first chain task P10 is situated, turning it to a prerequisite for the solution of the task M3. When the solution of task M3 is succeeded, students become able to achieve a correct solution for task P9. The second implicative chain indicates that if a student solves correctly task M2, he will proceed to a correct solution for tasks M1 and P9 respectively. Tasks M1 and P9 can be characterised as the easiest for students to solve.

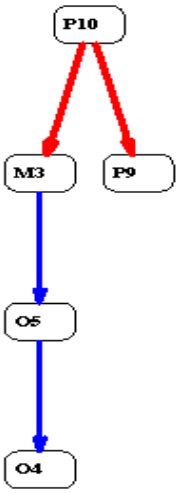

Figure 3: *Implicative Graph For Primary School Students*

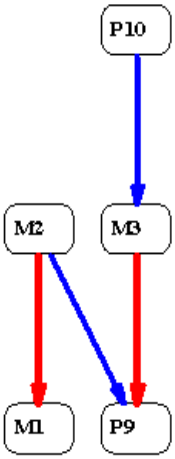

Figure 4: *Implicative Graph For Secondary School Students*

DISCUSSION

In this study, the role of various aspects of modifying a given figure, i.e., mereologic, optic and place ways, was investigated in the primary and secondary school students' (grades 6, 7, 8) operative apprehension. The results revealed that differences existed between the students' performance in mereologic, optic and place modification tasks. In particular, the sixth graders performance on the place modification tasks was similar to their performance on optic modification tasks. In contrast, their performance on the mereologic modification tasks was significantly lower than the other two types of modification tasks. The weak performance on these tasks may have been caused by the fact that they required more complex figural processes relative to most of the other tasks. That is, the students needed to understand the division of the given figure into parts and their combination in another figure and proceed to calculations of specific areas (e.g. M3) or estimations of the figures' perimeter (e.g. M2) in order to provide a solution to the corresponding tasks. As for seventh and eighth graders, they performed in a similar way on mereologic modification and optic modification tasks. On the other hand, a lower performance was shown on the place modification tasks. An explanation for this, according to Duval (1999), could be that the recognition of a figure is independent from its magnitude or its perimeter. It is possible that a conflict will appear to students, between measurements and what can be seen from the figure. In the case which students form a hypothesis based on measurements, operative apprehension is neutralised and the figure stands only as a picture.

The similarity diagrams showed a variation among students' consistency across the three types of geometrical figure modification. Although students exhibited consistency in the mereologic and the optic modification tasks, respectively, they applied the place way of modifying geometrical figures in a rather fragmentary way. A number of the place modification tasks (P9, P10) were approached similarly to the mereologic modification tasks, and the rest of the place modification tasks (P7, P8), were tackled similarly to the optic modification tasks. This finding suggests that although it is the place modification that gives insight to the solution of the corresponding tasks (Duval, 1995) some additional operations need to take place so that students successfully reach the ultimate solution. These additional operations may have common characteristics with the figural processing which is required in either the mereologic modification tasks or the optic modification tasks. Specifically, in the first case, the place modification tasks P9 and P10 did not require only the understanding of the position or orientation variation of the figures, but also the combination of figures in another figure (reconfiguration), which is a characteristic of the mereologic type of geometric figure modification. Moreover, both mereologic and place modification tasks (P9, P10), involved measurement or estimation concepts (e.g. perimeter) and processes in addition to the spatial processes. In the second case, both optic and place modification tasks entailed principally spatial skills and specifically the

comparison of figures of the same form which differed either in their position and orientation because of rotation (P7, P8), or in their magnitude (Fischbein & Nachlieli, 1998), because of enlargement (O4) or variation of distance from a reference point (O5, O6).

Despite the similarities revealed between primary and secondary students' consistency in the application of each type of figure modification, we noticed that primary school students' solutions on optic modification tasks were more coherent than secondary school students' ones. On the other hand, secondary school students displayed a greater consistency regarding their responses to the mereologic and the two place modification tasks modification tasks (P9, P10) that were located in the first cluster of the similarity diagram. This could be explained by the fact that secondary school students have acquired a development in measurement or estimation skills and in spatial processes than primary school students, because of maturation and as a teaching result.

Finally, it is obvious that the subject should be further investigated. It would be interesting and useful to examine whether the difficulties of secondary students and their limited consistency when applying a place modification remain invariant with development and learning at school. It would also be interesting to compare the strategies primary and secondary school students use in order to solve tasks of the three types of modifying a geometrical figure. The effects of intervention programs, which aim to develop students' abilities in modifying a figure, on the operative and other apprehensions for geometrical figures or in geometry problem solving, could also be investigated in future studies.

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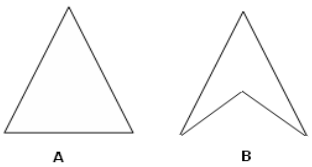
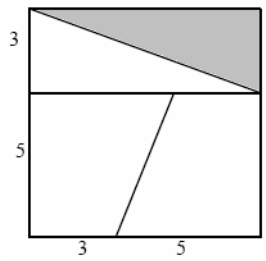
APPENDIX

1. Underline the right sentence: (M2)

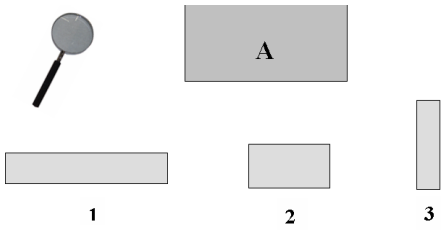


a) Fig. A has bigger perimeter than Fig. B
b) Fig. A has equal perimeter with Fig. B
c) Fig. A has smaller perimeter than Fig. B

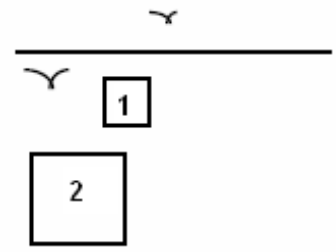
2. This figure is a square. Calculate the shaded area. Explain your answer. (M3)



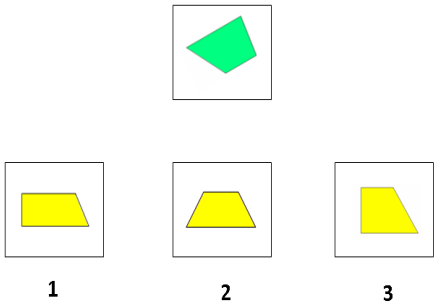
3. Vassilis constructed a rectangle in his writing book. Shape A is the rectangle as it looks through a magnifier. Circle the picture that shows the rectangle, as it is in Vassilis writing book. (O4)



4. Paris is looking the box 1 and 2 in the horizon. He says that the box 1 has exactly the same size with box 2. Is his opinion right? Explain your answer. (O5)



5. Maria must match the cards with the same shape. Circle the yellow card that has exactly the same shape with Maria's card. (P7)



6. Theodosis combines Triangle 1 and Triangle 2 making Figure A. Calculate the perimeter of Figure A. (P9)