METHODS FOR SOLVING SYSTEMS OF TRIGONOMETRIC EQUATIONS WITH TWO UNKNOWNS WITH AND WITHOUT A COMPUTER

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ABSTRACT

We consider several Systems of Trigonometric Equations with Two Unknowns (STETU), solve them graphically by PC and analytically "by paper-and-pen", compare the solving methods and make some comments on them, sharing our own experience.

1. METHODS FOR ANALYTICAL SOLVING (MAS) SYSTEMS OF TRIGONOMETRIC EQUATIONS WITH TWO UNKNOWNS (STETU)

Here we provide several "mini-types" of STETU:

Type 1 (reducible to $\left| \begin{array}{l} f_1(x) = g(y) \\ f_2(x) = g(y) \end{array} \right.$**):**

Problem 1: $\left| \begin{array}{l} Cos^2\left(3x+\dfrac{\pi}{4}\right) = 2Cos^2 y \\ Cos2y + 2Sin2x + \dfrac{3}{4} = 2Sin^3 2x \end{array} \right.$

To obtain the same function of *y* in both equations, we transform the first equation as $Cos^2\left(3x+\dfrac{\pi}{4}\right) - 1 = Cos2y$ and rearrange the second one:

$Cos2y = 2Sin^3 2x - 2Sin2x - \frac{3}{4}$. The two initial equations are "separable", i.e. they may be written in the form $f(x) = g(y)$. Here the STETU is reducible to

$$\left| \begin{array}{l} f_1(x) = g(y) \\ f_2(x) = g(y) \end{array} \right. .$$

By equating the left hand sides containing *x*, from the two equations we obtain an equation of one variable and combine it with one of the given ones to solve the system.

Problem 2: $\left| \begin{array}{l} 9^{2Tanx+Cosy} = 3 \\ 9^{Cosy} - 81^{Tanx} = 2 \end{array} \right.$

We may rearrange the STETU: $\left| \begin{array}{l} 9^{Cosy} = 3.9^{-2Tanx} \\ 9^{Cosy} = 9^{2Tanx} + 2 \end{array} \right. .$

Problem 3: $\left| \begin{array}{l} |Sin3x| = -\sqrt{2}Siny \\ Cos2y + 2Cos2x.Sin^2 2x = \frac{3}{4} \end{array} \right.$

By rising the equation with the absolute value to second power we obtain $2Sin^2 y$ and transform it to $1 - Cos2y$.

Problem 4: $\left| \begin{array}{l} |Cos3x| = Siny + Cosy \\ 2Sin^2 2x.Cos2x + \frac{3}{4} = -Sin2y \end{array} \right.$

Problem 5: $\left| \begin{array}{l} \left| Sin\left(3x + \frac{\pi}{4}\right) \right| = Siny - Cosy \\ Sin2y + 2Sin2x = \frac{3}{4} + 2Sin^3 2x \end{array} \right.$

By rising the equations with the absolute value in the previous two systems to second power we obtain $1 \pm Sin2y$. Some STETU of the type $\left| \begin{array}{l} f_1(x) = g(y) \\ f_2(x) = g(y) \end{array} \right.$ are also easily solvable by other concrete methods having in mind the rich set of trigonometric formulas.

Type 2: $\left|\begin{array}{l} f_1(x)=g_1(y) \\ f_2(x)=g_2(y) \end{array}\right.$ **not (easily) reducible to** $\left|\begin{array}{l} f_1(x)=g(y) \\ f_2(x)=g(y) \end{array}\right.$**:**

Problem 1: $\left|\begin{array}{l} x-y=-\dfrac{1}{3} \\ Cos^2\pi x-Sin^2\pi y=\dfrac{1}{2} \end{array}\right.$. Here $x=y-\dfrac{1}{3}$ (the explicit function) we substitute in the second equation and decrease the powers: $1+Cos\pi\left(y-\dfrac{1}{3}\right)-1+Cos2\pi y=\dfrac{1}{2}\Leftrightarrow Cos\pi\left(2y-\dfrac{1}{3}\right)=\dfrac{1}{2}\ldots$

Problem 2: $\left|\begin{array}{l} x+y=\dfrac{\pi}{4} \\ tgx.tgy=\dfrac{1}{6} \end{array}\right.$. We substitute y (inverse function, explicit function) from the first equation in the second one, simplify $Tan\left(\dfrac{\pi}{4}-x\right)$ and solve the obtained quadratic equation with respect to $Tanx$.

Problem 3: $\left|\begin{array}{l} tg\left(\dfrac{x}{2}\right)+tg\left(\dfrac{y}{2}\right)=\dfrac{2}{\sqrt{3}} \\ tgx+tgy=2\sqrt{3} \end{array}\right.$. Let $Tan\dfrac{x}{2}=a, Tan\dfrac{y}{2}=b$. Then we have

$$\left|\begin{array}{l} a+b=\dfrac{2}{\sqrt{3}} \\ \dfrac{2a}{1-a^2}+\dfrac{2b}{1-b^2}=2\sqrt{3} \end{array}\right.\ldots$$

As all the functions $f_i(x), g_i(x), i=1;2$ are invertible, these 3 systems might be transformed to $\left|\begin{array}{l} y=F(x) \\ y=G(x) \end{array}\right.$ (sub-type of Type 1), but via more transformations, and the obtained equations will “look strange”.

Type 3 (exactly one separable equation):

Problem 1: $\begin{vmatrix} x+y=\dfrac{5\pi}{6} \\ 5(Sin2x+Sin2y)=2(1+Cos^2(x-y)) \end{vmatrix}$.

The first equation is separable and its functions are invertible. Other different ways for solving it are based on different trigonometric formulas. **But they are not easier**.

Problem 2: $\begin{vmatrix} |x|+|y|=\dfrac{\pi}{4} \\ Cosx-Cosy=Sin(x+y) \end{vmatrix}$

We may transform the two sides of the second equation into products, simplify and solve it by considering the zeroes of the factors. The first equation is separable and its functions are invertible with pairs of inverse functions. Thus the solution via using inverse function **is not so complicated**.

Type 4 (two non-separable equations):

Problem 1: $\begin{vmatrix} 6Sinx.Cosy+2Cosx.Siny=-3 \\ 5Sinx.Cosy-3Cosx.Siny=1 \end{vmatrix}$. If $u=Sinx.Cosy$, $v=Cosx.Siny$, then $\begin{vmatrix} 6u+2v=-3 \\ 5u-3v=1 \end{vmatrix} \Leftrightarrow u=-\dfrac{1}{4}, v=-\dfrac{3}{4}$; $\begin{vmatrix} Sin(x+y)=u+v=-1 \\ Sin(x-y)=u-v=0,5 \end{vmatrix} \dots$

2. METHODS FOR GRAPHICAL SOLVING OF STETU WITH A COMPUTER (MGSC)

Type 1: $\begin{vmatrix} f_1(x)=g_1(y) \\ f_2(x)=g_2(y) \end{vmatrix}$

How to represent graphically a solution of one of these equations? We have a solution (*a*,*b*), when $f_1(a) = g_1(b)$ (along *Oz*). Their graphs are surfaces and their intersection in most cases is a curve. Solution of a system of two equations is the intersection of two curves. Solution with 3D-software requires knowledge and skills, which the ordinary schoolchildren do not have. We think it is rather for talented ones.

Is it possible to present a STETU on a plane? In the coordinate plane $f_1(x)$ takes values along the *y*-axis. If we construct the graph of $g_1(x)$ instead of $g_1(y)$, we will have the $g_1(y)$-values **along the *y*-axis too**. As the second function is of **another** variable, the points of intersection of the two graphs present symmetric solutions of the equation (a,a). The ordered pair of the *x*-coordinates of **any** two points **on the same height** (the first point on $f_1(x)$ and the second - on $g_1(x)$) is solution of the equation.

Let us consider problem 5 of Type 1. We rearrange the equations, thus:

$$\begin{cases} \left|Sin\left(3x+\frac{\pi}{4}\right)\right| = Siny - Cosy \\ \frac{3}{4} + 2Sin^3 2x - 2Sin2x = Sin2y \end{cases}.$$

The period of $g_1(x)$ is 2π and the period of $f_1(x)$ is $\pi/3$. Then a vertical strip on the screen of width 2π (between the lines *Oy* and *e* in **Fig. 1**) includes enough from the graphs.

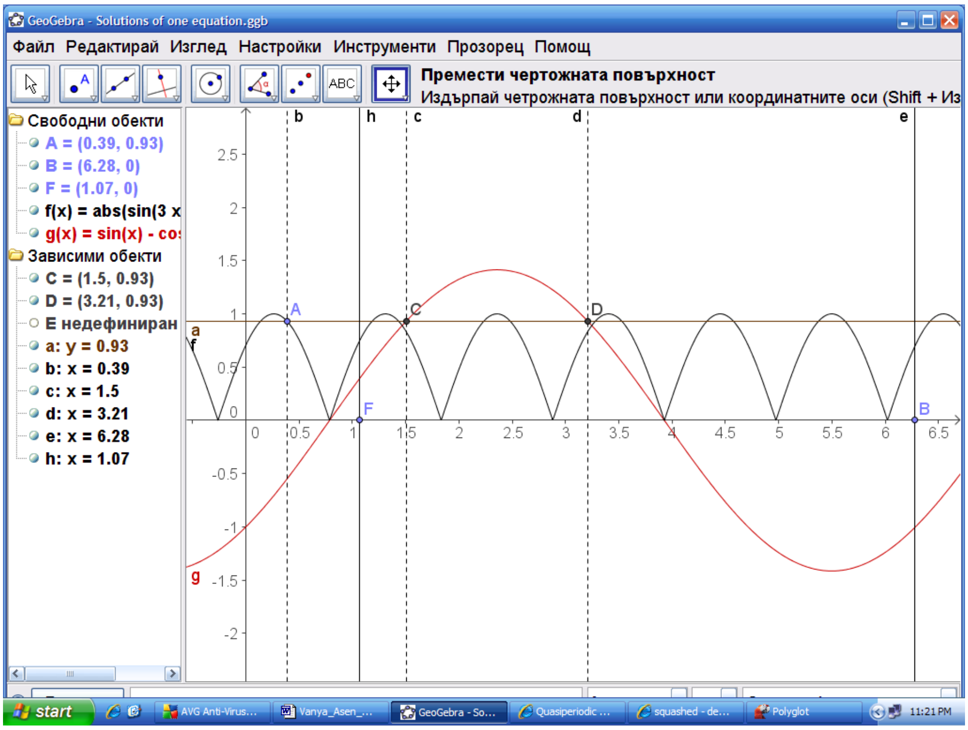

Fig. 1: Graphical presentation of the equation $\left|Sin\left(3x+\frac{\pi}{4}\right)\right| = Siny - Cosy$.

A is a slider point on $f_1(x)$, *a* is the horizontal line through *A*. *a* intersects $g_1(x)$ at several (or none) points (two on the screenshot – *C* and *D*). Hence the ordered pairs (x_A, x_C) and (x_A, x_D) of the *x*-coordinates of the points are solutions of the equation. It is enough to move the slider along the sector of the curve between the vertical lines *Oy* and *h*, which is one period of $f_1(x)$, in order to show with some approximation different solutions of the equation. Line *a* intersects the graph of $f_1(x)$ at two points on the sector between *Oy* and *h* in **Fig. 1**, but this does not provide new solutions, because the slider *A* will pass through the additional point and the intersection of *a* with $g_1(x)$ will be again {*C, D*}.

The system's solutions are **among** the solutions of the first equation. In order to find them, we use that the graphical representation of $\frac{3}{4}+2Sin^3 2x-2Sin2x=Sin2y$ is analogous. We plot the graphs of $f_2(x)$ and $g_2(x)$ (**Fig. 2**). If (*a*,*b*) is a solution of the system, then the points $A(a, f_1(a))$ and $C(b, g_1(b))$ have the same level (lie on the parallel to *Ox* line *a*) and the points $E(a, f_2(a))$ and $G(b, g_2(b))$ lie on another parallel to *Ox* line, i.e. *AEGC* is a rectangle. Else *AEGC* is a trapezium. (See **Fig. 2**). Hence the second equation will have a solution if line *EG* or *EH* is parallel to *Ox*. To find the solutions **more easily** we construct a horizontal line *i* through *E*, move the slider *A* and look **only** if **either** *G* or *H* will "step" on *i*. This will be a solution of the system.

We use GeoGebra – free software with good precision tested by us at school. Construction:

1) The four graphs
2) The slider *A* (on the graph of $f_1(x)$)
3) The horizontal line *a* and the vertical line *b* through *A*
4) *E* – the intersection of *b* with the graph of $f_2(x)$
5) The intersection points of *a* with the graph of $g_1(x)$
6) Vertical lines through them
7) Their points of intersection with the graph of $g_2(x)$
8) The horizontal line *i* through *E*.
9) The four graphs
10) The slider *A* (on the graph of $f_1(x)$)
11) The horizontal line *a* and the vertical line *b* through *A*

12) E – the intersection of b with the graph of $f_2(x)$

13) The intersection points of a with the graph of $g_1(x)$

14) Vertical lines through them

15) Their points of intersection with the graph of $g_2(x)$

16) The horizontal line i through E.

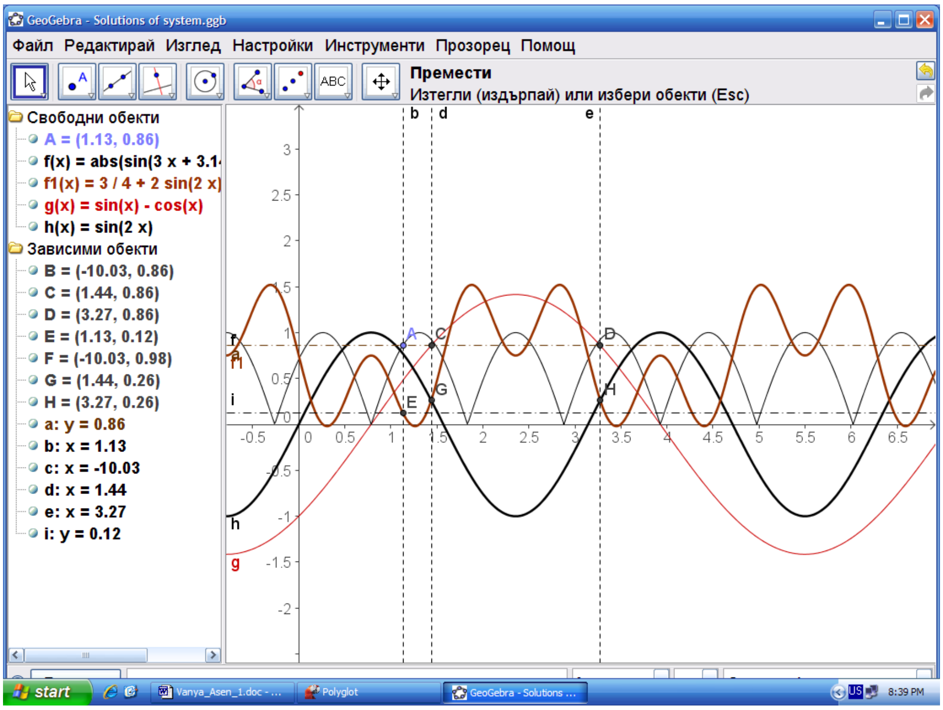

Fig. 2: Graphical presentation of the system.

Because the screen is overladen, we may make the verticals through A, C and D invisible after constructing E, G, H. The graph of $f_2(x)$ may also be hidden after the construction of E. The graph of $g_2(x)$ is needed as points of its intersection with additional verticals may become necessary to find in case of more points of intersection of a with the graph of $g_1(x)$. The graph of $f_1(x)$ is not necessary either, because the slider "knows" its trajectory as we pull it from left to right. An important thing is to check whether the period of $f_2(x) \backslash g_2(x)$ is longer than π/3 \ 2π. If so, then we take broader domain(s) for x or/and y (until they include the periods of all functions).

Type 2 (at least one separable equation with an (easily) invertible function):

Let the system be of type $\begin{vmatrix} (f(x))^2 = (g(x))^2 \\ F(x,y) = 0 \end{vmatrix}$. Then $f(x) = \pm g(y)$ and $y_{1,2} = g^{-1}(\pm f(x))$. The curves through B and C in **Fig. 3** are the graphs of $y_{1,2} = g^{-1}(\pm f(x))$ and the other curve is the graph of $F(x, g^{-1}(f(x)))$. The graph of $F(x, g^{-1}(-f(x)))$ is the same in our case: $\begin{vmatrix} Cos^2\left(3x + \dfrac{\pi}{4}\right) = 2Cos^2 y \\ Cos2y + 2Sin2x + \dfrac{3}{4} = 2Sin^3 2x \end{vmatrix}$. Every solution (a,b) of such STETU has these properties: $F(a, g^{-1}(f(a)))$ = $F(a, g^{-1}(-f(a)))$ = 0 and $b = g^{-1}(\pm f(a))$. And if $F(c, g^{-1}(f(c))) = 0$, then $(c, g^{-1}(f(c)))$ is solution of the system. That is, the co-ordinates of B and C in **Fig. 3** are solutions.

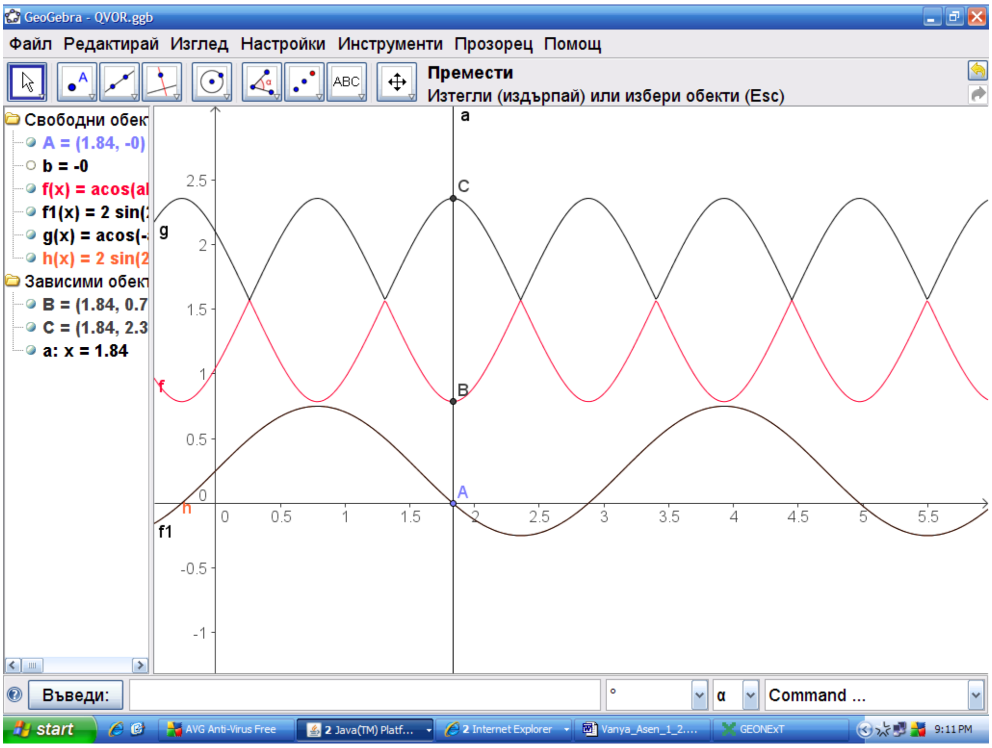

Fig. 3: MGSC of a STETU via using of inverse function.

3. COMPARISON OF THE METHODS

There are different MAS for the different types of STETU. We do not classify them. To deal successfully with an arbitrary STETU, students must have very solid mathematical training, such as handling trigonometry, knowing a set of MAS of algebraic and trigonometric equations, systems of equations; judging which transformation is equivalent etc; to be able to construct mentally a multiple-step strategy for solution of the problem and so on. We do not analyze these skills here.

MGSC of a STETU are **fewer** in number and each of them holds for **many types** of STETU. To succeed, a student does not need to know a large set of tools /methods. At the same time MGSC are really difficult even for the excellent students at the heuristic and construction steps of the solution.

The MGSC concept of a STETU might be introduced to the students via a step-by-step didactic technology (DT), which we plan to create in the future. A system of one unknown (STEOU) might be the first step in such a DT.

Articles [1] and [2] contain solutions of systems by the professional package MATHEMATIKA. Our methods are different and we use didactics software, because it is free, has an easier syntax and an option for Bulgarian interface (menus). Graphic files are conveniently exportable in different formats. The screenshots we have put here aim to enable the readers to see not only the graphics window, but also the algebraic one, the toolbar and the menus. The STETU provided here are from [3].

REFERENCES

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[2] Lazarov B., Mathematics and Informatics Quarterly, vol. 7, No. 4, 1997.

[3] Korolev, S. V. "Trigonometry in math exams", Moscow, 2006. (in Russian)