COMPUTER ALGEBRA - A GOOD REASON TO ENHANCE INTEREST AND ENTHUSIASM IN MATHEMATICS CLASSES

Elena A.Varbanova
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
8, Kliment Ohridski St., Sofia - 1000, Bulgaria
elvar@tu-sofia.bg

ABSTRACT

The learning environment is a bridge connecting the learner and teacher. The nature and value of the "objects and the net load" crossing the bridge are strongly dependent on the bridge capacity and quality. The latter could be improved integrating Computer Algebra (CA) for learning mathematics. CA used in an adequate way could enhance interest and enthusiasm in mathematics classes. Enthusiasm, however, is not a substitute for knowledge and imagination. It is teachers' duty not to reduce mathematics (for the sake of technology) to automatically obtained results and graphical images. To achieve this goal, teachers need adequate qualification and competency. Some ideas and educational experience in integrating CA into undergraduate mathematics are presented in the paper.

INTRODUCTION

> "As long as education can change, the world can change" [4]

Computer algebra systems (CAS) exist and mathematicians need to be involved in reflecting on their effect. Like any other technological tool, CAS print their mark in the form of instruction in mathematics education. The availability of such a powerful tool allows new avenues of approach. For instance, it allows the teacher not to spoon-feed his/her students and make them spectators only but to give them time and space to participate actively in the teaching-learning-assessment (TLA-) process. However, it has to be shown to the student that it is the knowledge and mental skills that can make a tool an effective and reliable instrument [1].

Since 1997, at the Technical University of Sofia we have tried to facilitate the learning of undergraduate mathematics and make the study come vital. CAS Derive [5] has been used to augment and to view concepts and statements in new ways. The range of questions and problems that can be handled successfully has been also extended. Our experience in combining traditional modes and CAS supported approaches is represented in [6-11]. The approaches have been developed having non-mathematics students in mind. They need a mathematics course that does not overwhelm them and is not unnecessarily complicated, but at the same time develops capabilities they need to be successful.

In the course of time, the author has developed a combined interactive model for polyvalent application of CAS in the TLA-process: delivering dynamic presentations in class, combining CAS-free and CAS-supported environment during the seminars, developing materials for project-based learning, creating assessment materials containing CAS-related components. The model has been applied in the mathematics courses for first-year students at the Technical University of Sofia and the International University College – Sofia (University of Portsmouth Programme). It is appreciated by the students and does boost their active part in the learning process. The latter is an essential ingredient in the learning of mathematics.

The next part of the paper presents some aspects of the Problem Solving activity aiming at particular educational goals and learning objectives. A number of representative examples associated with thought-provoking questions and ideas are presented. This paper is to be considered as an extension of the section "More of Problem Solving" in [11].

In the following examples the emphasis is on methodological principles (in application of CA in both pencil-based and software-related curricula) that could be helpful for teacher and useful to students [3]. Students are guided through such examples. Then they need to be given a set of problems in the same vein to solve. "In silence and solitude", they can make their own discoveries through experiments and through making their own mistakes.

Computer-based activities are not carried out in an ad hoc manner or in isolation: they are integrated into the mathematics course and related to other aspects of the curriculum. They are designed to further enrich and extend the existing scenarios for the activity Problem Solving.

"WORK SMARTER, NOT HARDER"

I was not surprised but I was very much impressed reading that the greatest enemy of people is boredom: not the boredom of doing nothing but the boredom of doing things that do not make them excited. What could the cure be against that "enemy" when it goes about mathematics classes? - Make things simple and interesting in order to provoke students' excitement and intellectual curiosity, to involve their full potential – thought, imagination, inspiration, and intuition.

An experienced teacher will construct appropriately structured systems of questions and problems for each topic and for the entire mathematics course. She/he will consider examples of the type below at the right time and place – in exercises on inverse functions, which precede exercises on differentiation.

Example 1. Find the first derivative of the function $F(x) = \arctan\left(\sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}}\right)$.

Comment: Is this function suitable for demonstrating or exercising the Chain Rule? Definitely not - because it will be tedious, time-wasting and could put students off mathematics for life. At the same time, it is suitable for technology-meets-tradition activity and it is here that CAS can come to the rescue. The visual presentation of the function (Figure 1) gives the teacher and the student the cue how to approach the problem: "First simplify!" – to the periodic piece-wise linear function, described by

$$(1) \qquad f(x) = \begin{cases} -x + \dfrac{\pi}{4}, & x \in (-\dfrac{\pi}{4}, \dfrac{\pi}{4}] \\ x - \dfrac{\pi}{4}, & x \in [\dfrac{\pi}{4}, \dfrac{3\pi}{4}) \end{cases},$$

and then differentiate.

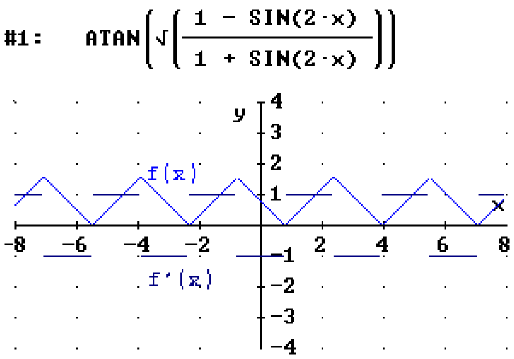

Fig.1 Example 1

Applying the Chain Rule here would develop the bad habit to solve the problem, i.e. to do whatever, somehow. Because mathematics teaching and learning is a skill-and-habit-forming process, hence, it makes its mark on the learners for life [3, 4]. Technology-meets-tradition activities can contribute to the enhancement of the results of this process.

In this case, CAS is used to get the function's graphical image which contains information of higher intensity about its analytical representation. Analyzing the graph, the student could build assumptions about appropriate steps for simplifying the initial expression. The model of Pap here helps define all the necessary consecutive steps.

As for the CAS available, none of them can simplify the initial function. Unfortunately, the very complicated expressions for the derivative obtained by them also need "No comment". A fruitful collaboration of developers of CAS and professionals in mathematics education will considerably improve this situation.

DO NOT KILL SPARROW WTH A CANNON

"Action is Enemy of Thought"

The aim of this section is to provide another example for illustrating one of the advantages of CAS in the development of students' habit and skills for doing things not just anyhow [2].

Example 2. Evaluate the integral: $\int_0^3 \sqrt{9-x^2}\, dx$.

Comment: In mathematics textbooks, either Substitution method or Integration by parts is applied. The authors of these approaches disregard important requirements like "Exact, clear and concise" [3] to the process of solving problems. As a result, students are "shown" how to make simple things complicated instead of making complicated things simple.

Actually, no calculation is needed in this particular case. By plotting the integrand and shading the area of interest (Figure 2), the student can compute the value in his/her head. Here is the role of the teacher to actively and interactively involve the students in creating "solution by observation".

To demand student reflection on the results obtained, i.e. on the work done by CAS, the students have to be able to confirm the result approaching the integral analytically. The Russian proverb "Trust but verify" has proved to be useful in doing mathematics with CAS.

To apply the principle "From simple to complex, from easy to more difficult, from known to unknown", the student is asked to experiment with the following types of integrals progressing in difficulty:

$$\int_0^r \sqrt{r^2-x^2}dx \;\Rightarrow\; \int_0^r \sqrt{2rx-x^2}dx \;\Rightarrow\; \int_a^{a+r} \sqrt{r^2-(x-a)^2}dx.$$

They help the student observe the phenomena of describing different objects by the same mathematics concept of definite integral. She/he becomes aware of the meaning and the importance of its "components": the integrand and the lower and upper limits of integration (Figure 2 - lines #2, #3, #4 and the plot in the upper right window).

The challenging question about the general case: $\int_{a}^{a+r}(b+\sqrt{r^2-(x-a)^2}\,)\,dx$

is appropriate for a Calculus project and certainly provokes a discussion among students (Figure 2 - line #5 and the lower right plot) on one hand, and students and teacher, on the other hand. During my 4-year experience in teaching Calculus, project-based learning has proved to be a meaningful activity for natural interaction of the student with peers and teacher, as well as for discovering mathematics through their own experiments. Actually, this learning strategy makes the TLA-process student-centered.

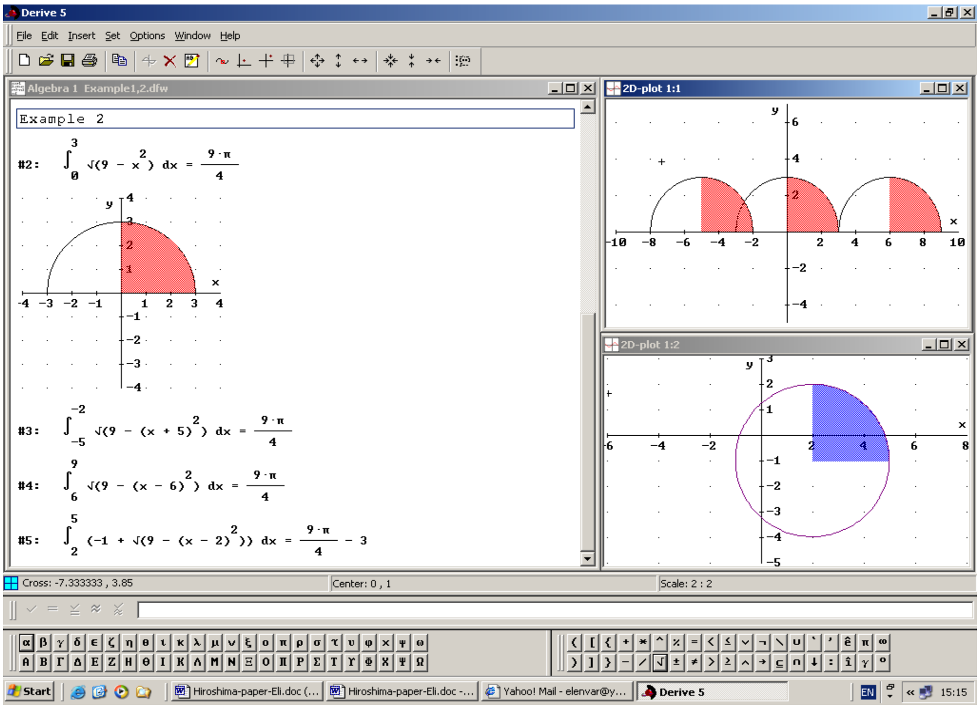

Fig. 2 Example 2

As far as the Substitution method is concerned, suitable integrals have been given that cannot be approached by “conventional weapons”: common sense, definitions, properties, statements, geometrical interpretations of concepts and statements, etc. There are two important guiding principles in the teaching-learning process – on the one hand, new knowledge/ideas/skills are usually built upon existing ones, and on the other hand, new knowledge/ideas/skills are introduced to take the student up on a higher stage, where from he/she could see more (“One can see as much as one knows”) and then could solve the problem more efficiently. The student will then be convinced in the necessity of acquiring extra knowledge and skills and will acquire them in a natural way. Simplicity and accessibility really boost student's interest and enthusiasm to participate in the learning process.

EXTEND EXISTING IDEAS

"Every picture tells a story"

As for solving given definite integrals, students become good sooner or later. However, when it comes to plotting and shading the area of interest, i.e. in giving geometrical meaning of the results obtained, students often face difficulties. To bridge this gap in their knowledge and skills, I decided to "reverse" the problem of solving integrals: students are asked to plot and shade the area first and then to construct and evaluate the corresponding integral (Figure 3). CAS is irreplaceable for this purpose.

Example 3. Find the area bounded by the straight lines $y=2$, $y=4$, $y=2x$ and the curve $y=8-x^2$.

Solution: The student plots the lines and the curve and shades the area:

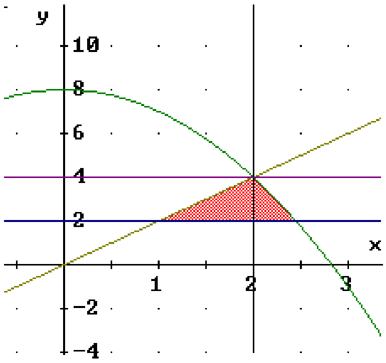

Fig. 3 Example 3

Then the type of the region is discussed and the right decision is made about its description in terms of double inequalities and the construction of the corresponding integral:

$$\#5: \quad 2 \le y \le 4 \wedge \frac{y}{2} \le x \le \sqrt{(8-y)}$$

$$\#6: \quad \int_2^4 \left(\sqrt{(8-y)} - \frac{y}{2} \right) dy = 4\cdot\sqrt{6} - \frac{25}{3}$$

The work done can be used for describing the same region as a union of two sub-regions. Using CAS, the student can easily find the points of intersection (the vertices of the region) and then compute the area of interest constructing the two integrals with respect to x:

$$\#7:\quad \text{SOLVE}(8 - x^2 = 2\cdot x,\ x,\ \text{Real})$$

$$\#8:\quad x = -4 \vee x = 2$$

$$\#9:\quad \text{SOLVE}(8 - x^2 = 2,\ x,\ \text{Real})$$

$$\#10:\quad x = -\sqrt{6} \vee x = \sqrt{6}$$

$$\#11:\quad \int_1^2 (2\cdot x - 2)\,dx + \int_2^{\sqrt{6}} (8 - x^2 - 2)\,dx = 1 + 4\cdot\sqrt{6} - \frac{28}{3} = 4\cdot\sqrt{6} - \frac{25}{3}$$

This approach to simple integration serves as a good preparation for introducing double integration.
The application of CAS allows the use of the three types of description or "prototypes" of the object - graphical, analytical and numerical. This is of great importance when solving real-world problems: it helps non-mathematics students see the relations of mathematics knowledge to everyday knowledge.

CHALLENGE EXISTING IDEAS

"Small things make perfection but
perfection is not a small thing" [4]

The example below is given to students to illustrate that CAS can make a valuable contribution to any aspect of mathematics teaching and learning.

Example 4. Solve the integral: $\int \frac{\cos x}{1+\cos x}\,dx$.

Comment: Substitution method or "...from known to unknown..."?
If the student has not a clear idea about the right first step to approach the solution, let him/her "ask" CAS and see the result:

$$\#1:\quad \int \frac{\text{COS}(x)}{1 + \text{COS}(x)}\,dx = \frac{x\cdot\text{COS}(x) - \text{SIN}(x) + x}{\text{COS}(x) + 1}$$

As the structure of the solution is still not helpful for building assumptions about the first step, students are recommended to apply the command Expand (in CAS Derive) with respect to x:

$$\#2:\quad \int \frac{\text{COS}(x)}{1 + \text{COS}(x)}\,dx = x - \text{TAN}\left(\frac{x}{2}\right)$$

Now let the students observe the new structure and reformulate the question: how to represent the integrand in the form $1-f(x)$, where $f(x)$ is the derivative of $\tan\frac{x}{2}$, i.e. $f(x)=\frac{1}{2\cos^2\frac{x}{2}}$.

Here is the right time and place for the teacher to explore the comparison principle: the given integrand has the same structure as those of the following integrals solved earlier:

$\int\frac{x}{1+x}\,dx = x-\ln|x+1|$ and $\int\frac{x^2}{1+x^2}\,dx = x-\tan^{-1}x$.

The only thing left is to add and subtract 1 to the numerator of the given integrand and reduce it to two basic integrals.

Then students can be asked to solve the definite integral $\int_0^{\pi/2}\frac{\cos x}{1+\cos x}\,dx$ by substituting $t=\tan\frac{x}{2} \Leftrightarrow x=2\arctan t$ with the only purpose that they become aware of the difference between solving problems somehow and solving problems not just anyhow.

Effective exercises are at the heart of any mathematics course. The feeling of being able to "control mathematics" rather than "being controlled by mathematics" proves to be important to student's motivation for participating in the learning process. CA used in an adequate way could support this process bringing interest and enthusiasm back to mathematics classes.

CONCLUSION

Good educational tradition enables its development through technology. Development implies disintegration of old forms and integration of new ones without completely destroying some of the beneficial old elements. But this process should by no means result in replacing an effective methodology by a pseudo-methodology.

When the students do mathematics both in a CAS-free and in a CAS-supported environment, they can appreciate the power of CAS and at the same time they will convince themselves that software is not a panacea. They have the opportunity to take advantage of CAS to

- obtain, check and compare solutions obtained by different approaches and techniques;
- get visual presentation of objects and results;
- support building assumptions and hypotheses.

The goal of CAS-supported undergraduate mathematics education is not only to learn how to use CAS, nor is it to show how quickly and easily CAS can obtain solutions. The goal to have students enhance their capabilities both in mathematics and in technology by means of appropriate questions and challenging tasks requiring more thought, imagination and intellectual curiosity.

I dare believe that this paper provides a starting point for a discussion on the questions: “What do we teach solving problems somehow?” and “What goals do we achieve solving problems not just anyhow?”. These questions require special attention by the authors of mathematics textbooks and of online delivery courses in mathematics, on one side, and the developers of CAS, on the other side. General methodological principles and proven practices in mathematics education have to be observed.

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