COMBINING NON-FORMAL AND FORMAL DISCUSSION HAS A PSYCHOLOGICAL EFFECT ON STUDENT PARTICIPATION IN SOLVING PROBLEMS

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ABSTRACT

Activating, exciting an interest, and using mathematical thinking in everyday activities were treated in **[**2**]** *and later in* **[**3, 4**]** *as aspects of my dialog teaching method. Another aspect is a psychological stimulation of the student participation. It can come by combining the formalization of the problem and its solution with the inverse word expression of a formalization and their correction. As earlier, the author sticks to the illustration on concrete and cited easy problems* **[**5, 6, 7**]** *which do not require special knowledge. He considers this way of exposition more effective for an apprehension than a general one. It gives enough premises for imaging how the approach can be applied in other situations. He has been applying it in all taught courses, including those in special math, since his early career.*

INTRODUCTION

The author of **[**1**]** writes “mathematics courses are not“, “traditionally“, “heralded as participation courses”. She indicates the expectation of active student participation and discusses its assessment. I have a great deal of experience in applying a dialog/activating method in teaching with very positive results. The indicated tradition is the reason why my new students seem initially surprised. In addition, this tradition leads to some natural fear of its application among the professors. My dialog/discussion method has many important aspects. The title of the paper shows one of the psychological ways to provoke student participation. We treat here these ways because without a beginning there is no development. Currently, there are no papers about this kind of psychological stimulus. Some other aspects of my method or its results could be found in **[**2,3,4**]**.

The best exposition of a methodological way is its illustration with examples. It was indicated in the abstract that the problems cited here are from **[**5, 6, 7**]** and therefore reading the text of this paper really does not need the references. The advantage of such an exposition is its concrete base and the possibilities of an effective imagination about how it could work at another concrete, complicated situation, be it a formal or a special one. It demonstrates the immediate reaction of student appreciation which shows both the psychological effectiveness and their involvement. The participation can be in different forms which depends on the concrete audience and could have different descriptions.

EXAMPLE 1

This example is my choice because it is not complex, it is applicable to any audience, and because it is expressive enough with its variety. It is from the textbook **[**5**]** of a Mathematical Inquiry course. It includes arithmetic, algebra, geometry, probability, and statistics chapters.

Problem. (**[**5**]**, p.198, solution **-** p. 202)**.** "*Fred has to get his parents' car back to their house by 5 o'clock. If it is now 3 o'clock and his parents' house is* ***240 miles*** *away, at what rate does Fred have to drive? (Remember* ***RxT = D****.)*"

The students are invited to give suggestions. It is written in the syllabus that this problem is one of the problems for the class session (a psychological effect, too), and some students have seen the solution. If not, the question, "what is **R**, what are **T** and **D**?" directs and almost always there is a student with the suggestion to substitute in the reminded formula the time **T** with **2** (**hours**) to get back and the distance **D** with **240** (**miles**). So we get the equation **2R = 240** exactly as it appears incorrectly in the solution from the book.

My psychological non-formal instruction is absolutely surprising "please re-read the text of the problem". It is so surprising that I was constrained to repeat it absolutely seriously and with an appropriate intonation. Usually, I was asked, "why?" An additional non-formal question of mine, "what is the most difficult part of Fred's task?" leads to the student answer, "… to get the car back by **5 o'clock**". The next non-formal and unexpected instruction is, "could you express that in other words?" This instruction psychologically leads to more active participation. Usually, the answer is almost correct, "… to get the car back before 5 o'clock."

"… not the exact expression!" Sometimes that is a student note. And finally,

"… not later than **5 o'clock**."

"How much time?"

"**2 hours**."

"Correction?"

"… In no more than **2 hours**."

Now there is a combination with a question/instruction about formalization, "how to translate the text of the problem into mathematical language?" This form ("translate") has an absolutely psychological impact for the explanation of the necessity of a sufficient rate and what that means. This rate must be sufficient to drive the distance of **240 miles** for no more than **2 hours** or in its equivalent form: For **2 hours** he has to drive not less than **240 miles** (**240÷R ≤ 2 ⇔ 240 ≤ 2R**) . The possible inaccuracies **2R > 240**(< **,** ≤) in the inequality should be corrected with non-formal or formal instructions like “a little bit more exactly” (“is that less than …?”, … “… less than or equal to …?”

So we come to the correct inequality **2R ≥ 240** and its solution **R ≥ 120**. Simultaneously, it is shown the solution in the book is incorrect. The students are non-formally convinced in this in a psychological way.

A very important psychological effect in problems like this one is the requirement always to write the solution in words, “… he must drive with a speed not less than **120 miles per hour**….” There are possible corrections due to the missing words “miles“ or “per hour”. An additional psychological effect can have the question, “what is the correct text of the problem in order it to match to the incorrect solution **R = 120 mph** from the book?” Analogically, for other inequalities.

There is sufficient variety for combining non-formal and formal discussion in this easy example which can be applied to other problems including formal courses. The task here is a demonstration of the psychological effect of its content. It was only marked there are different class forms for carrying this out.

MOTIVATION FOR THE SELECTION OF THE NEXT TWO EXAMPLES

The next two examples are from the introduction to the Theory of Groups. The first motive is we need to have examples about the psychological effect of combining formal and non-formal discussion from a Math major course but which again do not require very special knowledge except the definitions of a group, of an abelian (commutative) group, and of a group homomorphism. The solutions do not need any special previous results but it is shown below how we use the indicated definitions. Simultaneously, they give a possibility for a more exact mathematical expression and punctuality. A part of the discussion comes from their formulations and the written instruction for a solution. They are compared and annotated with corresponding conclusions below. Both of them are formulated in **[**6**]**, p. 11:

“…

Definition. *Two elements* ***a*** *and* ***b*** *in a semigroup* ***G*** *commute in case* ***ab = ba****. A semigroup is abelian (or commutative) in case every two elements in* ***G*** *commute.*

…

Exercises:

…

1.25. *Let* ***G*** *be a group in which the square of every element is identity. Prove that* ***G*** *is abelian.*

1.26. *A group* ***G*** *is abelian if and only if the function* $f\colon G \to G$ *defined by* $f(x) = x^{-1}$ *is a homomorphism.*”

The definition of a commutative group is formulated in **[**7**]**, p. 108, in the following way:

"*In a commutative group, if* ***a*** *and* ***b*** *are any members of the group,* ***ab = ba***."

Problem 1.25. From **[**6**],** it is formulated in **[**7**]**, p. 109, in the following way:

"…

Exercises:

…

43. a) *Prove that if the square of every member of a group is equal to the identity element, then the group is commutative. (Hint: First identify the inverse of each member.)* b) *Find in Chapter 7 an example of a group with more than two members where the square of every member is equal to the identity element.*"

It is shown in the next section how and why we compare the formulations 1.25 **[**6**]** and 43.**[**7**]** of the same problem. It is shown later (example 3) why and how the desire for a non-formal formulation of the second problem 1.26.**[**6**]** like the formulation of the first one is useful. A similar situation can happen in other sources. It seems the first problem was formulated in **[**8**]** with one formal part but with non-formal another part. Analogically, then the desire for equalizing these parts (making both non-formal or both formal) is well-grounded. Teaching follows usually one source but using the experience of different sources gives better results. The indicated comparison and desire are possible only for concrete examples. This note supports our sticking to exposition with illustration on concrete problems.

EXAMPLE 2

Our preferences are closer to the definition of a commutative group from **[**6**]** because it contains in itself definitely the sense for separating the three possibilities: there are groups in which only the elements inverse to each other commute, there are groups in which other elements, but not all, can commute, and there are groups in which all elements commute. On the other hand, our preferences about the formulation of the problem are closer to this one from **[**7**]** (with explanations added here in parentheses):

Problem. "*If the square of every member of a" (multiplicative) "group is the identity, then" this "group is commutative" (or abelian).*

The reason for choosing this formulation is it is short, absolutely of the form "if…, then…" of a statement and without unnecessary words or designations. The solution is symbolical of course. Students with a Math major usually come without special difficulties to its symbolic expression:

If <u>for each element</u> ***x*** *of a (multiplicative) group* $\boldsymbol{x^2 = e}$*, then (⇒) <u>for every two elements</u>* ***a*** *and* ***b*** *of its:* ***ab = ba*** .

Two parts of this symbolic expression are underlined because usually these parts are missing in the student writings. We will not indicate here the ways for the corrections and the requirements of explanations why the second part of the formulation is expressed in this way (according to the definition of a commutative group). We will not indicate absolutely formal writing these parts with limited quantifiers. Our goal is indication of instructions which have more psychological effect. I use another instruction instead the absolutely correct hint from the cited in the previous section problem 43. a) from **[**7**]**. My opinion is a non-formal hint has more effective influence on the student participation than the indicated one which is almost formal due to the word "inverse" in it. Such one can be:

"What do you dislike in the formulation of the left part of this statement?". It is absolutely non-formal, something like aesthetic one but it is not strange already for me after so much experience it leads to an absolutely correct answer like:

"We have on the left side in $\boldsymbol{x^2 = e}$ twice the variable $\boldsymbol{x}$ but there is a constant $\boldsymbol{e}$ only and no variable on the right one."

The students receive a next non-formal instruction about a formal expression:

"Then "eliminate" please this "injustice"!"

And consequently students come to the equivalent expression: $\boldsymbol{\ldots: x = x^{-1}}$. The psychological advantage of the indicated two non-formal hints instead of the almost formal one is building abilities in the student thinking for discovering ways of solving the problem, which simultaneously leads to more active participation. Then we start the very short solution of the problem:

$\boldsymbol{\ldots: ab = a^{-1} b^{-1} = (ba)^{-1} = ba}$.

The points above substitute "<u>for every element $\boldsymbol{x}$ of the group $\boldsymbol{G}$</u>", and "<u>for every two elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of its</u>". We will not pay attention here on all required explanations why the equalities are correct and their attaching to the corresponding of them. Usually, it is not difficult for the students to find the absolutely correct answers, except for the last one:

"So, $\boldsymbol{ab = ba}$. Why?"

The usual answer is:

"Due to the transitivity of the equality."

Then comes an absolutely surprising note of mine (and therefore it has an additional psychological effect):

"Excuse me, that is 50 percent of the entire explanation."

Not at once, but after some eventual additional instructions, the correct answer is formulated:

"Due to using twice the transitivity of the equality in a set."

So $\boldsymbol{\ldots: x^2 = e \Rightarrow \ldots: ab = ba}$ and therefore we have proved if the square of every member of a group is an identity, then this group is commutative. Very interesting situation comes after that when I ask another absolutely unexpected question:

"Could you formulate the same problem in other words?"

The first reaction is silence, of course, which reflects the indicated psychological effect. Really, except that the question is unexpected, it is a combination of non-formal and formal discussion. This combination passes through the formal result ***…:*** $x = x^{-1}$ after the formalization of the problem and insists a non-formal formulation back. This combination has a better effect after receiving the entire solution than eventually asking the same question immediately after the indicated result. Really, this result is already "forgotten" and its reminding brings better comparison of both equivalent properties ***…:*** $x^2 = e$ and ***…:*** $x = x^{-1}$. Sometimes, an additional instruction like

"Please get on the solution's track again!"

leads to the expected equivalent formulation of the problem:

"*If every member of a group is equal to its inverse, then this group is commutative.*"

I stick to the opinion that non-formal comments on a formal solution always bring better understanding. Such a better understanding of the meaning of the quantifiers can bring the comments about that how many times the property ***…:*** $x = = x^{-1}$ has been used during the solution.

An additional similar effect comes from the question:

"Is the inverse property true?"

I use this question instead of task b) from the formulation of the problem 43. **[**7**]**. It is more effective (including psychologically) because indicating a variety of examples for commutative groups which have elements whose squares are not identity, naturally requires a variety of examples which satisfy this condition.

EXAMPLE 3

is about **problem 1.26. [**6**]** which was cited above. I start with some improvements of its formulation. The first one is about the non-formal characterization of the "function $f(x) = x^{-1}$ " of ***G*** into ***G***. Instructions like

"What is the correlation between the original (the argument) of the function and its image?"

and after that

"can x^{-1} be an argument of this function and what is its image?"

lead to a characterization by the students:

"***f*** is a function which depicts every element of the group in its reverse one."

Simultaneously, they are becoming aware it is a one-to-one correspondence in each group but not only in such ones in which it has the property of preserving the operation. The last note is a motive due to which the combination of non-formal and formal formulations starts with the non-formal one. Independently, we will not need it for the (formal) solution, it gives a good orientation about the place of the parts of this problem in the system of used concepts and, in this way, ventures the students for a more active participation. So we come to a formulation of the problem in words:

"*A group is abelian if and only if the (one-to-one) correspondence which depicts every element in its inverse one is a homomorphism.*"

Then comes another question of mine which "sets" the concepts in a better order, due to which it has an analogous effect about the last part of the formulation:

"So, we have a (one-to-one) correspondence of a group into itself. Do you think the word "homomorphism" is the most exact expression which we can use about it?"

We come in this way to the best formulation of the problem:

"*A group is abelian if and only if the (one-to-one) correspondence which depicts every element in its inverse one is an automorphism of this group.*"

We would like to note this non-formal discussion is very useful and effective due to which we have used the expression "setting the concepts in a better order" above absolutely not accidentally. On the other hand, the necessary definitions are reminded, the students already keep in mind their symbolic writings, and as an effect, it is easier for them to come to the symbolic expression of this

Problem: "*Let* $f(x) = x^{-1}$ *be the given function of a group into itself. For every two elements* a *and* b *of its* $ab = ba$ *if and only if (*$\Leftrightarrow$*) for every two elements* x *and* y *of its* $f(xy) = f(x)f(y)$*."* or

"*Let* $f(x) = x^{-1}$ ... *in a group. … …:* $ab = ba \Leftrightarrow$ *…:* $f(xy) = f(x)f(y)$."

Proof. ($\Rightarrow$) Let *…:* $ab = ba$ and x, y be arbitrary elements of the group. Then $f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y)$. Each equality must be explained and explanations must be written over it: 1st - according to the definition of f, 2nd - according to the rule for the inverse of the product of two elements, 3rd - because every two elements of the group commute, the last 4th one - according to the applied twice definition of f (please remember the approach at the previous example).

($\Leftarrow$) **…** Then $ab = f((ab)^{-1}) = f(b^{-1}a^{-1}) = f(b^{-1})f(a^{-1}) = (b^{-1})^{-1}(a^{-1})^{-1} = ba$. Again each equality must be explained, explanations must be written over it, and corresponding corrections are possible. This task could be assigned for homework, too.

The (formal) proof is finished, but a "small" combination with the non-formal discussion comes. It is connected with the previous discussion about the non-formal formulations of the problem. We have indicated that the function which depicts each element of a group into its inverse one is a one-to-one correspondence and it is an automorphism in case of a commutative group. If we keep an eye on the proof, we can note we do not have to use the property that this correspondence is one-to-one. Then the question why we have paid a special attention to it, is absolutely well-founded. So, why? The reason is the one-to-one property is valid in any group, i.e. it is a universal group property and then it is not necessary to use it for a particular other property. But, simultaneously, we have a systematization of the group theory concepts whose psychological effect leads to a better correlation of these and, therefore, to more active ways for searching proofs.

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