# ON TEACHING AND LEARNING CALCULUS USING HISTORY OF MATHEMATICS: A historical approach of Calculus

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## ABSTRACT

*In this paper we present a didactical approach for the introduction in Differential Calculus that exploits the historical course of concepts and is inspired by the principals of Realistic Mathematics Education (RME). In accordance with the "Developmental Research" that is developed in the RME (Gravemeijer, 1994, 1998, 2004), this instructive proposal will include three phases: In the 1st phase we elaborate a "hypothetical learning trajectory". In the 2nd phase will follow experimental teaching in students of High School. While in the 3rd phase will be analyze the learning course in order to see in what degree is achieved the final objective of conceptual understanding and then will be revised the initial instructive scenario where it is judged necessary.*

**Keywords:** Calculus, RME, mathematics history, 97D40.

## INTRODUCTION

One of the basic questions in teaching Mathematics is the way with that teacher can help his students to construct ideas and concepts that mathematics community needed hundreds or thousands years to expound. The purely logical approach is criticized by a lot of mathematicians. According to Richard Skemp (1971)*: «It gives only the end-product of mathematical discovery, and fails to bring about in the learner those processes by which mathematical discoveries are made. It teaches mathematical thought, not mathematical thinking».*

Hans Freudenthal-who laid the foundations for **RME-**called this approach "*anti-didactic inversion*" because, deprives from student the right to re-invent himself "mathematics". Freudenthal (1973) proposes- as point of departure of teaching- the principle of guided re-invention, which is reported in the target for students to re-construct "mathematics" following their historical course and guided from their teacher.

## THE HISTORICAL RETROSPECTION: A VERY IMPORTANT PROCESS

The historical study of differential Calculus highlights the real problems that stimulated his growth and brings out the difficulty of concepts. The students should learn that errors, uncertainties, doubts, contestations and impasses, but also intuitive arguments, heuristics and alternative approaches in the problems are not only legitimately and legally, but the history reveal that all these constituting essential elements in the effort of construction and foundation of mathematics (Arcavi et al., 1987).

In the 17th century, the problems of maximum and minimum, which were connected more specifically with the consideration of celestial mechanics and ballistics as well as the problems of tangent, opened the path to differential calculus. The guided forces behind such growth were the need for the resolution of natural science problems (Artigue, 1991).

The study of history displays that should be avoided the introduction of a new concept straight via the definition of it. The limit, the derivative and the integral were concepts that mathematicians used for a lot of years intuitively, based on geometry, motion and the speed. The passage from intuitional approach in the definition was not a simple step, but a big jump. For a lot of years (often and centuries) mathematicians used the properties of concepts without they know which from them were predominant so that they can establish a definition. Thus and the student will be supposed engages with the basic properties of significances and afterwards we present to him a definition. Must we give a lot of examples in different frames before we give a definition, generalize and prove the theorems. And the proof precedes the discovery.

## A DIDACTICAL APPROACH

We propose the introduction in Differential Calculus becomes no with the formal definition of derivative, but with contexts problems that are reported in real situations or situations that imagine real in the brain of students (RME). The context aims to stimulate the interest of students and it shows that the mathematics is “the” tool for encounter problems of daily life. In the **1st Problem** be asked to calculate the maximum of function $f(x)=-x^3+2x^2$, in interval [0, 2]. The students do not allocate certain known method[1]. In the **2nd Problem** be asked the finding of tangent in graph of function $f(x =-x^3$, in the point A(-1.6, 4.1) and the students do

[1] We avoided parabola because it is known the way of finding extremum from the third Class of Gymnasium and if they do not remember it they could find the extremum from symmetry since they would see in figure the points of section with the horizontal axis.

not allocate certain known method. In this phase we cite a historical report on ***Fermat's rule for maxima and minima*** and ***Fermat's method of Tangent Lines.***

Then, we leave the students to study the historical report and based on this to try they answer the questions. In the next lesson we discuss the way that *Fermat's* method *can give solution in our problem* or the solutions that potentially have found certain students. If a right solution does not exist we direct them to find it, give feedback and correct where it needs. Later, in the next courses, when becomes comprehensible the definition of derivative and we study extremum with the help of derivatives we will point out how many formalization facilitates us in resolution of such problems.

## FIRST PROBLEM

***In the following figure we see the vertical cross-section of a hill. In the top of hill it is becomes a refreshment stall that will be connected with point A with cable car. Hill's outline is represented from graph of function $f(x)=-x^3+2x^2$, $x \in [0, 2]$ (the measure in two axes is 1Km). The engineer order to calculate distance AB should determine the coordinates of hill's top B (maximum point) but it does not know good mathematics. It could you help him finds maximum with the Fermat's rule for maxima?***

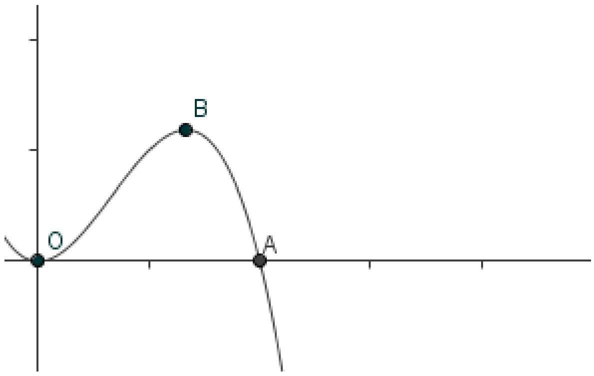


***Figure 1.***

### *1st Phase* **of instructive process*: Fermat's rule for maxima and minima***

The technique of differentiation was developed at 17th century as a method for solution of problems relative with extrema and constructions of tangents in various curves. Many mathematicians have studied the problem of finding the tangent to a curve. Archimedes explored the question in Antiquity. Pierre de Fermat (1601-1647) used the notion of maxima and the infinitesimal to find the tangent to a curve.

In order to understand Fermat's method, it is first essential we examine his technique for finding extrema. He studied polynomial curves $y = f(x)$ near in their extreme prices and observed that the prices of *f* are altered with big slowness round the points in which it presents very extremely. For example, if a price *f(x)* is maximal (or minimal) will be almost equal with price *f(x + E)*, where *x + E* is a neighboring point of *x*, namely *E* is very small. The prices *f (x)*, *f (x + E)*, although they are not precisely equal, become almost equal when E approaches in the zero. Fermat sets $f(x+E) \cong f(x)$, divides two members of equation by E and afterwards sets E=0. Thus he finds the *x-coordinate of extremum* (Boyer & Merzbach 1997, p. 389). Now we will cite one example that Fermat used in order explains his method.

***Example***: Divide a line-segment $\boldsymbol{\alpha}$ in two segments such that the product of the two new segments becomes maximum.

**Solution (The method of Fermat)**

Let a line segment of length $\alpha$ is divided into two segments. Those two segments are *x* and $(\alpha - x)$ and their product $x(\alpha-x)=x\alpha-x^2$. We seek the price x for that the function $f(x)=x\alpha-x^2$ takes maximum price. Fermat observed that prices of function, near in extremum, (for two very neighboring prices of *x)* are almost equal. Consequently if x is the price for which *f(x)* becomes biggest and *x+E* a neighboring price then: $f(x)\cong f(x+E)$. Therefore he had:

$$f(x+E)-f(x)=(x+E)\alpha-(x+E)^2-(x\alpha-x^2)=x\alpha+E\alpha-x^2-2xE-E^2-x\alpha+x^2=E\alpha-2xE-E^2$$

Fermat divides with E and results:

$\dfrac{f(x+E)-f(x)}{E}=\alpha-2x-E$ and then from $f(x+E)-f(x)\cong 0$ *we have:*

$\alpha-2x-E\cong 0$ *and therefore:* $\alpha\cong 2x+E$.

If in the last relation we omit the very small E results: $\alpha=2x$ and hence $x=\alpha/2$ is the price for which $f(x)=x\alpha-x^2$ becomes biggest, namely the line segment should be separated in two equal parts. (In terms of optimality, among the rectangles with certain perimeter the square includes the biggest area).

**Comment**: Although the previous conclusion is right, Fermat's method contains mysteries voids that are restored only with the modern knowledge. Fermat leaves simply E = 0, then, in the step where he divides by E, he would has division by zero. However, though he formulated his method saying E=0, in actual fact he considered the limit of E as this *approaches* to zero (that explains why his algebra works rightly).

Fermat's method amounts with finding the $\lim\limits_{E\to 0}\dfrac{f(x+E)-f(x)}{E}$ and equalizing with zero, namely, resolution of equation $f'(x)=0$ that gives the points of probable local extrema.[2] Provided that were absent a formal notion of a limit,

[2] As we know today, the nihilism of derivative is necessarily but no capable treaty on the existence most utmost, while Fermat considered obvious that the price that resulted from his method corresponds in the biggest price

Fermat was unable to properly justify his work. Nevertheless, examining his techniques, it is obvious that he occupy the method used in differentiation today, only that instead of *E* preferred symbol *h* or *Δx*. *Fermat's process of slight change of variable and regard of neighboring prices constitutes from then the substance of infinitesimal analysis* (Boyer and Merzbach 1997, p. 389). Although Fermat was never able to make a logically consistent formulation, his work can be interpreted as the definition of the differential (Edwards, 1979).

**2nd Phase of instructive process:**

We expect (or we guide) students to apply the method of Fermat as follows:

Let **x** the abscissa of B. A point on graph representation of *f* , neighboring B will have abscissa $x + E$ and ordinate $f(x+E)$ that will be almost equal with $f(x)$, that is $f(x+E) \approx f(x)$, *therefore* $f(x+E)- f(x) \approx 0$. We have:

$f(x+E)-f(x)=-(x+E)^3+2(x+E)^2-(-x^3+2x^2)=$

$-x^3-3x^2E-3xE^2-E^3+2x^2+4xE+2E^2+x^3-2x^2=-3x^2E-3xE^2-E^3+4xE+2E^2$

We divide with E and results:

$\frac{f(x+E)-f(x)}{E}=-3x^2-3xE-E^2+4x+2E$ and since $f(x+E)- f(x) \cong 0$ *we take:*

$(-3x^2-3xE-E^2+4x+2E) \approx 0$ .

If in the last relation we omit the very small E results: $-3x^2+4x=0$ *that is to say* $x(-3x+4)=0$ and so (since $x=0$ we have local minimal) it is: $-3x+4=0$, *therefore:* $x=4/3$ is the abscissa of B.

**Comment:** Later, in instructive course, we will find $f'(x)=(-x^3+2x^2)'=-3x^2+4x$ *and* setting $-3x^2+4x=0$ we will find easily the asked price x =4/3, but we believe that with proposed instructive trajectory we will have achieved the conceptual comprehension and then students will work consciously and no mechanically.

## SECOND PROBLEM

***It is become a street that will connect two villages A, B that is found in banks of a river, that are represented by function $f(x)=-x^3$ (Figure 2). The engineer proposed becomes street AOB, but the topographer informed him that segment OA goes through from rocky region and would have big cost. The topographer studied the morphology of region and made the following proposal: Becomes street AKB where segment AK is tangent of curve, which represent the river, in A and segment BK, vertical to AK. The segment AK pertains to smooth region and segment BK to rocky region. The cost of street is 250.000€ when it goes through smooth region and is doubled for boondocks regions. Examine which proposal is better. Taking into account that in the 1st proposal of street AOB should you calculate also the cost of Bridge that amounts in 150.000€.***

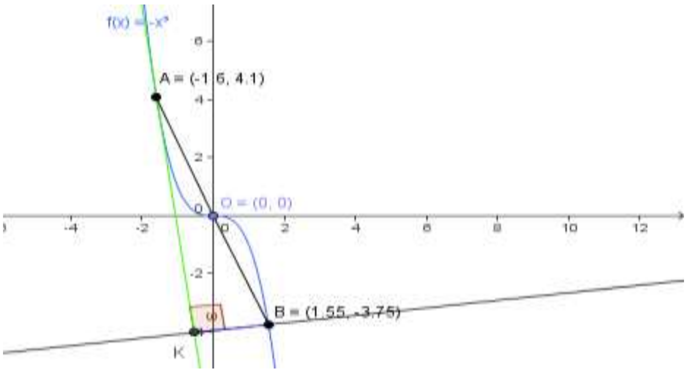


**Figure 2.**

***Comment-direction***

**1st Step:** Are given in figure 2 the coordinates of points A, B, hence students can calculate easily lengths OA and OB, and thereafter the cost of street AOB.

**2nd Step:** Students do not know any method in order to find the tangent's equation and thus be cited the following:

***Fermat's method of Tangent Lines***

Fermat discover how applies the mentioned before in extrema process of neighboring points, using the mysterious E, for finding tangent line of a curve *y=f(x)*. Let P*(a, b)* is a point of parabola and P′ a neighboring point in curve with coordinates *(a+E, f(a+E))* (Figure 3). If the P′ be found too much near the P then could one say that the secant PP' coincides with the tangent in the P (cf. Figure 3).

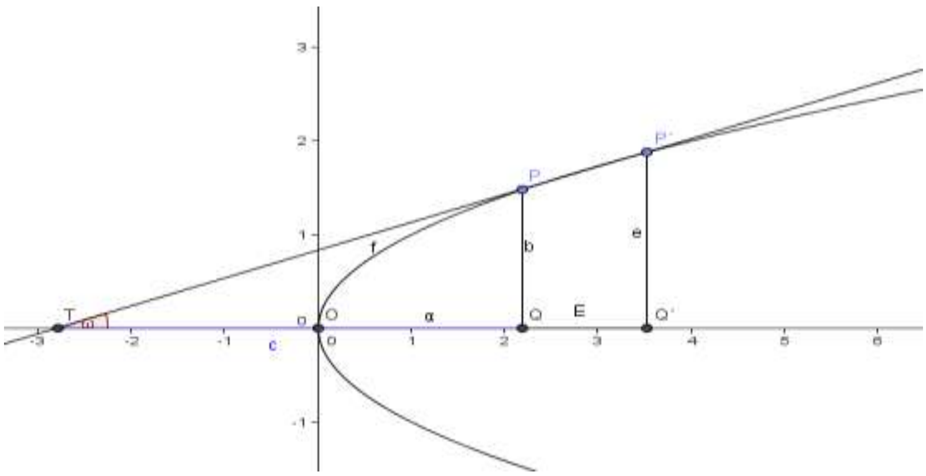


**Figure 3.**

Let Q, Q' the projections of P, P′ in x-axis, respectively and T the section of tangent with x-axis. Then *PQ=f(α)=b, P′Q′=f(α+E)=e, OQ=α, QQ′=E* and let TQ=c. The triangular TPQ and TP'Q' can be considered similar. Therefore: $\frac{PQ}{TQ}=\frac{P'Q'}{TQ'}$, namely $\frac{f(a)}{c}=\frac{f(a+E)}{c+E}$, from where we have:

$$f(a)\cdot(c+E)=c\cdot f(a+E)\Leftrightarrow f(a)\cdot E=c\cdot f(a+E)-c\cdot f(a).$$

We divide the two members by E and take:

$$f(a)=c\cdot\frac{f(a+E)-f(a)}{E}\Leftrightarrow\frac{f(a)}{c}=\frac{f(a+E)-f(a)}{E}\quad (1)$$

But $\frac{f(a)}{c}$ = $\frac{PQ}{TQ}$ = tan(ω), where ω the angle that shapes the secant **PP′** with horizontal axis. Fermat, placing in relation (1) E=0, calculated the slope of tangent, namely the slope of curve.

**Comment:** The process of Fermat amounts with finding of $\lim_{E\to 0}\frac{f(a+E)-f(a)}{E}$ that gives the slope of secant PP′ when P′ is very near in P, namely, when the secant PP′ is identified with the tangent (Boyer and Merzbach 1997, p. 390).

Fermat again lets the quantity E *become 0* (with modern term, he took the limit as E approaches to 0) and recognized that the quotient $\frac{f(a+E)-f(a)}{E}$ was identical to his differential in his method of extrema. Consequently, in order to find the slope of a curve, it is sufficient to find $\frac{f(x)}{c}$ and set E=0.

**3rd Step:**

We expect (or we guide) students to apply the method of Fermat as follows:

We have the functio $f(x)=-x^3$. Let *(x, y)* the coordinates of point P of Cf and P′ a neighboring point in curve with coordinates *(x + E, f (x + E)).* If the P′ is found too much near in P then could one say that the secant PP′ coincides with the tangent at P. Let T the section of tangent with x-axis and let TQ = c. According to previous example we have:

$$\frac{f(x)}{c}=\frac{f(x+E)-f(x)}{E}=\frac{-(x+E)^3+x^3}{E}=\frac{-(3x^2+3xE+E^2)E}{E}=$$

$=-3x^2-3xE-E^2$,

and setting E=0 we take: $\frac{f(x)}{c}=-3x^2$

But $\frac{f(x)}{c}$=tan(ω), where ω the angle that shapes the secant **PP′** with x-axis. Hence, the slope of tangent (namely the slope of curve) in the point A(-1.6, 4.1) is: l = tan(ω)=$-3x^2=-3(-1.6)^2=-3*(2.56)=-7.68$, and consequently we can now calculate the equation of tangent from the type: $\mathbf{y-y_0=l(x-x_0)}$ and then the rests questions.

## CONCLUDING REMARKS

The problems we have selected aim to motivate the interest of students and retract their weakness to answer in certain questions and cause the curiosity to learn which solution was given at the historical course in similar problems. The use of geometry and history gives a concrete meaning in the notion of derivative - that will be imported later with the formal definition- and can help student to overcome this epistemological obstacle. The instructive sequence could be supplemented with the presentation of different ideas and expressions of infinitesimal methods at the historical course, more specifically, relative with the notion of limit. The report in the symbolism that Leibniz developed, ***dx*** for the infinitesimal change of x (differential of x) and ***dy*** for the corresponding change of y (differential of y), or the report in the symbolism *f '(x)* of Lagrange, helps students realize how a good symbolism renders the life easier for us in the study of mathematics. Then, it can become a comparison of approaches of Newton and Leibniz in the Differential Calculus and finally are presented the steps of rigorous foundation of Differential Calculus.

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