METHODICAL ARITHMETICAL JUSTIFICATION OF RISING THE CHROMATIC SCALE AND INVERSELY OF THE DIATONIC ONE FROM IT, OF RULES FOR THE MUSICAL CIRCLE OF THE FIFTHS AND FOR ROTATION IN THE BASIC CHORDS

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ABSTRACT

*Simple properties of the number **12** methodically justify the formation of exactly **12** semitone intervals (why?) in the chromatic musical scale and then (inversely) of its already in one way determined part (why?) of the diatonic scale. **7x** gives the circle of the fifths **F , C, G, D, A, E, B, F#, C#, G#, D#, A#** from **F#, G, G#, A, A#, B, C, C#, D, D# , E, F mod 12** but **5x** gives it (and therefore the **7** diatonic circle) in an inverse order due to **7 ≡-5(mod12)**; **x** gives the consecutive chromatic performance up of an octave but **11x** gives the same down due to **11 ≡ -1(mod12)**. The basic major and minor chords can be reordered by the sums **8-12**, **14** of the halftones of their internal intervals. The intuitively built in the intellect rules for their variations is a left-right rotation for sums **8-11** but other ones for **12** and **14**.*

Keywords: Prime numbers, Arithmetical properties, Methodical justification, Chromatic musical scale, Diatonic musical scale, Residues at a given modulo, Circle of the fifths, Sum of the internal intervals in a musical chord, Justification of the rotation rules for their variations, Ascendancy of the mathematical rules.

1. SIMPLE ARITHMETICAL PROPERETIES OF THE NUMBER 12 METHODICALLY JUSTIFY THE RISE OF THE CHROMATIC SCALE AND INVERSELY OF THE DIATONIC ONE FROM IT

The usual chromatic musical scale with **12** semitone intervals is **C, C#, D, D#, E, F, F#, G, G#, A, A#, B, C↑**. The question is: why the intervals are **12**? We show simple arithmetical natural properties which explain why human intellect has chosen this construction due the naturally built in it similar ways in the development of human knowledge and art. Two (**2**) is the first (smallest) and only one even prime number in the infinite series {**1, 2, 3, 4, … ,11, 12, …**} of the natural numbers. All other prime numbers **3, 5, 7, 11, 13, 17, …** are odd ones. They are infinitely many. We ask ourselves the *Question*: Which is the smallest number with two natural properties: 1) It to have a presentation $\mathbf{2^k + 2^{k+1}}$ (or $\mathbf{(3 \times 2^{k-1}) \times 2, k = 1, 2, 3, 4, \ldots}$) and 2) It to be a sum of two neighbor (consecutive) prime numbers?

The smallest number which satisfies both natural properties is **12** and this is reflected in the natural formation of the **12** intervals in the chromatic scale. So, we have already the chromatic scale. The only (only!) one presentation **12 = 5 + 7** of **12** as a sum of two consecutive prime numbers is reflected in the formation of its diatonic part and we will explain now why and how. The prime number **7** has only one presentation as a sum **7 = 2 + 5** of the smallest (and only one even) prime number **2** and the other such one **5**. Therefore we have received the only one presentation **12 = 5 + 7 = 5 + (2 + 5) = 5 + (2) + 5**. This presentation has equal and symmetrical left and right sides with respect to its part **(2)**. Therefore the intellect has been directed to distribute in one and the same way **5** semitones on both sides of this **2** semitone interval (It will be later the interval **F - G**.) Now: **5** has a presentation $\mathbf{5 = 2^2 + 1}$ (which is similar to the presentation $\mathbf{2^{n-1}}$, where **n** is a prime, of Mercenn's prime numbers). On other hand $\mathbf{4 = 2^2 = 2 \times 2}$ is the only one (again only one!) different of **2** natural number which satisfies the natural equation $\mathbf{2^k = 2k, k = 1, 2, 3, 4, \ldots}$. Therefore **5 = 2 + 2 + 1** must be presented in this pointed only one way. The last conclusion has been reflected by the intellect with a distribution of the **5** semitone interval in the order **2**, after that again **2**, and finally **1** of them. That means the diatonic scale has been created for its aesthetic delight as it follows:

$$\mathbf{C \underset{1d}{\overset{2chr}{—}} D \underset{1d}{\overset{2chr}{—}} E \underset{1d}{\overset{1chr}{—}} F \underset{1d}{\overset{(2chr)}{—}} G \underset{1d}{\overset{2chr}{—}} A \underset{1d}{\overset{2chr}{—}} B \underset{1d}{\overset{1chr}{—}} C^{\uparrow}}$$

. Both parts **C —- D —-- E —-- F** and **G — A — B — C**$^{\uparrow}$ around **F — G** are distributed in one and the same way and they have one and the same sound.

We have explained how very simple arithmetical properties of the number **12** are reflected in the formation of the exactly **12** semitone intervals of the chromatic musical scale and then **-** of its in one way determined parts of the diatonic scale. The number of the tones and the number of the intervals of the last one are a reflection of obvious simple arithmetical properties of **8** and **7** and of an ascendancy of the mathematical rules in the human activity which is similar to this one from (TARKALANOV, 2000). Last note concerns to all parts of this paper of course.

2. JUSTIFICATION OF ARITHMETICAL RULES FOR FORMATION OF THE MUSICAL CIRCLE OF THE FIFTS

The construction of the clockwise circle of the fifths **F, C, G, D, A, E, B, F**$^{\#}$**, C**$^{\#}$**, G**$^{\#}$**, D**$^{\#}$**, A**$^{\#}$ (**12** tones) can be seen in (CM, 1985, 1986) as a consecutive adding of **7** semitones. The paper (SW, 2006) appears like some correction of the use of the diatonic circle of its first **7** tones (a permutation of the basic diatonic scale **C, D, E, F, G, A, B**) for a proof of a more general mathematical property. We will not need it here but we will motivate the reflection of the elementary theory of numbers property: “If ***a*** is coprime with the **modulo m** and ***x*** runs through a complete residue system **modulo m**, then ***ax* + *b*** runs through an complete residue system this **modulo**” (ROSEN, 2005, Theorem 4.6.). The authors (CM, 1985, 1986) use a stronger property for constructing the generalized circle of the fifths. We mark especially the indicated here residue property is sufficient and we don’t need their stronger property in the particular case of the usual musical circle of the fifths.

Due to working with **modulo 12** we will use the usual **12**-tone chromatic musical scale **C, C$^{\#}$, D, D$^{\#}$, E, F, F$^{\#}$, G, G$^{\#}$, A, A$^{\#}$, B** in the form: **F$^{\#}$, G, G$^{\#}$, A, A$^{\#}$, B, C, C$^{\#}$, D, D$^{\#}$, E, F**. We will correlate so ordered tones to the smallest positive residues **modulo12**: ***x* = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12** which we will express in the following way: $\frac{1}{\text{F\#}},\frac{2}{\text{G}},\frac{3}{\text{G\#}},\frac{4}{\text{A}},\frac{5}{\text{A\#}},\frac{6}{\text{B}},\frac{7}{\text{C}},\frac{8}{\text{C\#}},\frac{9}{\text{D}},\frac{10}{\text{D\#}},\frac{11}{\text{E}},\frac{12}{\text{F}}$. All coprime with **modulo 12** natural numbers are **1, 5, 7, 11**. Then ***ax*** transforms these residues in the following way:

1*x* is the residue: $\frac{1}{\text{F\#}},\frac{2}{\text{G}},\frac{3}{\text{G\#}},\frac{4}{\text{A}},\frac{5}{\text{A\#}},\frac{6}{\text{B}},\frac{7}{\text{C}},\frac{8}{\text{C\#}},\frac{9}{\text{D}},\frac{10}{\text{D\#}},\frac{11}{\text{E}},\frac{12}{\text{F}}$;

5*x* is the residue: $\frac{5}{\text{A\#}},\frac{10}{\text{D\#}},\frac{3}{\text{G\#}},\frac{8}{\text{C\#}},\frac{1}{\text{F\#}},\frac{6}{\text{B}},\frac{11}{\text{E}},\frac{4}{\text{A}},\frac{9}{\text{D}},\frac{2}{\text{G}},\frac{7}{\text{C}},\frac{12}{\text{F}}$;

7*x* is the residue: $\frac{7}{\text{C}},\frac{2}{\text{G}},\frac{9}{\text{D}},\frac{4}{\text{A}},\frac{11}{\text{E}},\frac{6}{\text{B}},\frac{1}{\text{F\#}},\frac{8}{\text{C\#}},\frac{3}{\text{G\#}},\frac{10}{\text{D\#}},\frac{5}{\text{A\#}},\frac{12}{\text{F}}$;

11*x* is the residue: $\frac{11}{\text{E}},\frac{10}{\text{D\#}},\frac{9}{\text{D}},\frac{8}{\text{C\#}},\frac{7}{\text{C}},\frac{6}{\text{B}},\frac{5}{\text{A\#}},\frac{4}{\text{A}},\frac{3}{\text{G\#}},\frac{2}{\text{G}},\frac{1}{\text{F\#}},\frac{12}{\text{F}}$.

We see the residues **1-11** on the first and forth lines are in an inverse order. The same about residues in the second and third lines. And that is not accidentally but - the first one because **1 ≡ –11(mod 12)** or **11 ≡ –1(mod 12)**; the second one - because **5 ≡ –7(mod 12)** or **7 ≡ –5 (mod 12)**. The indicated formal changes of the signs **modulo12** lead to a real performance of the (formally received) music in an inverse order: chromatically up for line 1, chromatically down for line 4, and clockwise (up) performance of the circle of the fifths for line 3, its counterclockwise performance (down) for line 2. Starting from **F modulo12** doesn’t matter due to the rotation this **modulo**.

There are other incorporated simple arithmetic properties in our table above which reflect the same result. For example, all consecutive differences **1 – 2, 2 – 3, … , 11 – 12** in the first row are **– 1**, but the same ones **11 – 10, 10 – 9, … , 2 – 1** in the last row are **+1**; all differences **3 – 5, 8 – 10, 1 – 3, 6 – 8,** (jumping **11**)**, 4 – 6, 9 – 11, 2 – 4, 7 – 9** in the second row are **–2** but all differences **9 – 7, 4 – 2, 11 – 9, 6 – 4,** (jumping **1**)**, 8 – 6, 3 – 1, 10 – 8, 5 – 3, 12 – 10** in the third row are **+2**. All these simple arithmetical properties in the presentation **12 = 5 + 7** were motivated about the choice of the intellect in the first section. We have shown in this section to what they lead the musical performance and we have received the inverse musical circles of the fifths using weaker property of the congruence’s a given modulo which is a methodically motivated trend at any research.

3. JUSTIFICATION OF ARITHMETICAL RULES FOR ROTATION OF THE BASIC CHORDS

We explore in this section simple common arithmetical rules for using rotation at performance of variations of major and minor musical chords. They are coming again from natural properties of the number **12**, its small dividers, and coprime numbers. We will demonstrate these rules on *C Major*, *C Minor*, and their variations because all other chords are received from them by simple shifting. The Web site source (FAST&SOFT, 1999) has been chosen in the references because it gives a possibility for marking the tones on the piano keyboard and their listening.

The basic chords are disposed there in 6 rows: *Major - Minor, Major 7⁻ - Minor 7, Major 7 - Minor 7⁺, Major 6 - Minor 6, Augmented - Diminished, Major 9 - Minor 9.*

We will not indicate why we disagree with some signs ± or their absence. We will make some changes in this order receiving a more appropriate disposition for indicating rules about their variations. Simultaneously we will mark like in the numerators the chromatic distances between their internal tones, the sums of these distances, and the distances from the last to the first tones. A "**.**" will stay like in the denominators because the corresponding diatonic distances are clear or they do not exist.

C Major	$C \overset{4}{\underset{.}{—}} E \overset{3}{\underset{.}{—}} G \overset{5}{\underset{.}{—}} C \overset{0}{\underset{.}{—}}$	**12**	**12**	$C \overset{3}{\underset{.}{—}} E^b \overset{4}{\underset{.}{—}} G \overset{5}{\underset{.}{—}} C \overset{0}{\underset{.}{—}}$	***C Minor***
C Major7	$C \overset{4}{\underset{.}{—}} E \overset{3}{\underset{.}{—}} G \overset{4}{\underset{.}{—}} B \overset{1}{\underset{.}{—}}$	**11**	**11**	$C \overset{3}{\underset{.}{—}} E^b \overset{4}{\underset{.}{—}} G \overset{4}{\underset{.}{—}} B \overset{1}{\underset{.}{—}}$	**CMinor7+**
C Major7⁻	$C \overset{4}{\underset{.}{—}} E \overset{3}{\underset{.}{—}} G \overset{3}{\underset{.}{—}} B^b \overset{2}{\underset{.}{—}}$	**10**	**10**	$C \overset{3}{\underset{.}{—}} E^b \overset{4}{\underset{.}{—}} G \overset{3}{\underset{.}{—}} B^b \overset{2}{\underset{.}{—}}$	***CMinor7***
CMajor9	$C \overset{4}{\underset{.}{—}} E \overset{3}{\underset{.}{—}} G \overset{3}{\underset{.}{—}} B^b \overset{4}{\underset{.}{—}} D \overset{-2}{—}$	**10/14**	**10/14**	$C \overset{3}{\underset{.}{—}} E^b \overset{4}{\underset{.}{—}} G \overset{3}{\underset{.}{—}} B^b \overset{4}{\underset{.}{—}} D \overset{-2}{\underset{.}{—}}$	***Minor9***
CMajor6	$C \overset{4}{\underset{.}{—}} E \overset{3}{\underset{.}{—}} G \overset{2}{\underset{.}{—}} A \overset{3}{\underset{.}{—}}$	**9**	**9**	$C \overset{3}{\underset{.}{—}} E^b \overset{3}{\underset{.}{—}} G^b \overset{3}{\underset{.}{—}} A \overset{3}{\underset{.}{—}}$	***CDimin.***
***CAugm*.**	$C \overset{4}{\underset{.}{—}} E \overset{4}{\underset{.}{—}} G^\# \overset{4}{\underset{.}{—}} C \overset{0}{\underset{.}{—}}$	**12**	**8**	$C \overset{3}{\underset{.}{—}} E^b \overset{4}{\underset{1}{—}} G \overset{1}{\underset{.}{—}} A^b \overset{4}{\underset{.}{—}}$	***C Minor6***

A first note about this different order of ours with respect to the traditional one is we have switched second and third rows and we have place the last row with the 9 chords at a fourth position. This allows to order the first three rows by the sums **12, 11, 10** of their internal chromatic intervals. Next one: we can mark the first **4** tones of the **9** chords from the 4-th row repeat the tones of the chords in the third row and therefore the sum **10** of their internal intervals. The last note **D** of **9** chords increases this sum to **14** but the indicated before that coincidence gives a reason for this disposition and the situation is marked by **10/14** in the table. Next 5^{th} row of its decreases the sum of the internal intervals to **9** after we have taken its two chords from different rows in the traditionally accepted table. Therefore we have taken the chords of the last row of our table again from the same different traditional rows. The indicated changes have allowed us to order the sums of the internal intervals in the Minor chord column decreasingly: **12, 11, 10, 9, 8** except the indicated 4-th ***Minor9*** chord with its characteristic **10/14**. The augment chord in the left Major chords column however returns to a sum **12** after **9**.

Our idea to pay attention to the sum of the internal intervals brocks their consecutive distribution **(4, 3, 5 → 2)** of the intervals in the Major chords only at the last row with **(4, 4, 4)** and the naturally inverse in the beginning for the Minor chords **(3, 4, 5 3)**, **(3, 4, 1)** with **(3, 3, 3)** only at the Diminished chord. The indicated below rules for receiving chords' variations are valid for the sums of the internal intervals independently of the indicated inverse distributions, i.e. for both Major and Minor scales. All sums of the internal intervals in the main chords are clearly indicated in the middle columns of our table and we can now formulate the desired *Rules:*

The **3** variations of all chords for which this sum is **8, 9, 10** or **11** can be received by a simple left-right rotation as in the following example for *C Diminished*: Var. 1. is $\mathbf{C — E^b — G^b — A}$, Var. 2. is $\mathbf{E^b — G^b — A — C}$, Var. 3. $\mathbf{G^b — A — C — E^b}$. This rule is valid for 7 chords: *Major7*, *Minor7*$^{+}$, *Major7*$^{-}$, *Minor7*, *Major6*, *Diminished*, *Minor6*.

We can not apply this rotation *rule* to the three chords with the sum of their internal intervals **12**: *Major*, *Minor*, *Augmented*. The following *rule* performs their variations: Eliminating the first note and repeating the first one of the remaining three ones after them at the next octave level. We give an example for *C Augmented*: Var.1. $\mathbf{C — E — G^\# — C}$, Var. 2. $\mathbf{E — G^\# — C — E}$, Var. 3. $\mathbf{G^\# — C — E — G^\#}$.

Two remaining chords *Major9* and *Minor9* have a sum **14** of their internal intervals. The following *rules* are valid for performance of the variations of the both of them: Eliminate the first tone of Var. 1. and put it at the next octave level before its last tone. That is Var. 2. Rotate the first tone of Var. 2. at the next octave level in order to receive Var. 3.

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