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IDENTITIES IN SOME SPECIAL LIE ALGEBRAS

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Abstract

For some Lie algebras we study special identities which are of type provoked by an approach of Formanek and Bergman for investigating matrix identities by means of commutative algebra.

We consider the matrix algebra of fourth order with symplectic involution and the Lie algebra of the skew-symmetric to the involution variables with a new multiplication Lie commutator. We investigate if there are Lie identites which are polynomials of the considered type.

Estimations for the least possible degree of such identities lead to a general conclusion. We prove that the considered polynomials are not Lie identities for this special Lie algebra as an application of results obtained by the author for matrix algebras with symplectic involution.

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I. Introduction

The special Lie algebras play an important role in the theory of Lie algebras with polynomial identities (PI-algebras) as seen from [1] and its good bibliography.

We recall some facts concerning them needed in the sequel.

An algebra L over a commutative ring R with identity is called special Lie algebra if it is isomorphic to a subalgebra of a Lie algebra of type $[A]$, where A is an associative PI-algebra over R and its multiplication $\cdot$ defines the multiplication in $[A]$ by the Lie commutator $[a,b] = a \cdot b - b \cdot a$.

If R is a field every finite dimensional Lie algebra L is special. Due to the Ado-Iwasawa theorem [1, p.249] L is imbedded in a Lie algebra of type $[End_R V]$, where V is a finitely dimensional vector space. As a finite associative algebra satisfies a standard identity of appropriate degree the Lie algebra L is special. In [1] the following is proved:

Proposition 1 [1, p.254] *Let L be a special Lie algebra. Then L satisfies a nontrivial identity as a Lie algebra.*

The present paper gives details on the nature of such identities. We investigate special types of identities for one special Lie algebra - the Lie algebra $so(4, K, *)$ of the skew-symmetric to the symplectic involution $*$ variables of the matrix algebra of fourth order $M_4(K, *)$, where K is a field of characteristic 0 and $*$ is the symplectic involution in $M_4(K)$ defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix},$$

where A, B, C, D are 2×2 matrices and t is the usual transpose.

These identities are polynomials in skew-symmetric variables of the following type considered for the first time by Formanek [4] and Bergman [2] and later on in a series of papers by the author and Drensky [3, 7, 5, 6].

To a homogeneous polynomial in commuting variables

$$g(t_1, \dots, t_{n+1}) = \sum \alpha_p t_1^{p_1} \dots t_{n+1}^{p_{n+1}} \in K[t_1, \dots, t_{n+1}] \tag{1}$$

we relate a polynomial $v(g)$ from the free associative algebra
$K\langle x, y_1, \dots, y_n \rangle$

$$v(g) = v(g)(x, y_1, \dots, y_n) = \sum \alpha_p x^{p_1} y_1 \dots x^{p_n} y_n x^{p_{n+1}}. \tag{2}$$

Any homogeneous and multilinear in $y_1, \dots, y_n$ polynomial $f(x, y_1, \dots, y_n)$ can be written as

$$f(x, y_1, \dots, y_n) = \sum_{i=(i_1,\dots,i_n) \in Sym(n)} v(g_i)(x, y_{i_1}, \dots, y_{i_n}), \tag{3}$$

where $g_i \in K[t_1, \dots, t_{n+1}]$.

A detailed survey on polynomial identities is [8]. For $*$-polynomials in skew-symmetric variables one could see [5].

Bergman shows in [2] that no polynomials of type (2) could be identities for $M_n(K)$. An easy observation affirms the same for the considered polynomials in skew-symmetric variables for the algebra $M_{2n}(K, *)$ as well. That is why we consider polynomials of type (3) calling them for simplicity Bergman polynomials.

We are interested in their minimal degree for the algebra $so(4, K, *)$, which could be suggested from the following statement.

Proposition 2 [7, Theorem 1] *Let a polynomial $f(x, y_1, \dots, y_n)$ of type* (3) *be a $*$-identity in skew-symmetric variables for $M_{2n}(K, *)$. Then*

$$\prod_{\substack{1 \le p < q \le n+1 \\ (p,q) \neq (1, n+1)}} (t_p^2 - t_q^2)(t_1 - t_{n+1})$$

divides the polynomials g_i from (1) *for all $i = (i_1, \dots, i_n)$.*

Thus this degree could be 7. But it is not so. The way of proving it works in the general case as well. In fact we diminish the possibilities for the Lie identities in the considered special algebra and prove the following statement.

Theorem 1 *No Bergman polynomials are Lie identities for the Lie algebra $so(4, K, *)$.*

Before proving the theorem we show the connection between the polynomials from the free Lie algebra $L\{X\}$ and the corresponding commutative polynomials.

The basis of $L\{X\}$ on a countable set X is studied in [1, p.68, 2.3]. We use the Hall basis of $L\{X\}$ [1, p.73, 2.3.7], which means that for $X = \{x, y_2, y_1\}$ we consider the following elements (of a given degree $k+2$): $[y_{i_1}, \underbrace{x, \dots, x}_{k}, y_{i_2}]$ and $[[y_{i_1}, \underbrace{x, \dots, x}_{l}], [y_{i_2}, \underbrace{x, \dots, x}_{k-l}]]$, where (i_1, i_2) is any permutation on $\{1, 2\}$ and $l = 1, \dots, k-1$.

We point that the commutators in the sequel are left normed. We write them as elements of the free associative algebra $K\langle X\rangle$ and thus the commutative polynomials are uniquely defined as seen by the following examples: For

$$\begin{aligned} f &= [y_1, x, y_2] = y_1xy_2 - xy_1y_2 - y_2y_1x + y_2xy_1 \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) \end{aligned}$$

we have $g_1 = t_2 - t_1$ and $g_2 = -(t_3 - t_2)$. For

$$\begin{aligned} f &= [[y_1, x], [y_2, x, x]] = y_1xy_2x^2 - 2y_1x^2y_2x + y_1x^3y_2 \\ &- xy_1y_2x^2 + 2xy_1xy_2x - xy_1x^2y_2 - y_2x^2y_1x + y_2x^3y_1 \\ &+ 2xy_2xy_1x - 2xy_2x^2y_1 - x^2y_2y_1x + x^2y_2xy_1 \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) \end{aligned}$$

one gets $g_1 = (t_2 - t_1)(t_3 - t_2)^2$ and $g_2 = -(t_2 - t_1)^2(t_3 - t_2)$.

Proof of Theorem 1:

Let l denote the degree of the Bergman polynomial f. We write its explicit form due to the parity of l.

(i) Let $l = 2k + 2$. Then

$$\begin{aligned} f &= A[y_1, \underbrace{x, \dots, x}_{l-2}, y_2] + B[y_2, \underbrace{x, \dots, x}_{l-2}, y_1] \\ &+ a_1[[y_1, x], [y_2, \underbrace{x, \dots, x}_{l-3}]] + a_2[[y_1, x, x], [y_2, \underbrace{x, \dots, x}_{l-4}]] \\ &+ \dots + a_k[[y_1, \underbrace{x, \dots, x}_{k}], [y_2, \underbrace{x, \dots, x}_{k}]] \\ &+ b_1[[y_2, x], [y_1, \underbrace{x, \dots, x}_{l-3}]] + b_2[[y_2, x, x], [y_1, \underbrace{x, \dots, x}_{l-4}]] \\ &+ \dots + b_{k-1}[[y_2, \underbrace{x, \dots, x}_{k-1}], [y_1, \underbrace{x, \dots, x}_{k+1}]] \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) = f_1 + f_2. \end{aligned}$$

The polynomial g_1 corresponding to f_1 has the form

$$\begin{aligned} g_1 &= A(t_2 - t_1)^{l-2} - B(t_3 - t_2)^{l-2} \\ &+ a_1(t_2 - t_1)(t_3 - t_2)^{l-3} + a_2(t_2 - t_1)^2(t_3 - t_2)^{l-4} \\ &+ \dots + a_k(t_2 - t_1)^k(t_3 - t_2)^k \\ &- b_1(t_2 - t_1)^{l-3}(t_3 - t_2) - b_2(t_2 - t_1)^{l-4}(t_3 - t_2)^2 \\ &- \dots - b_{k-1}(t_2 - t_1)^{k+1}(t_3 - t_2)^{k-1}. \end{aligned}$$

According to Proposition 2 the polynomials $t_1 - t_2$, $t_2 - t_3$ and $t_1 + t_2$ are factors of g_1. Thus $f_1(t_1 = t_2) = 0$ leads to $B = 0$ and $f_1(t_2 = t_3) = 0$ gives $A = 0$. Then we consider the polynomial $f_1(t_1 = -t_2) = 0$. Its explicit form is

$$\begin{aligned} & 2a_1t_2(t_3-t_2)^{l-3} + 4a_2t_2^2(t_3-t_2)^{l-4} + \cdots \\ + \quad & 2^k a_k t_2^k (t_3-t_2)^k - 2^{l-3}b_1t_2^{l-3}(t_3-t_2) \\ - \quad & 2^{l-4}b_2t_2^{l-4}(t_3-t_2)^2 - \cdots \\ - \quad & 2^{k+1}b_{k-1}t_2^{k+1}(t_3-t_2)^{k-1} = 0. \end{aligned}$$

Dividing it by $2t_2(t_3 - t_2)$ we get

$$\begin{aligned} & a_1(t_3-t_2)^{l-4} + 2a_2t_2(t_3-t_2)^{l-5} + \cdots \\ + \quad & 2^{k-1}a_kt_2^{k-1}(t_3-t_2)^{k-1} - 2^{l-4}b_1t_2^{l-4} - 2^{l-5}b_2t_2^{l-5}(t_3-t_2) \\ - \quad & \cdots - 2^k b_{k-1}t_2^k(t_3-t_2)^{k-2} = 0. \end{aligned}$$

Equating to zero the coefficients of t_3^{l-4}, $t_2t_3^{l-5}$,..., $t_2^{k-1}t_3^{k-1}$ we get $a_1 = 0$, $a_2 = 0$, ..., $a_k = 0$, respectively. Then we do the same for the coefficients of $t_2^kt_3^{k-2}$, $t_2^{k+1}t_3^{k-3}$, ..., $t_2^{l-5}t_3$, t_2^{l-4} and get $b_{k-1} = 0$, $b_{k-2} = 0$, ..., $b_2 = 0$, $b_1 = 0$, respectively.

(ii) Let $l = 2k + 3$. Then

$$\begin{aligned} f \quad = \quad & A[y_1, \underbrace{x,\dots,x}_{l-2}, y_2] + B[y_2, \underbrace{x,\dots,x}_{l-2}, y_1] \\ + \quad & a_1[[y_1,x],[y_2,\underbrace{x,\dots,x}_{l-3}]] + a_2[[y_1,x,x],[y_2,\underbrace{x,\dots,x}_{l-4}]] \\ + \quad & \cdots + a_k[[y_1,\underbrace{x,\dots,x}_{k}],[y_2,\underbrace{x,\dots,x}_{k+1}]] \\ + \quad & b_1[[y_2,x],[y_1,\underbrace{x,\dots,x}_{l-3}]] + b_2[[y_2,x,x],[y_1,\underbrace{x,\dots,x}_{l-4}]] \\ + \quad & \cdots + b_k[[y_2,\underbrace{x,\dots,x}_{k}],[y_1,\underbrace{x,\dots,x}_{k+1}]] \\ = \quad & v(g_1)(x,y_1,y_2) + v(g_2)(x,y_2,y_1) = f_1 + f_2. \end{aligned}$$

The polynomial g_1 has the form

$$\begin{aligned} g_1 \quad = \quad & A(t_2-t_1)^{l-2} - B(t_3-t_2)^{l-2} \\ + \quad & a_1(t_2-t_1)(t_3-t_2)^{l-3} + a_2(t_2-t_1)^2(t_3-t_2)^{l-4} \\ + \quad & \cdots + a_k(t_2-t_1)^k(t_3-t_2)^{k+1} \\ - \quad & b_1(t_2-t_1)^{l-3}(t_3-t_2) - b_2(t_2-t_1)^{l-4}(t_3-t_2)^2 \\ - \quad & \cdots - b_k(t_2-t_1)^{k+1}(t_3-t_2)^k. \end{aligned}$$

As in (i) $f_1(t_1 = t_2) = 0$ leads to $B = 0$ and $f_1(t_2 = t_3) = 0$ gives $A = 0$. Then we consider the polynomial $f_1(t_1 = -t_2) = 0$. Its explicit form is

$$2a_1t_2(t_3-t_2)^{l-3} + 4a_2t_2^2(t_3-t_2)^{l-4} + \cdots$$

$$
\begin{aligned}
&+ \quad 2^k a_k t_2^k (t_3 - t_2)^{k+1} - 2^{l-3} b_1 t_2^{l-3} (t_3 - t_2) \\
&- \quad 2^{l-4} b_2 t_2^{l-4} (t_3 - t_2)^2 - \cdots - 2^{k+1} b_k t_2^{k+1} (t_3 - t_2)^k = 0.
\end{aligned}
$$

Dividing by $2t_2(t_3 - t_2)$ and equating to zero the coefficients of t_3^{l-4}, $t_2 t_3^{l-5}$,..., $t_2^{k-1} t_3^k$ we get $a_1 = 0$, $a_2 = 0$, ..., $a_k = 0$, respectively. Then the same procedure for the coefficients of $t_2^k t_3^{k-1}$, $t_2^{k+1} t_3^{k-2}$, ..., $t_2^{l-5} t_3$, t_2^{l-4} gives $b_k = 0$, $b_{k-1} = 0$, ..., $b_2 = 0$, $b_1 = 0$, respectively.

We stress on the efficiency of the approach considered by Bergman and Formanek. Once getting necessary conditions on the type of the possible identities (Proposition 2) one works only in the commutative algebra without concerning skew-symmetric variables at all.

Theorem 1 could be formulated in another way too.

Theorem 2 *For the special Lie algebra $[M_4(K, *)]$ no Bergman polynomials are Lie identities in skew-symmetric variables.*

A statement concerning the minimal degree of the Bergman polynomials as Lie identities for $so(6, K, *)$ could be formulated as well. Its proof follows the same pattern however the technical difficulties are much more.

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