METHOD FOR ANALYTICAL REPRESENTATION OF THE MAXIMUM INACCURACIES OF INDIRECTLY MEASURABLE VARIABLE
(Survey)

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Abstract. Let us have an indirectly measurable variable which is a function of directly measurable variables. In this survey we present the introduced by us method for analytical representation of its maximum absolute and relative inaccuracy as functions, respectively, of the maximum absolute and of the relative inaccuracies of the directly measurable variables. Our new approach consists of assuming for fixed variables the statistical mean values of the absolute values of the coefficients of influence, respectively, of the absolute and relative inaccuracies of the directly measurable variables in order to determine the analytical form of the maximum absolute and relative inaccuracies of an indirectly measurable variable. Moreover, we give a method for determining the numerical values of the maximum absolute and relative inaccuracies. We define a sample plane of the ideal perfectly accurate experiment and using it we give a universal numerical characteristic – a dimensionless scale for determining the quality (accuracy) of the experiment.

1. Introduction

Let us have a real function $f = f(x_1, x_2, ..., x_n) \neq 0$ of n real independent variables $x_1, x_2, ..., x_n$, which can be used in order to model directly measurable (using measuring tools or methods) variables. Then the function f models an indirectly measurable variable. Moreover, let f has continuous first partial derivatives in respect to all its variables.

In order to compute the maximum absolute and relative inaccuracies of an indirectly measurable variable f in the classic theory of the inaccuracies [1, 2, 3, 4, 5] one has to find the full differential

$$(1) \qquad df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

of the function f. When the inaccuracies of the directly measurable variables $x_1, x_2, ..., x_n$ are small enough, in formula (1) the differential d can be replaced by the finite difference Δ, and in this substitution every sign minus is replaced by plus in order for the value of the inaccuracy to be maximum. Thus we get the maximum absolute inaccuracy

$$(2) \qquad \Delta f = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| . \left| \Delta x_i \right|,$$

where Δx_i is the maximum absolute inaccuracy of the directly measurable variable x_i ($i = 1, 2, ... n$). Then the maximum absolute inaccuracy f_r is determined by

$$(3) \qquad f_r = \frac{\Delta f}{|f|} = \frac{1}{|f|} \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| . \left| \Delta x_i \right|.$$

In [6, 7, 8] we develop the classical theory of inaccuracies by defining a new method for representing the maximum absolute and relative inaccuracies of an indirectly measurable variable $f(x_1, x_2, ..., x_n)$ as functions, respectively, of the maximum absolute and relative inaccuracies of the directly measurable variables $x_1, x_2, ..., x_n$.

2. Analytical representation of the maximum absolute and relative inaccuracies of an indirectly measurable variable

According to formula (2) and (3) the evaluations of the maximum absolute and relative inaccuracies of an indirectly measurable variable $f(x_1, x_2, ..., x_n)$ depend not only on, respectively, the absolute and relative inaccuracies with which the directly measurable variables $x_1, x_2, ..., x_n$ are determined, but also on the analytical form of the functional dependency f itself.

Our new approach [6, 7, 8] towards the determining the analytical form of the maximum absolute and relative inaccuracies of the indirectly measurable variable $f(x_1, x_2, ..., x_n)$ consists in the fact that in formula (2) and (3) we consider the mean values, respectively, $\overline{\left| \frac{\partial f}{\partial x_1} \right|}, \overline{\left| \frac{\partial f}{\partial x_2} \right|}, ..., \overline{\left| \frac{\partial f}{\partial x_n} \right|}$ and

$$\overline{\left|\frac{x_1}{f} \cdot \frac{\partial f}{\partial x_1}\right|}, \overline{\left|\frac{x_2}{f} \cdot \frac{\partial f}{\partial x_2}\right|}, \ldots, \overline{\left|\frac{x_n}{f} \cdot \frac{\partial f}{\partial x_n}\right|}$$ of the absolute values of the coefficients of

influence of the absolute $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ and relative $\dfrac{\Delta x_1}{x_1}, \dfrac{\Delta x_2}{x_2}, \ldots, \dfrac{\Delta x_n}{x_n}$

inaccuracies in the indirectly measurable variable f for being fixed variables, and the maximum absolute $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ and relative $\dfrac{\Delta x_1}{x_1}, \dfrac{\Delta x_2}{x_2}, \ldots, \dfrac{\Delta x_n}{x_n}$ inaccuracies of the directly measurable variables $x_1, x_2, \ldots, x_n$ we consider to be variables. Therefore:

The maximum absolute inaccuracy Δf of the indirectly measurable variable f is a lineal function

(4)
$$\Delta f = \sum_{i=1}^{n} \overline{\left|\frac{\partial f}{\partial x_i}\right|} \cdot \left|\Delta x_i\right|$$

of the maximum absolute inaccuracies $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ of the directly measurable variables $x_1, x_2, \ldots, x_n$.

The maximum relative inaccuracy f_r of the indirectly measurable variable f is a linear function

(5)
$$f_r = \sum_{i=1}^{n} \overline{\left|\frac{x_i}{f} \cdot \frac{\partial f}{\partial x_i}\right|} \cdot \left|\frac{\Delta x_i}{x_i}\right|$$

of the maximum relative inaccuracies $\dfrac{\Delta x_1}{x_1}, \dfrac{\Delta x_2}{x_2}, \ldots, \dfrac{\Delta x_n}{x_n}$ of the directly measurable variables $x_1, x_2, \ldots, x_n$.

If we consider $\Delta x_1, \Delta x_2, \ldots, \Delta x_n, \pm \Delta f$ to be a system of generalised orthogonal coordinates, then we get an $n+1$-dimensional metric hyperspace F_{n+1}, in which the equation (4) is an equation of a hyperplane, passing through the origin of the coordinate system. The hyperspace F_{n+1} we call *a space of the absolute inaccuracy of f*, and Δf we call *a plane of the absolute inaccuracy of f*. The term space of the absolute inaccuracy is introduced by us for the first time in [6].

On the other hand, if we consider $\dfrac{\Delta x_1}{x_1}, \dfrac{\Delta x_2}{x_2}, \ldots, \dfrac{\Delta x_n}{x_n}, \pm f_r$ to be a system of generalised orthogonal coordinates, then analogically we get $n+1$-

dimensional metric hyperspace F_r^{n+1}, which we call *a space of the relative inaccuracy of f*, and f_r from equation (5) we call *a plane of the relative inaccuracy of f*. The term space of the relative inaccuracy is introduced by us for the first time in [8].

3. Determining the numerical values of the maximum absolute and relative inaccuracies of an indirectly measurable variable

In [7, 8] we show how the numerical values of the maximum inaccuracies of an indirectly measurable variable can be determined.

Namely, let in an experiment k measurements have been made of the directly measurable variables $x_1, x_2, ..., x_n$. On the m-th measurement ($m = 1, 2, ..., k$) the absolute values of the partial derivatives $\left|\dfrac{\partial f}{\partial x_1}\right|_m, \left|\dfrac{\partial f}{\partial x_2}\right|_m, ..., \left|\dfrac{\partial f}{\partial x_n}\right|_m$ and the absolute inaccuracies $\left|\Delta x_1\right|_m, \left|\Delta x_2\right|_m, ..., \left|\Delta x_n\right|_m$ are computed. Following, the mean values $\overline{\left|\dfrac{\partial f}{\partial x_j}\right|} = \dfrac{1}{k}\sum_{m=1}^{k}\left|\dfrac{\partial f}{\partial x_j}\right|_m$ ($j = 1, 2, ..., n$) are found and from formula (4) the analytical representation (equation) of the plane of the inaccuracies is given.

Furthermore, if $\overline{\left|\Delta x_i\right|} = \dfrac{1}{k}\sum_{m=1}^{k}\left|\Delta x_i\right|_m$, then according to formula (4) the numerical value of the maximum absolute inaccuracy $\Delta f = \sum_{i=1}^{n}\overline{\left|\dfrac{\partial f}{\partial x_i}\right|}.\overline{\left|\Delta x_i\right|}$ is determined as the point $\left(\overline{\left|\Delta x_1\right|}, \overline{\left|\Delta x_2\right|}, ..., \overline{\left|\Delta x_n\right|}, \Delta f\right)$ lies in the place of the absolute inaccuracy.

According to formula (5) analogically the numerical value of the maximum relative inaccuracy f_r is determined as the point $\left(\overline{\left|\dfrac{\Delta x_1}{x_1}\right|}, \overline{\left|\dfrac{\Delta x_2}{x_2}\right|}, ..., \overline{\left|\dfrac{\Delta x_n}{x_n}\right|}, f_r\right)$ lies in the plane of the relative inaccuracy.

4. Scale characterising the quality of the experiment

Let us consider the hyperplane α of the absolute inaccuracy of f:

$$\alpha : \sum_{i=1}^{n} A_i \cdot |\Delta x_i| - \Delta f = 0, \text{ where } A_i = \left| \overrightarrow{\frac{\partial f}{\partial x_i}} \right| = const \geq 0.$$

Let us also consider the hyperplane $\varepsilon : \Delta f = 0$. Obviously, the equation $\Delta f = 0$ is possible if and only if $\dfrac{\partial f}{\partial x_1} = \dfrac{\partial f}{\partial x_2} = ... = \dfrac{\partial f}{\partial x_n} = 0$. Thus we consider [7, 9] ε to be a *sample plane* in the space of the absolute inaccuracy – in the sense that it corresponds to an imaginary ideal perfectly accurate experiment. Strictly speaking, such an experiment is impossible and the sample plane ε is unreachable. However, by increasing the accuracy of the real experiment the plane α approximates ε. Thus the smaller the deviation of the plane α of the experiment from the sample plane ε of the ideal experiment is, i.e. the smaller the angle between these two planes is, the more accurate the experiment is.

This angle is equal to the angle between the normal vectors $\overrightarrow{n_\alpha}(A_1, A_2, ..., A_n, -1)$ of the plane α and $\overrightarrow{n_\varepsilon}(0, 0, ..., 0, -1)$ of the plane ε. Thus the value of the cosine

$$k_\alpha = \cos \angle \left(\overrightarrow{n_\alpha}, \overrightarrow{n_\varepsilon} \right) = \frac{1}{\sqrt{A_1^2 + A_2^2 + ... + A_n^2 + 1}}$$

of this angle we choose to be a *coefficient of accuracy in a dimensionless scale,* i.e. for *numerical characteristic of the quality of the experiment.*

The scale for determining the quality of the experiment is the interval $[0,1]$. An experiment is as accurate as the value of the coefficient of accuracy k_α is closer to 1 and is as inaccurate as the value of the coefficient of accuracy k_α is closer to 0. The value $k_\alpha = 1$ represents the ideal perfectly accurate experiment and the value $k_\alpha = 0$ – the ideal absolutely inaccurate experiment.

Analogically in [8] we introduce a dimensionless scale in the space of the relative inaccuracy as well.

5. Inaccuracies of second order

Let the function $f(x_1, x_2, ..., x_n)$ is at least twice differentiable with respect to all its variables. Let us consider its differential of second order, namely $d^2 f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$. Analogically to the classic theory of inaccuracies in the case when the values of the absolute inaccuracies of the directly measurable variables are small enough, we substitute d^2 with Δ^2. Moreover, again, all minus signs are replaced with plusses. Thus, analogically to formula (2) we get the variable $\Delta^2 f = \sum_{i,j=1}^{n} \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| . |\Delta x_i| . |\Delta x_j|$.

Having in mind formula (3), we directly get the variable $f_r^2 = \frac{\Delta^2 f}{f^2} = \sum_{i,j=1}^{n} \left| \frac{x_i}{f} . \frac{x_j}{f} . \frac{\partial^2 f}{\partial x_i \partial x_j} \right| . \left| \frac{\Delta x_i}{x_i} \right| . \left| \frac{\Delta x_j}{x_j} \right|$ as well. Then using formula (4) we can determine the variable

$$(6) \qquad \Delta^2 f = \sum_{i,j=1}^{n} \overline{\left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right|} . |\Delta x_i| . |\Delta x_j|,$$

and using formula (5) – the variable

$$(7) \qquad f_r^2 = \frac{\Delta^2 f}{f^2} = \sum_{i,j=1}^{n} \overline{\left| \frac{x_i}{f} . \frac{x_j}{f} . \frac{\partial^2 f}{\partial x_i \partial x_j} \right|} . \left| \frac{\Delta x_i}{x_i} \right| . \left| \frac{\Delta x_j}{x_j} \right|.$$

In the expansion of the classical theory of inaccuracies, which we make in [10], we introduce the following terminology. The maximum absolute and relative inaccuracy, defined respectively by formula (4) and (5) we call *maximum absolute inaccuracy of first order* and *maximum relative inaccuracy of first order*. The variables, defined respectively by formulas (6) and (7) we call *maximum absolute inaccuracy of second order* and *maximum relative inaccuracy of second order*.

The main contribution in the numerical value of the maximum inaccuracies is given by the maximum inaccuracies of first order, but when these values are practically equal, the maximum inaccuracies of second order give additional and more precise information.

6. Characteristic of the method

The advantages of the described method for analytical representation of the maximum absolute and relative inaccuracies of an indirectly measurable variable can be summarised in the following general directions.

(i) More adequate to the objective reality quantitative value of the maximum absolute and relative inaccuracies of an indirectly measurable variable.

(ii) Universality, because it can be applied in different fields of science; considering experiments, carried out with different tools using different methods; considering mathematical models, described even by indifferentiable functions.

In [8] we show how the method can be applied even to experiments, modelled by functions which are continuous, but are not differentiable with respect to some of its arguments in some points.

7. Conclusion

The function $f(x_1, x_2, ..., x_n)$ can be considered as random variable of random independent variables. In that sense our method for computing Δf and f_r is more adequate to the objective reality because the statistical mean value of a random variable is in fact its most probably value. Again in that sense the planes of the absolute and the relative inaccuracies of f are *stochastic* planes.

While in the classical method arithmetic mean values of the indirectly measurable variable f are used, we use the statistic mean values of the random variables which compose it. This way we get the most probably value of f.

The suggested by us method has significant importance in any experimental field of science – chemistry, physics, biology, medicine, sociology, economics, etc. in which the studied processes are described by differentiable functions.

Acknowledgements

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