CONFORMALLY INVARIANT TENSORS ON ALMOST CONTACT MANIFOLDS WITH B-METRIC

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Abstract. In this paper, we study the main classes of almost contact manifolds with B-metric. A canonical connection on these classes is introduced with respect to which the structure tensors are covariantly constant. Conformally invariant tensors with respect to groups of contactly conformal transformations of the considered classes are found. The zero Bochner curvature tensor of the canonical connection is proved to be an integrability condition of a geometrical system of partial differential equations and a characterization condition of a contactly conformally flat manifold.

1. Almost contact manifolds with B-metric.

Let (M,φ,ξ,η,g) be a $(2n+1)$-dimentional almost contact manifold with B-metric, i.e. (φ,ξ,η) is the almost contact structure [2] and g is a metric on M so that :

$$\varphi^2 = -id + \eta \otimes \xi \quad ; \quad \eta(\xi) = 1 \quad ; \quad g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y) \tag{1.1}$$

for all vector fields X, Y on M. The associated metric $\tilde{g}$ of the manifold is given by $\tilde{g}(X,Y) = g(X,\varphi Y) + \eta(X)\eta(Y)$. Both metrics are necessarily of signature $(n+1,n)$ [4].

Further, $X,\ Y,\ Z,\ W$ will stand for arbitrary differentiable vector fields on M and $x,\ y,\ z,\ w$— arbitrary vectors in the tangential space T_pM to M at an arbitrary point p in M. The Levi-Civita connection of g will be denoted by ∇. The tensor field F of type (0,3) on the manifold is defined by $F(X,Y,Z) = g((\nabla_X \varphi)Y, Z)$ and it has the following properties :

$$\begin{aligned} F(X,Y,Z) &= F(X,Z,Y) \ ; \\ F(X,\varphi Y,\varphi Z) &= F(X,Y,Z) - \eta(Y)F(X,\xi,Z) - \eta(Z)F(X,Y,\xi) \ . \end{aligned} \tag{1.2}$$

If $\{e_i,\xi\}$ $(i = 1,2,\ldots,2n)$ is an arbitrary basis of T_pM , g^{ij} are the components of the inverse matrix of g, then the 1- forms $\theta,\ \theta^*$ and ω associated with F are defined by

$$\theta(x) = g^{ij}F(e_i,e_j,x) \ , \ \theta^*(x) = g^{ij}F(e_i,\varphi e_j,x) \ , \ \omega(x) = F(\xi,\xi,x) \ . \tag{1.3}$$

Let Q be the corresponding vector to θ by g, i.e. $\theta(x) = g(x,Q)$.

A classification of the almost contact manifolds with B-metric with respect to the tensor F is given in [4], where the basic classes $\mathcal{F}_i$ $(i = 1,2,\ldots,11)$ of almost contact manifolds with B-metric are defined. In this paper we consider the following classes arised from the main components of F :

$$(1.4)\qquad \mathcal{F}_1\ :\ F(X,Y,Z)=\frac{1}{2n}\{g(X,\varphi Y)\theta(\varphi Z)+g(X,\varphi Z)\theta(\varphi Y)+g(\varphi X,\varphi Y)\theta(\varphi^2 Z)+g(\varphi X,\varphi Z)\theta(\varphi^2 Y)\}\ ,$$

$$(1.5)\qquad \mathcal{F}_4\ :\ F(X,Y,Z)=-\frac{\theta(\xi)}{2n}\{g(\varphi X,\varphi Y)\eta(Z)+g(\varphi X,\varphi Z)\eta(Y)\}\ ,$$

$$(1.6)\qquad \mathcal{F}_5\ :\ F(X,Y,Z)=-\frac{\theta^*(\xi)}{2n}\{g(X,\varphi Y)\eta(Z)+g(X,\varphi Z)\eta(Y)\}\ .$$

The class $\mathcal{F}_0$ is defined by the condition $F(X,Y,Z)=0$. This special class belongs to each of the basic classes. The 1-form ω is zero on $\mathcal{F}_i$ $(i=0,1,4,5)$. An almost contact manifold with B-metric in the class $\mathcal{F}_i$, for the sake of brevity, we call $\mathcal{F}_i$-manifold $(i=0,1,4,5)$.

Let $\mathcal{F}_i^0$ is subclass of $\mathcal{F}_i$ $(i=1,4,5)$ that is defined by the next conditions for the 1-forms θ and θ^* :

(1.7)
a) (M,φ,ξ,η,g) is an $\mathcal{F}_1^0$-manifold iff θ and θ^* are closed ;
b) (M,φ,ξ,η,g) is an $\mathcal{F}_4^0$-manifold iff θ is closed ;
c) (M,φ,ξ,η,g) is an $\mathcal{F}_5^0$-manifold iff θ^* is closed .

The transformation

$$(1.8)\quad c:\bar{g}(X,Y)=e^{2u}\cos 2v g(X,Y)+e^{2u}\sin 2v g(X,\varphi Y)+(1-e^{2u}\cos 2v)\eta(X)\eta(Y)$$

is a contactly conformal transformation, where u and v are arbitrary differentiable functions on M. The manifolds (M,φ,ξ,η,g) and $(M,\varphi,\xi,\eta,\bar{g})$ are contactly conformally related by c . The group of the contactly conformal transformations on (M,φ,ξ,η,g) will be denoted by C [5].

Further, we shall use the symbols $\alpha=du=g(\cdot,p)$ and $\beta=dv=g(\cdot,q)$. The following subgroups of the contactly conformal group C are introduced in [5] :

$$C_0=\{c\in C\mid \alpha=\beta\circ\varphi,\beta=-\alpha\circ\varphi\}\ ,\ C_1=\{c\in C\mid \alpha(\xi)=\beta(\xi)=0\}\ ,$$

$$C_4=\{c\in C\mid \alpha=\beta\circ\varphi\}\ ,\ C_5=\{c\in C\mid \beta=-\alpha\circ\varphi\}\ ,$$

$$C_{45}=\{c\in C\mid \alpha\circ\varphi=\beta\circ\varphi^2\}\ ,\ C_{45}^*=\{c\in C_{45}\mid c\notin C_4, c\notin C_5\}\ ,$$

$$C_{15}=\{c\in C\mid \alpha(\xi)=0\}\ ,\ C_{15}^*=\{c\in C_{15}\mid c\notin C_1, c\notin C_5\}\ ,$$

$$C_{14}=\{c\in C\mid \beta(\xi)=0\}\ ,\ C_{14}^*=\{c\in C_{14}\mid c\notin C_1, c\notin C_4\}\ ,$$

$$(1.9)\quad C_1^0=\{c\in C_1\mid \alpha(\xi)=d(\alpha\circ\varphi)=0\ ,\ \beta(\xi)=d(\beta\circ\varphi)=0\},\ C_1^{0'}=\{c\in C_1^0\mid \beta=0\}$$

$$C_4^0=\{c\in C_4\mid \alpha(\xi)=d(\alpha\circ\varphi)=0\}\ ,\ C_4^{0\prime}=\{c\in C_4^0\mid \alpha=0\}\ ,$$

$$C_5^0=\{c\in C_5\mid \beta(\xi)=d(\beta\circ\varphi)=0\}\ ,\ C_5^{0\prime}=\{c\in C_5^0\mid \beta=0\}\ ,$$

$$C_{45}^{*0}=\{c\in C_{45}^*\mid c=c_2\circ c_1, c_1\in C_4^0, c_2\in C_5^0\}\ ,$$

$$C_{15}^{*0}=\{c\in C_{15}^*\mid c=c_2\circ c_1, c_1\in C_1^0, c_2\in C_5^0\}\ ,$$

$$C_{14}^{*0}=\{c\in C_{14}^*\mid c=c_2\circ c_1, c_1\in C_1^0, c_2\in C_4^0\}\ .$$

In this paper, we shall use the following propositions.

Theorem A. (Theorem 2.2 and Theorem 2.5 in [5]) *The classes* $\mathcal{F}_0$, $\mathcal{F}_i^0$ *are closed with respect to the groups* C_0, C_i^0 $(i = 1, 4, 5)$, *respectively.*

Theorem B. (Theorems 2.6 — 2.8 in [5]) *The class* $\mathcal{F}_i^0$ *is the class of the manifolds, which are contactly conformally equivalent to the* $\mathcal{F}_0$*-manifolds with respect to the group* C_i^0, *i.e.* $\mathcal{F}_i^0 = C_i^0(\mathcal{F}_0)(i = 1, 4, 5)$.

Theorem C. (Theorem 2.9 in [5]) *The classes* $\mathcal{F}_i^0$ *and* $\mathcal{F}_j^0$ *are contactly conformally equivalent with respect to the group* C_{ij}^{*0}, *i.e.* $\mathcal{F}_j^0 = C_{ij}^{*0}(\mathcal{F}_i^0)(i, j = 1, 4, 5; i \neq j)$.

2. The Bochner curvature tensor of φ-holomorphic type in the class $\mathcal{F}_0$

Definition 2.1 [3] . *A tensor* T *of type (0,4) on* M *is called a curvature-like tensor if it satisfies the conditions :*

$$T(X, Y, Z, W) = -T(Y, X, Z, W) \ ; \tag{2.1}$$

$$T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W) = 0 \ ; \tag{2.2}$$

$$T(X, Y, Z, W) = -T(X, Y, W, Z) \ . \tag{2.3}$$

In analogy with the definition in [3] of Kaehler tensor on an almost complex manifold with B-metric, we give the following

Definition 2.2. *A curvature-like tensor* T *we call a Kaehler tensor if it satisfies the condition*

$$T(X, Y, Z, W) = -T(X, Y, \varphi Z, \varphi W) \ . \tag{2.4}$$

Let S be a tensor of type (0,2). We consider the following fundamental tensors

$$\begin{aligned} \Psi_1(S)(X, Y, Z, W) &= g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\ &\quad + g(X, W)S(Y, Z) - g(Y, W)S(X, Z) \ , \\ \Psi_2(S)(X, Y, Z, W) &= g(Y, \varphi Z)S(X, \varphi W) - g(X, \varphi Z)S(Y, \varphi W) \\ &\quad + g(X, \varphi W)S(Y, \varphi Z) - g(Y, \varphi W)S(X, \varphi Z) \ . \end{aligned} \tag{2.5}$$

We have

Lemma 2.1. *A)* $\Psi_1(S)$ *is a curvature-like tensor iff* $S(X,Y) = S(Y,X)$;
B) $\Psi_2(S)$ *is a curvature-like tensor iff* $S(X,\varphi Y) = S(Y,\varphi X)$.

The tensors π_i $(i = 1,2,3,4,5)$ are defined as follows :

$$\begin{aligned}
&\pi_1(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \ ; \\
&\pi_2(X,Y,Z,W) = g(Y,\varphi Z)g(X,\varphi W) - g(X,\varphi Z)g(Y,\varphi W) \ ; \\
&\pi_3(X,Y,Z,W) = -g(Y,Z)g(X,\varphi W) + g(X,Z)g(Y,\varphi W) \\
&\qquad\qquad - g(X,W)g(Y,\varphi Z) + g(Y,W)g(X,\varphi Z) \ ; \\
&\pi_4(X,Y,Z,W) = g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\
&\qquad\qquad + g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z) \ ; \\
&\pi_5(X,Y,Z,W) = g(Y,\varphi Z)\eta(X)\eta(W) - g(X,\varphi Z)\eta(Y)\eta(W) \\
&\qquad\qquad + g(X,\varphi W)\eta(Y)\eta(Z) - g(Y,\varphi W)\eta(X)\eta(Z) \ .
\end{aligned} \tag{2.6}$$

It is easy to verify that

$$\Psi_1(g) = 2\pi_1 \ ; \ \Psi_2(g) = 2\pi_2 \ ; \ \Psi_1(\tilde{g}) = -\pi_3 + \pi_4 \ ; \ \Psi_2(\tilde{g}) = \pi_3 + \pi_5 \ . \tag{2.7}$$

Lemma 2.2. *The tensors* π_i $(i = 1,2,3,4,5)$ *are curvature-like tensors and the tensors* $(\pi_1 - \pi_2)\circ\varphi$ *and* $\pi_3\circ\varphi$ *are Kaehler tensors.*

The proof is trivial.

Let R be the curvature tensor field of type (1,3) of ∇, i.e.

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ . \tag{2.8}$$

The corresponding tensor of type (0,4) is denoted by the same letter and is given by

$$R(X,Y,X,W) = g\big(R(X,Y)Z,W\big) \ . \tag{2.9}$$

Let (M,φ,ξ,η,g) be an $\mathcal{F}_0$-manifold. Since φ,ξ,η are covariantly constant on M with respect to ∇, then

$$R(X,Y,\varphi Z,\varphi W) = R(X,\varphi Y,\varphi Z,W) = -R(X,Y,Z,W) \ ; \tag{2.10}$$

$$R(X,Y)\xi = 0 \ . \tag{2.11}$$

From (2.10) it is seen, that R is a Kaehler tensor on $M \in \mathcal{F}_0$. This properties of the curvature tensor R on an $\mathcal{F}_0$-manifold M gives us a reason for defining on M a tensor field $\tilde{R}$ of type (0,4) by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,\varphi W) = R(X,Y,\varphi Z,W) \ . \tag{2.12}$$

We verify immediately, that $\tilde{R}$ has the properties (2.1)—(2.4). Therefore $\tilde{R}$ is a Kaehler tensor.We remark that if ρ (respectively $\tilde{\rho}$) is the Ricci tensor and τ (respectively $\tilde{\tau}$)—the scalar curvature of R (respectively $\tilde{R}$), then

$$\begin{aligned} &\rho(\tilde{R})(y,z) = \tilde{\rho}(R)(y,z) = \rho(R)(y,\varphi z)\ ; \\ &\tau(\tilde{R}) = \tilde{\tau}(R) \qquad ; \qquad \tilde{\tau}(\tilde{R}) = -\tau(R)\ . \end{aligned} \tag{2.13}$$

The Bochner curvature tensor of B-manifold is introduced in [2] by

$$B(R) = R - \frac{1}{2(n-2)}\big[\Psi_1(\rho) - \Psi_2(\rho)\big] + \frac{1}{4(n-1)(n-2)}\big[\tau(\pi_1 - \pi_2) + \tilde{\tau}\pi_3\big]\ .$$

Now, for $\mathcal{F}_0$-manifold $M(dimM \geq 7)$ we define a tensor field $B(R)$ of type (0,4) , corresponding to R whereby

$$\begin{aligned} B(R)(x,y,z,w) = {} & R(x,y,z,w) \\ & -\frac{1}{2(n-2)}\big[\Psi_1(\rho)(\varphi x,\varphi y,\varphi z,\varphi w) - \Psi_2(\rho)(x,y,z,w)\big] \\ & +\frac{1}{4(n-1)(n-2)}\big[\tau\big(\pi_1(\varphi x,\varphi y,\varphi z,\varphi w) - \pi_2(x,y,z,w)\big) \\ & \qquad + \tilde{\tau}\pi_3(\varphi x,\varphi y,\varphi z,\varphi w)\big]\ , \end{aligned} \tag{2.14}$$

that we call *Bochner curvature tensor on M of φ-holomorphic type.* From the properties of R and π_i $(i = 1,2,3)$ it follows that the Bochner curvature tensor $B(R)$ is a Kaehler tensor.

3. The canonical connection on almost contact manifold with B-metric

Definition 3.1. *A linear connection on an almost contact manifold with B-metric we call canonical connection if the structure tensors φ, ξ, η and g on M are covariantly constant with respect to this connection.*

Let the linear connection D on (M,φ,ξ,η,g) if defined by

$$D_X Y = \nabla_X Y + \frac{1}{2}\big\{(\nabla_X\varphi)\varphi Y + (\nabla_X\eta)Y.\xi\big\} - \eta(Y)\nabla_X\xi\ . \tag{3.1}$$

Now, we verify immediately

$$D\varphi = D\xi = D\eta = Dg = 0\ . \tag{3.2}$$

Consequently D is a canonical connection. We remark, that according to Definition 3.1 , the Levi-Civita connection ∇ on an $\mathcal{F}_0$-manifold is a canonical connection. The equalities (3.1), (1.4), (1.5) and (1.6) give us a possibility to determine the particular kind of the canonical connection D in each of the considered classes $\mathcal{F}_1$, $\mathcal{F}_4$ and $\mathcal{F}_5$. We have got the validity of the next

Lemma 3.1. *a) If $M \in \mathcal{F}_1$, then*

$$D_X Y = \nabla_X Y + \frac{1}{4n}\Big[-\theta(\varphi Y)\varphi^2 X - \theta(Y)\varphi X + g(\varphi X, \varphi Y)\varphi Q + g(X, \varphi Y)Q\Big] \ ,$$

where $\theta = g(\cdot, Q)$;

b) If $M \in \mathcal{F}_4$, then $D_X Y = \nabla_X Y + \dfrac{\theta(\xi)}{2n}\Big[g(X, \varphi Y)\xi - \eta(Y)\varphi X\Big]$;

c) If $M \in \mathcal{F}_5$, then $D_X Y = \nabla_X Y - \dfrac{\theta^*(\xi)}{2n}\Big[g(\varphi X, \varphi Y)\xi - \eta(Y)\varphi^2 X\Big]$.

We shall use the following proposition.

Lemma A. (Lemma 1.1 in [5]) *a) If $M \in \mathcal{F}_1$, then θ (respectively θ^*) is closed iff $(\nabla_X \theta)Y = (\nabla_Y \theta)X$ $\big($respectively $(\nabla_X \theta)\varphi Y = (\nabla_Y \theta)\varphi X\big)$;*
b) If $M \in \mathcal{F}_4$, then θ is closed iff $X\theta(\xi) = \eta(X)\xi\theta(\xi)$;
c) If $M \in \mathcal{F}_5$, then θ^* is closed iff $X\theta^*(\xi) = \eta(X)\xi\theta^*(\xi)$.

Further, K will stand for the curvature tensor field of type (1,3) of D, i.e.

$$K(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \ . \tag{3.3}$$

The corresponding tensor field K of type (0,4) is denoted by the same letter and is given by

$$K(X, Y, Z, W) = g\big(K(X, Y)Z, W\big) \ . \tag{3.4}$$

Theorem 3.1. *Let M be an $\mathcal{F}_i^0$-manifold $(i = 1, 4, 5)$. Then K is a Kaehler tensor and it is determined respectively by*

$$\begin{aligned} K = R + \frac{1}{4n}\Big\{\varphi\big(\Psi_1(S)\circ\varphi\big) + \Psi_2(S) + \frac{1}{2n}\varphi\big(\Psi_1(L)\circ\varphi\big)\Big\} \\ - \frac{1}{16n^2}\Big\{\theta(t)\big[3\varphi(\pi_1\circ\varphi) + \pi_2\big] - \theta(\varphi t)\varphi(\pi_3\circ\varphi)\Big\} \quad \text{for } i = 1 \ , \end{aligned} \tag{3.5}$$

where $S(Y,Z) = S(Z,Y) = -S(\varphi Y, \varphi Z) = -(\nabla_Y \theta)\varphi Z - \frac{1}{4n}\left[\theta(Y)\theta(Z) - \theta(\varphi Y)\theta(\varphi Z)\right]$,

$L(Y,Z) = L(Z,Y) = L(\varphi Y, \varphi Z) = \theta(Y)\theta(Z) + \theta(\varphi Y)\theta(\varphi Z)$;

$$K = R + \frac{\xi\theta(\xi)}{2n}\pi_5 + \frac{\theta^2(\xi)}{4n^2}[\pi_2 - \pi_4] \quad \text{for } i = 4 \ ; \tag{3.6}$$

$$K = R + \frac{\xi\theta^*(\xi)}{2n}\pi_4 + \frac{\theta^{*2}(\xi)}{4n^2}\pi_1 \quad \text{for } i = 5 \ . \tag{3.7}$$

Proof. Using Lemma 3.1 and Lemma A, from the equalities (3.3), (3.4) and (1.7) , we get the equalities (3.5), (3.6) and (3.7), respectively. Hence according to Lemma 2.1 and Lemma 2.2, it follows that K is a curvature-like tensor . From the conditions $D\varphi = D\xi = 0$, it is clear, that K is a Kaehler tensor on an $\mathcal{F}_i^0$-manifold $(i = 1, 4, 5)$.

This proposition gives us a reason to define an associated tensor $\tilde{K}$ of K on an $\mathcal{F}_i^0$-manifold $(i = 1, 4, 5)$ by

$$\tilde{K}(X, Y, Z, W) = K(X, Y, Z, \varphi W) \ . \tag{3.8}$$

We remark that $\tilde{K}$ satisfies the conditions (2.1)—(2.4), hence $\tilde{K}$ is a Kaehler tensor .

Theorem 3.2. *Let (M,φ,ξ,η,g) and $(M,\varphi,\xi,\eta,\bar{g})$ be contactly conformally related $\mathcal{F}_i$-manifolds by transformations of the groups C_i $(i=0,1,4,5)$. Then*

$$(3.9)\quad \bar{\nabla}_X Y=\nabla_X Y-\alpha(X)\varphi^2Y-\alpha(Y)\varphi^2X-\alpha(\varphi X)\varphi Y-\alpha(\varphi Y)\varphi X + g(\varphi X,\varphi Y)p+g(X,\varphi Y)\varphi p \qquad \text{for } i=0\ ;$$

$$(3.10)\quad \bar{D}_X Y=D_X Y-\alpha(X)\varphi^2Y-\frac{1}{2}[\alpha(Y)+\beta(\varphi Y)]\varphi^2X +\beta(X)\varphi Y-\frac{1}{2}[\alpha(\varphi Y)-\beta(Y)] +\frac{1}{2}(p+\varphi q)(g(\varphi X,\varphi Y)+\frac{1}{2}(\varphi p-q)g(X,\varphi Y) \qquad \text{for } i=1\ ;$$

$$(3.11)\quad \bar{D}_X Y=D_X Y-\beta(\varphi X)\varphi^2Y-\beta(\varphi Y)\varphi^2X+\beta(X)\varphi Y-\beta(\varphi^2Y)\varphi X +g(\varphi X,\varphi Y)\varphi q+g(X,\varphi Y)\varphi^2 q \qquad \text{for } i=4\ ;$$

$$(3.12)\quad \bar{D}_X Y=D_X Y-\alpha(X)\varphi^2Y-\alpha(\varphi^2Y)\varphi^2X-\alpha(\varphi X)\varphi Y-\alpha(\varphi Y)\varphi X -g(\varphi X,\varphi Y)\varphi^2 p+g(X,\varphi Y)\varphi p \qquad \text{for } i=5\ .$$

Proof. Let $\bar{\nabla}$ (respectively ∇) be the Levi-Civita connection of the metric $\bar{g}$ (respectively g) on $(M,\varphi,\xi,\eta,\bar{g})$ (respectively (M,φ,ξ,η,g)). Then using for $\bar{g}$ and $\bar{\nabla}$ the well known equality

$$2\bar{g}(\bar{\nabla}_X Y,Z)=X\bar{g}(Y,Z)+Y\bar{g}(X,Z)-Z\bar{g}(X,Y) +\bar{g}([X,Y],Z)+\bar{g}([Z,X],Y)+\bar{g}([Z,Y],X)$$

and the analogical equality for g and ∇, we obtain [1]:

(3.9) for $i=0$;

$$(3.13)\quad \bar{\nabla}_X Y=\nabla_X Y-\alpha(X)\varphi^2Y-\alpha(Y)\varphi^2X+\beta(X)\varphi Y+\beta(Y)\varphi X +\Big[\cos^2 2v\ p+\sin^2 2v\Big(\frac{\varphi Q}{2n}+\varphi q\Big)-\sin 2v\cos 2v\Big(\frac{Q}{2n}+\varphi p+q\Big)\Big]g(\varphi X,\varphi Y) -\Big[\cos^2 2v\ q-\sin^2 2v\Big(\frac{Q}{2n}+\varphi p\Big) -\sin 2v\cos 2v\Big(\frac{\varphi Q}{2n}-p+\varphi q\Big)\Big]g(X,\varphi Y) \qquad \text{for } i=1\ ;$$

$$(3.14)\quad \bar{\nabla}_X Y=\nabla_X Y-\beta(\varphi X)\varphi^2Y-\beta(\varphi Y)\varphi^2X+\beta(X)\varphi Y+\beta(Y)\varphi X +\Big[\varphi q-e^{2u}\sin 2v\Big(\frac{\theta(\xi)}{2n}+\beta(\xi)\Big)\xi\Big]g(\varphi X,\varphi Y) +\Big[\varphi^2 q-e^{2u}\cos 2v\Big(\frac{\theta(\xi)}{2n}+\beta(\xi)\Big)\xi+\frac{\theta(\xi)}{2n}\xi\Big]g(X,\varphi Y) \qquad \text{for } i=4\ ;$$

$$(3.15)\quad \bar{\nabla}_X Y=\nabla_X Y-\alpha(X)\varphi^2Y-\alpha(Y)\varphi^2X-\alpha(\varphi X)\varphi Y-\alpha(\varphi Y)\varphi X -\Big[\varphi^2 p-e^{2u}\cos 2v\Big(\frac{\theta^*(\xi)}{2n}+\alpha(\xi)\Big)\xi+\frac{\theta^*(\xi)}{2n}\xi\Big]g(\varphi X,\varphi Y) +\Big[\varphi p-e^{2u}\sin 2v\Big(\frac{\theta^*(\xi)}{2n}+\alpha(\xi)\Big)\xi\Big]g(X,\varphi Y) \qquad \text{for } i=5\ ;$$

By means of Lemma 3.1, from the equalities (3.13), (3.14) and (3.15), we find the equalities (3.10), (3.11) and (3.12), respectively .

We remark, that the groups C_i ($i = 0, 1, 4, 5$) of $\mathcal{F}_i$-manifolds ($i = 0, 1, 4, 5$) give rise to contactly conformal groups of transformations of the canonical connection D on the corresponding $\mathcal{F}_i$-manifolds, respectively . Formulas (3.9), (3.10), (3.11) and (3.12) express analitically contactly conformal transformations of the canonical connection D on a manifold M in the classes $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_4$ and $\mathcal{F}_5$, respectively. We notice, that for $i = 0$ the Levi-Civita connection is a canonical connection .

Using Theorem 3.2 and Lemma A, we get the following

Theorem 3.3. *Let $(M, \varphi, \xi, \eta, g)$ and $(M, \varphi, \xi, \eta, \bar{g})$ be contactly conformally related $\mathcal{F}_i^0$-manifolds ($i = 0, 1, 4, 5$; as $\mathcal{F}_0^0 \equiv \mathcal{F}_0$) by transformations of the groups C_i^0 ($i = 0, 1, 4, 5$; as $C_0^0 \equiv C_0$), respectively. Then*

$$\bar{R} = R - \varphi\big(\Psi_1(S) \circ \varphi\big) + \Psi_2(S) \quad \text{for } i = 0 \ , \tag{3.16}$$

where

$$\begin{aligned} S(Y, Z) = {} & (\nabla_Y \alpha)Z + \alpha(\varphi Y)\alpha(\varphi Z) - \alpha(Y)\alpha(Z) \\ & - \frac{1}{2}\alpha(p)g(\varphi Y \varphi Z) - \frac{1}{2}\alpha(\varphi p)g(Y, \varphi Z) \ ; \end{aligned} \tag{3.17}$$

$$\bar{K} = K - \varphi\big(\Psi_1(S) \circ \varphi\big) + \Psi_2(S) \quad \text{for } i = 1, 4, 5 \ , \tag{3.18}$$

where

$$\begin{aligned} S(Y, Z) = {} & (\nabla_Y \sigma)Z + \sigma(\varphi Y)\sigma(\varphi Z) - \sigma(Y)\sigma(Z) \\ & - \frac{1}{2}\sigma\Big(s + \frac{\varphi Q}{2n}\Big)g(\varphi Y \varphi Z) - \frac{1}{2}\sigma\Big(\varphi s - \frac{\varphi^2 Q}{2n}\Big)g(Y, \varphi Z) \\ & + \frac{1}{4n}\Big\{\big[\theta(Y) + \theta(\xi)\eta(Y)\big]\sigma(\varphi Z) + \big[\theta(Z) + \theta(\xi)\eta(Z)\big]\sigma(\varphi Y) \\ & \quad + \big[\theta^*(Y) + \theta^*(\xi)\eta(Y)\big]\sigma(Z) + \big[\theta^*(Z) + \theta^*(\xi)\eta(Z)\big]\sigma(Y)\Big\} \ , \end{aligned} \tag{3.19}$$

as we specify $\sigma = g(\cdot, s)$ respectively : $\sigma = \frac{1}{2}(\alpha + \beta \circ \varphi)$ for $i = 1$, $\sigma = \beta \circ \varphi$ for $i = 4$, $\sigma = -\alpha \circ \varphi^2$ for $i = 5$.

Theorem B and Theorem 3.3 imply

Corollary 3.3.1. *Let $(M, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_0$- manifold and $(M, \varphi, \xi, \eta, \bar{g})$ be the contactly conformally related $\mathcal{F}_i^0$-manifold to $(M, \varphi, \xi, \eta, g)$ by a transformation of the group C_i^0 ($i = 1, 4, 5$). Then*

$$\bar{K} = R - \varphi\big(\Psi_1(S) \circ \varphi\big) + \Psi_2(S) \quad \text{for } \ i = 1, 4, 5 \ , \tag{3.20}$$

where

$$\begin{aligned} S(Y, Z) = {} & (\nabla_Y \sigma)Z + \sigma(\varphi Y)\sigma(\varphi Z) - \sigma(Y)\sigma(Z) \\ & - \frac{1}{2}\sigma(s)g(\varphi Y \varphi Z) - \frac{1}{2}\sigma(\varphi s)g(Y, \varphi Z) \end{aligned} \tag{3.21}$$

as we specify $\sigma = g(\cdot, s)$ respectively : $\sigma = \frac{1}{2}(\alpha + \beta \circ \varphi)$ for $i = 1$, $\sigma = \beta \circ \varphi$ for $i = 4$, $\sigma = -\alpha \circ \varphi^2$ for $i = 5$.

4. Contactly conformal invariants

Let (M,φ,ξ,η,g) be an arbitrary almost contact manifold with B-metric and $dimM \geq 7$. If K is the Kaehler curvature tensor of the canonical connection D over $T_pM,\ p\in M$ and $\{e_1,\dots,e_{2n+1}\}$ is a basis of T_pM then $\rho(K)$ will stand for the Ricci tensor of K and $\tau(K),\ \tilde{\tau}(K)$ — curvatures of K. The Bochner curvature tensor of φ-holomorphic type $B(K)$ we define on the analogy of (2.14) by

$$
\begin{aligned}
(4.1)\ B(K)(x,y,z,w)=&K(x,y,z,w)\\
&-\frac{1}{2(n-2)}\Big[\Psi_1\big(\rho(K)\big)(\varphi x,\varphi y,\varphi z,\varphi w)-\Psi_2\big(\rho(K)\big)(x,y,z,w)\Big]\\
&+\frac{1}{4(n-1)(n-2)}\Big\{\tau(K)\big[\pi_1(\varphi x,\varphi y,\varphi z,\varphi w)-\pi_2(x,y,z,w)\big]\\
&\qquad+\tilde{\tau}(K)\pi_3(\varphi x,\varphi y,\varphi z,\varphi w)\Big\}
\end{aligned}
$$

Theorem 4.1. *Let (M,φ,ξ,η,g) $(dimM \geq 7)$ be an $\mathcal{F}_i^0$-manifold . Then the Bochner curvature tensor $B(K)$ is a contactly conformal invariant of the group C_i^0 $(i=1,4,5)$.*

Proof. Let $(M,\varphi,\xi,\eta,\bar{g})$ by the contactly conformally related $\mathcal{F}_i^0$-manifold to (M,φ,ξ,η,g) by a transformation c of C_i^0 $(i=1,4,5)$, respectively. If $\bar{D}$ and $\bar{K}$ are the canonical connection and its curvature tensor of $(M,\varphi,\xi,\eta,\bar{g})$, then (3.18) implies

$$
\begin{aligned}
(4.2)\ S(y,z)=&\frac{1}{2(n-2)}\Big[\rho(y,z)-\bar{\rho}(y,z)\Big]\\
&+\frac{1}{8(n-1)(n-2)}\Big[\tau g(\varphi y,\varphi z)+\tilde{\tau}g(y,\varphi z)-\bar{\tau}\bar{g}(\varphi y,\varphi z)+\tilde{\bar{\tau}}\bar{g}(y,\varphi z)\Big].
\end{aligned}
$$

where $\rho,\tau,\tilde{\tau}$ (respectively $\bar{\rho},\bar{\tau},\tilde{\bar{\tau}}$) are the Ricci tensor and the scalar curvatures of K (respectively $\bar{K}$). Substituting S into (3.18) and taking into account (4.1) and (3.8) , we obtain

$$
(4.3)\qquad B(\bar{K})=e^{2u}\big[\cos 2v\ B(K)+\sin 2v\ B(\tilde{K})\big]\ .
$$

Thus, if $B(K)$ and $B(\bar{K})$ are the corresponding tensors of type (1,3), then (4.3) and (1.8) imply $B(K)=B(\bar{K})$, i.e. $B(K)$ is a invariant tensor with respect to the contactly conformal group C_i^0 $(i=1,4,5)$.

Since the group C_0 is a subgroup of each group C_i^0 $(i=1,4,5)$, then Theorem 4.1 implies, that the Bochner curvature tensor $B(K)$ is contactly conformal invariant of the group C_0, too.

According to Theorem B and Theorem 4.1, we obtain

Corollary 4.1.1. *Let $(M,\varphi,\xi,\eta,\bar{g})$ be an $\mathcal{F}_i^0$- manifold $(i=1,4,5)$ contactly conformally equivalent to the $\mathcal{F}_0$- manifold (M,φ,ξ,η,g) by transformation of the group C_i^0 $(i=1,4,5)$. If $B(R)$ and $B(\bar{K})$ are the Bochner curvature tensor of (M,φ,ξ,η,g) and $(M,\varphi,\xi,\eta,\bar{g})$, respectively, then $B(R)=B(\bar{K})$.*

Since the group C_0 is a subgroup of C_i^0 $(i=1,4,5)$ and the class $\mathcal{F}_0$ is contained in the class $\mathcal{F}_i^0$ $(i=1,4,5)$, then from Theorem A and Corollary 4.1.1 we obtain

Corollary 4.1.2. *The Bochner curvature tensor $B(R)$ on an $\mathcal{F}_0$-manifold is a contactly conformal invariant of the group C_0.*

Theorem 4.2. *Let (M,φ,ξ,η,g) be an $\mathcal{F}_i^0$-manifold with Bochner curvature tensor $B(K)$ of D. Then every almost contact manifold with B-metric $(M,\varphi,\xi,\eta,\bar{g})$ contactly conformally equivalent to (M,φ,ξ,η,g) by a transformation of C_{ij}^{*0} $(i,j = 1,4,5;\ i \neq j)$ is an $\mathcal{F}_j^0$-manifold with the same Bochner curvature tensor $B(K)$ of $\bar{D}$.*

Proof. The statement follows from Corollary 4.1.1 .

Let (M,φ,ξ,η,g) and $(M,\varphi,\xi,\eta,\bar{g})$ be conformally related $\mathcal{F}_i^0$-manifolds by transformation $c \in C_i^{0\prime}$ $(i = 4,5)$. According to the definition conditions of $C_4^{0\prime}$ (respectively $C_5^{0\prime}$), we have that $\sigma = dv \circ \varphi = 0$ (respectively $\sigma = -du \circ \varphi^2 = 0$). Therefore, from (3.19) we obtain $S = 0$ for i=4,5.Hence (3.18) takes the shape $\bar{K} = K$. So we ascertain the truthfulness of the following important

Theorem 4.3. *Let (M,φ,ξ,η,g) and $(M,\varphi,\xi,\eta,\bar{g})$ be contactly conformally related $\mathcal{F}_i^0$-manifolds by transformations of the group $C_i^{0\prime}$ $(i = 4,5)$. Then K is a contactly conformal invariant of the group $C_i^{0\prime}$ $(i = 4,5)$.*

Having in mind Theorem B, we obtain

Theorem 4.4. *Let $(M,\varphi,\xi,\eta,\bar{g})$ be an $\mathcal{F}_i^0$-manifold contactly conformally equivalent to the $\mathcal{F}_0$-manifold (M,φ,ξ,η,g) by a transformation of the group $C_i^{0\prime}$ $(i = 4,5)$. If R and $\bar{K}$ are the curvature tensors of (M,φ,ξ,η,g) and $(M,\varphi,\xi,\eta,\bar{g})$, respectively, then $R = \bar{K}$.*

5. The geometric interpretation of the Bochner curvature tensor

A fundamental topic of the metric differential geometry is the geometric characterization of the tensor invariants. In this chapter we consider the geometric interpretation of the received contactly conformal tensor invariants in the previous chapter .

Now we prove the next

Theorem 5.1. *Let $(M,\varphi,\xi,\eta,g)(dim M \geq 9)$ be an $\mathcal{F}_0$-manifold with vanishing Bochner tensor $B(R)$. Then (M,φ,ξ,η,g) is contactly conformally related to an $\mathcal{F}_0$ -manifold (M,φ,ξ,η,g) by a transformation of C_0, so that the Levi-Civita connection $\bar{\nabla}$ of $(M,\varphi,\xi,\eta,\bar{g})$ is flat.*

Proof. Let we have got the transformation
$c : \bar{g}(X,Y){=}e^{2u}\cos 2v\ g(X,Y){+}e^{2u}\sin 2v\ g(X,\varphi Y){+}(1{-}e^{2u}\cos 2v)\eta(X)\eta(Y)$ of C_0, where the unknown functions u and v are a φ-holomorphic pair, i.e. $du = dv \circ \varphi$ and $dv = -du \circ \varphi$. According to Theorem A, as we take an account of (3.16) and (3.17), we ascertain the connection $\bar{\nabla}$ on the $\mathcal{F}_0$-manifold $(M,\varphi,\xi,\eta,\bar{g}) = c(M,\varphi,\xi,\eta,g)$ is flat iff

$$(5.1)\quad (\nabla_Y\alpha)Z+\alpha(\varphi Y)\alpha(\varphi Z)-\alpha(Y)\alpha(Z)-\frac{1}{2}\alpha(p)g(\varphi Y,\varphi Z)-\frac{1}{2}\alpha(\varphi p)g(Y,\varphi Z)$$
$$= \frac{1}{2(n-2)}\rho(Y,Z)+\frac{1}{8(n-1)(n-2)}\Big[\tau g(\varphi Y,\varphi Z)+\tilde{\tau}(Y,\varphi Z)\Big],$$

where $\rho, \tau, \tilde{\tau}$ are the Ricci tensor and the scalar curvatures of the curvature tensor R. Now, we will show $B(R) = 0$ is a integrability condition for the system (5.1). We denote the right hand side of (5.1) by $L(Y,Z)$. Since the curvature tensor R on an $\mathcal{F}_0$-manifold is a Kaehler tensor, then we have $\rho(\varphi Y, \varphi Z) = -\rho(Y,Z)$. Therefore $L(Y,Z) = L(Z,Y) = -L(\varphi Y, \varphi Z)$. Applying the Ricci identity $(\nabla_X \nabla_Y \alpha)Z - (\nabla_Y \nabla_X \alpha)Z = -\alpha(R(X,Y)Z)$ for the 1-form α in the left hand side of (5.1) and using $B(R) = 0$, we find that the system is integrable iff

$$(5.2) \qquad (\nabla_X L)(Y,Z) = (\nabla_Y L)(X,Z)\ .$$

To prove (5.2), we have in mind the fact that if $B(R) = 0$, then $R = \varphi(\Psi_1(L) \circ \varphi) - \Psi_2(L)$, where

$$(5.3) \qquad L(Y,Z) = \frac{1}{2(n-2)}\rho(Y,Z) + \frac{1}{8(n-1)(n-2)}\big[\tau g(\varphi Y, \varphi Z) + \tilde{\tau}(Y, \varphi Z)\big].$$

We enforce the second Bianchi identity for R. After a contraction we obtain

$$(2n-5)\Big[(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)\Big] + \Big[(\nabla_{\varphi X} L)(\varphi Y, Z) - (\nabla_{\varphi Y} L)(\varphi X, Z)\Big] = 0$$

and by the substitution $X \to \varphi X,\ Y \to \varphi Y$ in the previous equality we have

$$(2n-5)\Big[(\nabla_{\varphi X} L)(\varphi Y, Z) - (\nabla_{\varphi Y} L)(\varphi X, Z)\Big] + \Big[(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)\Big] = 0$$

From the last two equalities it follows that

$$(n-2)(n-3)\Big[(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)\Big] = 0\ .$$

Hence (5.2) is a consequence of $B(R) = 0$ and the system (5.1) is integrable for $n > 3$. Then the contactly conformal transformation c $\big(u$ is a solution of (5.1) and v — of $dv = du \circ \varphi\big)$ gives rise to an $\mathcal{F}_0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ with flat Levi-Civita connection $\bar{\nabla}$.

Thus, we establish, that if $dimM \geq 9$ the connection $\bar{\nabla}$ is flat iff $B(R) = 0$. If $dimM = 7$, then $\bar{\nabla}$ is a flat connection, when $B(R) = 0$ and (5.2) is satisfied, too.

Theorem 5.2. *Let $(M, \varphi, \xi, \eta, g)(dimM \geq 9)$ be an $\mathcal{F}_0$-manifold with vanishing Bochner curvarure tensor $B(R)$. Then $(M, \varphi, \xi, \eta, g)$ is contactly conformally related to an $\mathcal{F}_1^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ by a transformation c of $C_1^{0\prime}$, so that the canonical connection $\bar{D}$ of $(M, \varphi, \xi, \eta, \bar{g})$ is flat .*

Proof. Let $\bar{g} = e^{2u}g + (1 - e^{2u})\eta \otimes \eta$ be a transformation of $C_1^{0\prime}$, where u is an unknown differentiable function on M. Using Theorem B, the equality (3.20) and the equality (3.21) for $i = 1$ in the case $v = 0$, we obtain $\bar{D}$ on the $\mathcal{F}_i^0$-manofold $(M, \varphi, \xi, \eta, \bar{g})$ is a flat connection iff

$$(5.4) \quad (\nabla_Y \alpha)Z + \frac{1}{2}\alpha(\varphi Y)\alpha(\varphi Z) - \frac{1}{2}\alpha(Y)\alpha(Z) - \frac{1}{4}\alpha(p)g(\varphi Y, \varphi Z) - \frac{1}{4}\alpha(\varphi p)g(Y, \varphi Z)$$
$$= \frac{1}{n-2}\rho(Y,Z) + \frac{1}{4(n-1)(n-2)}\Big[\tau g(\varphi Y, \varphi Z) + \tilde{\tau}(Y, \varphi Z)\Big]\ .$$

On the analogy of the proof of Theorem 5.1, we assert $B(R) = 0$ is a integrability condition of the system (5.4) for $dimM \geq 9$. Then, according to Theorem B, the contactly conformal transformation $\bar{g} = e^{2u}g + (1 - e^{2u})\eta\otimes\eta$ $\big(u$ — a solution of (5.4)$\big)$ gives rise to an $\mathcal{F}_i^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ with flat canonical connection $\bar{D}$.

Thus, we ascertain the fact that if $dimM \geq 9$ the canonical connection $\bar{D}$ is flat iff $B(R) = 0$. In case of $dimM = 7$, $\bar{D}$ is flat when $B(R) = 0$ and the condition $(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z) = 0$ is satisfied, too $\big(L(Y,Z)$ denotes the right hand side of (5.4) $\big)$.

Let $c_1 \in C_0$ and $c_2 \in C_1^{0\prime}$ be transformations, determined by the functions (u', v) and u'', respectively. Then the transformation $c = c_2 \circ c_1$ belongs to C_1^0 and it is determined by the functions $(u = u' + u'', v)$. It is obvious $c \notin C_0$ and $c \notin C_1^{0\prime}$.

Theorem 5.3. *Let $(M, \varphi, \xi, \eta, g)(dimM \geq 9)$ be an $\mathcal{F}_0$ -manifold with vanishing Bochner curvature tensor $B(R)$. Then $(M, \varphi, \xi, \eta, g)$ is contactly conformally related to an $\mathcal{F}_1^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ by transformation c of C_1^0 $(c \notin C_1^{0\prime}$ and $c \notin C_0$) , so that the canonical connection $\bar{D}$ of $(M, \varphi, \xi, \eta, \bar{g})$ is flat .*

Proof. Let $(M, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_0$-manifold and
$c_1 :\ g'(X,Y) = e^{2u'}\cos 2v\ g(X,Y) + e^{2u'}\sin 2v\ g(X,\varphi Y) + (1 - e^{2u'}\cos 2v)\eta(X)\eta(Y)$
be a transformation of C_0. Then Corollary 4.1.2 implies that the manifold $(M, \varphi, \xi, \eta, g') = c_1(M, \varphi, \xi, \eta, g)$ is an $\mathcal{F}_0$-manifold, so that $B(R) = B(R')$, where R' is the curvature tensor of ∇' of g'.

Let us consider the transformation $c_2 : \bar{g} = e^{2u''}g' + (1 - e^{2u''})\eta\otimes\eta$ belonging to $C_1^{0\prime}$. Then the transformation $c = c_2 \circ c_1$, determinated by the functions $(u = u' + u'', v)$, belongs to C_1^0 and according to Theorem B this transformation generates an $\mathcal{F}_1^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ contactly conformally related to $(M, \varphi, \xi, \eta, g)$. The condition $B(R) = 0$ implies $B(R') = 0$ and according to Theorem 5.2, we obtain $\bar{K} = 0$.

So we ascertain the existence of φ-pluriharmonic functions u and v determining transformation $c \in C_1^0$, which arises an $\mathcal{F}_1^0$- manifold with flat canonical connection iff $B(R) = 0,\ dimM \geq 9$.

Theorem 5.4. *Let $(M, \varphi, \xi, \eta, g)(dimM \geq 9)$ be an $\mathcal{F}_0$-manifold with vanishing Bochner curvature tensor $B(R)$. Then $(M, \varphi, \xi, \eta, g)$ is contactly conformally related to an $\mathcal{F}_i^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ by a transformation of C_i^0 $(i = 4, 5)$, so that the canonical connection $\bar{D}$ of $(M, \varphi, \xi, \eta, \bar{g})$ is flat .*

Proof. Let $(M, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_0$-manifold with $B(R) = 0$. According to Theorem 5.1, we have the functions u' and v' such that the transformation $c_1 :\ g'(X,Y) = e^{2u'}\cos 2v'\ g(X,Y) + e^{2u'}\ \sin 2v' g(X,\varphi Y) + (1 - e^{2u'}\cos 2v')\eta(X)\eta(Y)$ of C_0 , gives rise to an $\mathcal{F}_0$-manifold $(M, \varphi, \xi, \eta, g')$ with flat connection ∇' of g', i.e. $R' = 0$. Let us consider the transformation $c_2 :\ \bar{g}(X,Y) = \cos 2v''\ g'(X,Y) + \sin 2v''\ g'(X,\varphi Y) + (1 - \cos 2v'')\eta(X)\eta(Y)$ of $C_4^{0\prime}$ which according to Theorem 4.4 gives rise to an $\mathcal{F}_4^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$, so that $R' = \bar{K} = 0$. Then the transformation $c = c_2 \circ c_1$ belongs to C_4^0 and it is determinated by the functions $u = u'$ and $v = v' + v''$. Thus the transformation c gives rise an $\mathcal{F}_4^0$-manifold $(M, \varphi, \xi, \eta, \bar{g}) = c(M, \varphi, \xi, \eta, g)$ with flat canonical connection $\bar{D}$ for $dimM \geq 9$.

In the similar way, if $c_2 : \bar{g} = e^{2u''}g' + (1 - e^{2u''})\eta \otimes \eta$ is a transformation of $C_5^{0\prime}$, then according to Theorem 4.4, c_2 gives rise to an $\mathcal{F}_5^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$, so that $R' = \bar{K} = 0$. Then the functions $u = u' + u''$ and $v = v'$ set the transformation $c = c_2 \circ c_1$ of C_5^0 and c gives rise to an $\mathcal{F}_5^0$-manifold $(M, \varphi, \xi, \eta, \bar{g}) = c(M, \varphi, \xi, \eta, g)$ with flat canonical connection $\bar{D}$.

Having in mind the invariability of the Bochner curvatures tensor $B(K)$ on an $\mathcal{F}_i^0$-manifold with respect to the group C_i^0 $(i = 1, 4, 5)$ the contactly conformal equivalence of $\mathcal{F}_0$ and $\mathcal{F}_i^0$ $(i = 1, 4, 5)$ and Theorems 4.1, 4.3, 4.4, 5.1, 5.2, 5.3, 5.4, we get the following main

Theorem 5.5. *Let $(M, \varphi, \xi, \eta, g)(dimM \geq 9)$ be an $\mathcal{F}_i^0$-manifold $(i = 1, 4, 5)$ with vanishing Bochner curvature tensor $B(K)$. Then $(M, \varphi, \xi, \eta, g)$ is contactly conformally related to an $\mathcal{F}_i^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ by a transformation of C_i^0 $(i = 1, 4, 5)$, so that the canonical connection $\bar{D}$ of $(M, \varphi, \xi, \eta, \bar{g})$ is flat.*

Theorem 5.6. *Let $(M, \varphi, \xi, \eta, g)(dimM \geq 9)$ be an $\mathcal{F}_i^0$ -manifold $(i = 1, 4, 5)$ with vanishing Bochner curvatures tensor $B(K)$. Then $(M, \varphi, \xi, \eta, g)$ is contactly conformally related to an $\mathcal{F}_j^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ by a transformation of C_{ij}^{*0} $(i, j = 1, 4, 5; i \neq j)$, so that the canonical connection $\bar{D}$ of $(M, \varphi, \xi, \eta, \bar{g})$ is flat.*

Proof. Since $B(K) = 0$ for the $\mathcal{F}_i^0$-manifold $(M, \varphi, \xi, \eta, g)$, according to Corollary 4.1.1, $(M, \varphi, \xi, \eta, g)$ is contactly conformally equivalent to an $\mathcal{F}_0$-manifold $(M, \varphi, \xi, \eta, g')$ with respect to $c_1 \in C_i^0$ $(i = 1, 4, 5)$ and $B(R) = B(K) = 0$. From Theorem 5.3 (respectively Theorem 5.4) it follows that $(M, \varphi, \xi, \eta, g')$ is contactly conformally related to an $\mathcal{F}_j^0$-manifold $(M, \varphi, \xi, \eta, \bar{g})$ with respect to $c_2 \in C_j^0$ $(j = 1, 4, 5; j \neq i)$ and the canonical connection $\bar{D}$ on $(M, \varphi, \xi, \eta, \bar{g})$ is flat.

Since the transformation $c = c_2 \circ c_1$ belongs to the group C_{ij}^{*0}, then it follows $\bar{D}$ on the $\mathcal{F}_j^0$-manifold $(M, \varphi, \xi, \eta, \bar{g}) = c(M, \varphi, \xi, \eta, g)$ is flat iff $B(K) = 0$ for $dimM \geq 9$.

REFERENCES

1.S.Kobayashi,K.Nomizu. Foundations of differential geometry.Vol I.Wiley.New York.1963.

2.D.E.Blair. Contact manifolds in Riemannian geometry.Lecture Notes in Math. 509.1976.Springer Verlag.

3.G.Ganchev,K.Gribachev,V.Mihova. B-connections and their conformal invariants on conformally Kaehler manifolds with B-metric.Publ.de L'inst.Math.42(56). 1987. 107-121.

4.G.Ganchev,V.Michova,K.Gribachev. Almost contact manifolds with B-metric (to appear).

5.M.Manev,K.Gribachev. Contactly conformal transformations on almost contact manifolds with B-metric(to appear).

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