CONTACTLY CONFORMAL TRANSFORMATIONS OF GENERAL TYPE OF ALMOST CONTACT MANIFOLDS WITH *B*-METRIC. APPLICATIONS

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Contactly conformal transformations of general type of almost contact manifolds with *B*-metric are introduced. Some special subgroups of such transformations are given. Conformally invariant tensors with respect to considered groups of transformations of some basic classes of manifolds are found. An example of a manifold, belonging one of basic classes, is constructed.

1. Introduction

On an almost complex manifold (M,J) there can be considered a metric g, compatible with the almost complex structure J, which indices an antiisometry in each tangent fibre. Then (M,J,g) is said to have a structure of an almost complex Riemannian manifold (almost complex manifold with *B*-metric). The conformal geometry of these manifolds is studied in [2].

Geometry of almost contact manifolds with *B*-metric is a natural extension of geometry of almost complex manifolds with B-metric to the odd dimensional case. The almost contact manifolds with *B*-metric are considered in [3]. Contactly conformal transformations and their invariants are found in [4,5]. Contactly conformally equivalent classes to some basic classes are got there.

A group of contactly conformal transformations of more general type are introduced in this paper, which allow us to study the conformal geometry of one of the other basic classes. Contactly conformal equivalence of this class to the class of manifolds with covariantly constant structure tensors gives us the possibility to construct an example of a manifold of the considered class.

The conformal transformations introduced in this paper are an analogue of the conformal transformations of almost contact metric manifolds, investigated in [6].

2. Preliminaries

Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional almost contact manifold with *B*-metric, i.e. (φ,ξ,η,g) is an almost contact structure [1] determined by a tensor field φ of type (1,1), a vector field ξ and 1-form η, and g is a metric on M so that:

(2.1) $\varphi^2 = -id + \eta\otimes\xi\ ;\ \eta(\xi) = 1\ ;\ g(\varphi X,\varphi Y) = -g(X,Y)+\eta(X)\eta(Y),$

where $X,Y \in \mathfrak{X}M$ - the Lie algebra of differentiable vector fields on M.

The associated with g *B*-metric $\tilde{g}$ on the manifold is given by

$\tilde{g}(X,Y) = g(X,\varphi Y)+\eta(X)\eta(Y).$

Both metrics are necessarily of signature $(n+1,n)$ [3].

Further, capitals will stand for the elements of $\mathfrak{X}M$ and small letters - for arbitrary vectors in the tangential space T_pM to M at an arbitrary point p in M.

The Levi-Civita connection of g will be denoted by ∇. The tensor F of type (0,3) on T_pM is defined by $F(X,Y,Z) = g((\nabla_X \varphi)Y,Z)$ and it has the following properties:

$$F(x,y,z) = F(x,z,y);$$

(2.2)

$$F(x,\varphi y,\varphi z) = F(x,y,z) - \eta(y)\, F(x,\xi,z) - \eta(z)\, F(x,y,\xi).$$

The following 1-forms are associated with F:

(2.3) $\theta(x) = g^{ij}\, F(e_i,e_j,x),\ \ \theta^*(x) = g^{ij}\, F(e_i,\varphi e_j,x),\ \ \omega(x) = F(\xi,\xi,x),$

where $\{e_i,\xi\}$ $(i = 1,2,...,2n)$ is a basis of T_pM, and (g^{ij}) is the inverse matrix of (g_{ij}).

A classification of the almost contact manifolds with B-metric with respect to the tensor F is given in [3], where the basic classes $\mathcal{F}_i$ $(i = 1,2,...,11)$ of almost contact manifolds with B-metric are defined. The class $\mathcal{F}_0$ is defined by the condition F=0. This special class belongs to each of the basic classes.

Classes $\mathcal{F}_0$ and $\mathcal{F}_{11}$ are subject of study in this paper. According to [3] the characteristic conditions of these classes are:

(2.4) $\mathcal{F}_0$: $\nabla\varphi = \nabla\xi = \nabla\eta = 0$;

(2.5) $\mathcal{F}_{11}$: $F(x,y,z) = \eta(x)\,\{\eta(y)\omega(z) + \eta(z)\omega(y)\}$

Let us point out that the 1-form ω isn't zero only on manifolds of the class $\mathcal{F}_{11}$ or its direct sums with others classes.

We shall say that M is an $\mathcal{F}_{11}^{\,0}$-manifold if M belongs to $\mathcal{F}_{11}$ and the 1-form $\tilde{\omega} = \omega\circ\varphi$ is closed.

Since ∇ is a symmetric linear connection, then $\tilde{\omega}$ is closed iff $(\nabla_x\tilde{\omega})y = (\nabla_y\tilde{\omega})x$. Having in mind the condition (2.5), we obtain

Lemma 2.1 The 1-form $\omega\circ\varphi$ is closed iff $(\nabla_x\omega)\varphi y = (\nabla_y\omega)\varphi x$ for arbitrary $x,y \in T_pM$.

3. Contactly conformal transformations of general type

Definition. Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric. The transformation f: $f(\varphi,\xi,\eta,g) = (\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, such that

$$\bar{\varphi} = \varphi\ ;\ \bar{\xi} = e^{-w}\,\xi\ ;\ \bar{\eta} = e^{w}\,\eta\ ;$$

(3.1)

$$\bar{g} = e^{2u}\cos 2v\, g + e^{2u}\sin 2v\, \tilde{g} + (e^{2w} - e^{2u}\cos 2v - e^{2u}\sin 2v)\, \eta\otimes\eta,\ (u,v,w \in \mathcal{F}M),$$

will be called *a contactly conformal transformation of general type* of the structure (φ,ξ,η,g).

Let us note, that the contactly conformal transformation c, introduced in [4], where $c(\varphi,\xi,\eta,g) = (\varphi,\xi,\eta,\bar{g})$, is of general type at $w = 0$.

Since $\bar{g}(\bar{\xi},\bar{\xi}) = 1$, $\bar{g}(X,\bar{\xi}) = \bar{\eta}(X)$, and $\bar{g}(\varphi X,\varphi Y) = -\bar{g}(X,Y) + \bar{\eta}(X)\bar{\eta}(Y)$, then the structure $(\varphi,\bar{\xi},\bar{\eta},\bar{g})$ is an almost contact structure with B-metric on M, too.

Lemma 3.1. The contactly conformal transformations of general type on an almost contact manifold with B-metric form a group.

This group will be denoted by G.

It's clear that the group of the contactly conformal transformations C, considered in [4], is a subgroup of G.

From [4] it is clear that there exists a subgroup of C whit respect to which the class $\mathcal{F}_{11}$ is closed, but it is impossible to find such a subgroup of C, with respect to which the classes $\mathcal{F}_0$ and $\mathcal{F}_{11}$ are conformally equivalent. The introduced group here gives us this possibility. It allows us to construct an example of an $\mathcal{F}_{11}$-manifold.

Lemma 3.2. Let (M,φ,ξ,η,g) and $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ be contactly conformally related by a transformation of G, while θ, θ^*, ω and $\overline{\theta}, \overline{\theta}^*, \overline{\omega}$ are the corresponding 1-forms associated with F and $\overline{F}$. Then

$$
\begin{aligned}
&2\overline{F}(x,y,z) = 2e^{2u}\cos 2v\, F(x,y,z)\\
&+ e^{2u}\sin 2v\,[\eta(z)\,F(x,\varphi y,\xi) + \eta(y)\,F(x,\varphi z,\xi)\\
&+ F(\varphi y,z,x) - F(y,\varphi z,x) + F(\varphi z,x,y) - F(z,x,\varphi y)]\\
(3.2)\quad &+ (e^{2w} - e^{2u}\cos 2v)\,\{\eta(x)\,[F(y,z,\xi) + F(\varphi z,\varphi y,\xi) + F(z,y,\xi) + F(\varphi y,\varphi z,\xi)]\\
&+ \eta(y)\,[F(x,z,\xi) + F(\varphi z,\varphi x,\xi)] + \eta(z)\,[F(x,y,\xi) + F(\varphi y,\varphi x,\xi)]\}\\
&- \lambda(z)\,g(\varphi x,\varphi y) - \lambda(y)\,g(\varphi x,\varphi z) - \tilde{\lambda}(z)\,g(x,\varphi y) - \tilde{\lambda}(y)\,g(x,\varphi z)\\
&+ 2e^{2w}\,\eta(x)[\eta(y)dw(\varphi z) + \eta(z)dw(\varphi y)],\\
&(\lambda = d(e^{2u}\cos 2v)\circ\varphi + d(e^{2u}\sin 2v),\ \tilde{\lambda} = d(e^{2u}\cos 2v) - d(e^{2u}\sin 2v)\circ\varphi);
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\theta} = \theta + 2n[du\circ\varphi + dv],\\
(3.3)\quad &\overline{\theta}^* = \theta^* + 2n[du - dv\circ\varphi],\\
&\overline{\omega} = \omega + dw\circ\varphi.
\end{aligned}
$$

Proof. The equality (3.2) is obtained from the well known conditions [7] for the Levi-Civita connections ∇ and $\overline{\nabla}$, corresponding to g and $\overline{g}$, using (3.1) and (2.2). By (3.2), having in mind the definition equalities (2.3) for the associated 1-forms to F and $\overline{F}$, respectively, we get (3.3).

We define some subgroups of G using the following conditions for $u, v, w \in \mathcal{F}M$:

$$
\begin{aligned}
&G_0 = \{f \in G \mid du\circ\varphi = dv\circ\varphi^2,\ du(\xi) = dv(\xi) = dw\circ\varphi = 0\},\\
(3.4)\quad &G_{11} = \{f \in G \mid du\circ\varphi = dv\circ\varphi^2,\ du(\xi) = dv(\xi) = 0\},\\
&G_{11}^0 = \{f \in G_{11} \mid d(dw(\xi))\circ\varphi = 0\},\\
&G_{11}^{0'} = \{f \in G_{11}^0 \mid du = 0\}.
\end{aligned}
$$

From Theorem 3.2, taking into account the characteristic conditions (2.4) and (2.5) for $\mathcal{F}_0$ and $\mathcal{F}_{11}$, Lemma 2.1, as well as the definition conditions (3.4) for G_0 and G_{11}, we establish the truthfulness of the following

Proposition 3.3. The class $\mathcal{F}_i$ (respectively $\mathcal{F}_{11}{}^0$) is closed with respect to the transformations of the group $G_i (i = 0,11)$(respectively G_{11}^0), which is the maximal such subgroup of G (respectively G_{11}) and for the corresponding 1-forms are valid the equalities:

a) $\overline{\theta} = \theta = 0,\ \overline{\theta}^* = \theta^* = 0,\ \overline{\omega} = \omega = 0,$ for $i = 0$;

b) $\overline{\theta} = \theta = 0,\ \overline{\theta}^* = \theta^* = 0,\ \overline{\omega} = \omega + dw\circ\varphi,$ for $i = 11$;

Since the class $\mathcal{F}_0$ is contained in $\mathcal{F}_{11}{}^0$, then Proposition 3.3 implies $G_{11}^0(\mathcal{F}_0) \subset \mathcal{F}_{11}{}^0$.

The inverse inclusion is also valid. Thus we obtain a characterization of the class $\mathcal{F}_{11}{}^{0}$ stated in the next

Theorem 3.4. The class $\mathcal{F}_{11}{}^{0}$ is the class of manifolds, which are equivalent by the transformations of the group G_{11}^{0} to the $\mathcal{F}_0$-manifolds, i.e. $\mathcal{F}_{11}{}^{0}=G_{11}^{0}(\mathcal{F}_0)$.

Proof. It is sufficient to prove that $\mathcal{F}_{11}{}^{0}\subset G_{11}^{0}(\mathcal{F}_0)$. Let $(M,\varphi,\xi,\eta,g)\in\mathcal{F}_{11}{}^{0}$, i.e. $\omega_{\circ}\varphi$ is a closed 1-form on M. We consider the equation

(3.5) $\mathrm{d}w'=\omega_{\circ}\varphi$ for $w'\in\mathcal{F}M$,

whence we have

(3.6) $\mathrm{d}w'(\xi)=0$

and

(3.7) $\mathrm{d}w'_{\circ}\varphi=-\omega$.

Solving locally the equation (3.5), we find the function w', satisfying the conditions (3.6) and (3.7).

Let $f'\in G$, determined by the triad of functions (u' = const, v' = const, w'). According to $\mathrm{d}u'=\mathrm{d}v'=0$, the equalities (3.6) and (3.7), the transformation f' belongs to $G_{11}^{0'}$, but $f'\notin G_0$. From the inclusion $G_{11}^{0'}\subset G_{11}^{0}$, the equality (3.7) and Proposition 3.3 it follows that the manifold $(M,\varphi,\xi',\eta',g')=f'(M,\varphi,\xi,\eta,g)$ belongs to $\mathcal{F}_0$.

Let $f''\in G_0$ be determined by the triad of functions (u'',v'',w''). Proposition 3.3 implies that $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})=f''(M,\varphi,\xi',\eta',g')$ belongs to $\mathcal{F}_0$. Since $G_0\subset G_{11}^{0}$ and $f'\notin G_0$, then the transformation $f=f''_{\circ}f'$ belongs to G_{11}^{0}, but $f\notin G_{11}^{0'}$, i.e. $G_{11}^{0}(\mathcal{F}_{11}{}^{0})\subset\mathcal{F}_0$. Hence, having in mind that G_{11}^{0} is a group, we conclude that $\mathcal{F}_{11}{}^{0}\subset G_{11}^{0}(\mathcal{F}_0)$.

4. Contactly conformal invariants on an $\mathcal{F}_{11}$-manifold

Definition [5]. A linear connection on an almost contact manifold with B-metric is said to be a canonical connection if the structure tensors φ, ξ, η and g on M are covariantly constant with respect to this connection.

A canonical connection on the considered manifolds is introduced in [5] by

(4.1) $$D_XY=\nabla_XY+\frac{1}{2}\{(\nabla_X\varphi)\varphi Y+(\nabla_X\eta)Y.\xi\}-\eta(Y)\nabla_X\xi.$$

If $M\in\mathcal{F}_{11}$, then according to (2.5) we have

(4.2) $D_XY=\nabla_XY+\eta(X)[\omega(\varphi Y)\xi-\eta(Y).\varphi\Omega]$,

where Ω is the corresponding vector of the 1-form ω. Let us remark the vector Ω is the important vector $-\varphi\nabla_{\xi}\xi$ on the manifold.

Let K be the curvature tensor field of type (1,3) of D, i.e.

(4.3) $K(X,Y)Z=D_XD_YZ-D_YD_XZ-D_{[X,Y]}Z,$

and the corresponding tensor field of type (0,4) is given by

(4.4) $K(X,Y,Z,T)=g(K(X,Y)Z,T)$.

In [5] it is shown that K is a Kaehler tensor, i.e. $K(X,Y,Z,T)=-K(X,Y,\varphi Z,\varphi T)$, in some special subclasses of the basic classes. Then the tensor $\widetilde{K}(X,Y,Z,T) = K(X,Y,Z,\varphi T)$ is also a Kaehler tensor associated with K.

Let S be a tensor of type (0,2). We define the following fundamental tensor for S:

(4.5) $\psi_4(S)(X,Y,Z,T) = \eta(Y)\,\eta(Z)S(X,T) - \eta(X)\eta(Z)S(Y,T) + $
$+ \eta(X)\eta(T)S(Y,Z) - \eta(Y)\eta(T)S(X,Z).$

It is easy to ascertain that $\psi_4(S)$ is a curvature-like tensor [2] iff S is symmetric.

Let M be an $\mathcal{F}_{11}{}^{0}$- manifold. Then from (4.3) and (4.4), using (4.2), we obtain

(4.6) $K(X,Y,Z,T) = R(X,Y,Z,T) - \psi_4(S)(X,Y,Z,T),$

where $S(Y,Z) = (\nabla_Y\omega)\,\varphi Z - \omega(\varphi Y)\,\omega(\varphi Z)$. Lemma 2.1 and the equality (4.6) imply that K is a curvature-like tensor. Since D is a cannonical connection, then K is a Kaehler tensor.

Lemma 4.1. Let (M,φ,ξ,η,g) and $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ be contactly conformally related $\mathcal{F}_{11}$- manifolds by a transformation from the group G_{11}. Then

(4.7) $\overline{D}_XY = D_XY - \alpha(X)\varphi^2Y - \alpha(Y)\varphi^2X - \alpha(\varphi X)\varphi Y- \alpha(\varphi Y)\varphi X + g(\varphi X,\varphi Y)p$
$+ g(X,\varphi Y)\varphi p + \gamma(X)\eta(Y)\xi,$

where D and $\overline{D}$ are the corresponding canonical connections to (M,φ,ξ,η,g) and $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$, and $\alpha = du = g(\bullet,p)$, $\gamma = dw$.

Proof. Let $\overline{\nabla}$ (respectively ∇) be the Levi-Civita connection of the metric $\overline{g}$ (respectively g). Using the well known Levi-Civita condition for ∇ and $\overline{\nabla}$, we get the relation between them. Having in mind (4.2) for D and ∇, as well as the respective to (4.2) equality for $\overline{D}$ and $\overline{\nabla}$, we obtain the equality (4.7).

Let (M,φ,ξ,η,g) and $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ be contactly conformally related $\mathcal{F}_{11}{}^{0}$-manifolds by a transformation of the group G^{0}_{11}. Using the last proposition and Lemma 2.1, we get

(4.8) $\overline{K} = K - \varphi(\psi_1(L)_\circ\varphi) + \psi_2(L),$

where

(4.9) $L(Y,Z) = (\nabla_Y\alpha)Z + \alpha(\varphi Y)\,\alpha(\varphi Z) - \alpha(Y)\,\alpha(Z) - \frac{1}{2}\,\alpha(p)\,g(\varphi Y,\ \varphi Z)$
$-\frac{1}{2}\,\alpha(\varphi p)\,g(Y,\ \varphi Z) + \eta(Y)\,\eta(Z)\,\alpha(\varphi\Omega).$

Let (M,φ,ξ,η,g) be an $\mathcal{F}_0$-manifold and $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ be an $\mathcal{F}_{11}{}^{0}$-manifold and both are contactly conformally related by a transformation of the group G^{0}_{11}. Then Theorem 3.4 and the equality (4.8) imply immediately

(4.10) $\overline{K} = R - \varphi(\psi_1(L)_\circ\varphi) + \psi_2(L),$

where $L(Y,Z) = (\nabla_Y\alpha)Z + \alpha(\varphi Y)\,\alpha(\varphi Z) - \alpha(Y)\,\alpha(Z) - \frac{1}{2}\,\alpha(p)\,g(\varphi Y,\ \varphi Z) - \frac{1}{2}\,\alpha(\varphi p)\,g(Y,\ \varphi Z).$

Let us consider an arbitrary almost contact manifold with B-metric (M,φ,ξ,η,g) with $\dim M \geq 7$ and let $B(K)$ be the Bochner curvature tensor of φ-holomorphic type introduced in [5] by analogy of the Bochner curvature tensor $B(R)$ known from [2], i.e.

(4.11) $$B(K) = K - \frac{1}{2(n-2)}[\psi_1(\rho(K))_o\varphi - \psi_2(\rho(K))] + \frac{1}{4(n-1)(n-2)}\{\tau(K)[\pi_{1o}\varphi - \pi_2] + \tilde{\tau}(K)\,\pi_{3o}\varphi\}.$$

Theorem 4.2. Let (M,φ,ξ,η,g) (dim $M \geq 7$) be an $\mathcal{F}_{11}{}^{0}$-manifold. Then the Bochner curvature tensor $B(K)$ is a contactly conformal invariant of the group G_{11}^{0}.

Proof. Let $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ be the contactly conformally related $\mathcal{F}_{11}{}^{0}$-manifold to (M,φ,ξ,η,g) by a transformation f of G_{11}^{0}. If $\bar{D}$ and $\bar{K}$ are the canonical connection and its curvature tensor on $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$, respectively, then (4.8) implies

(4.12) $$L(y,z) = \frac{1}{2(n-2)}[\rho(y,z) - \bar{\rho}(y,z)] + \frac{1}{8(n-1)(n-2)}[\tau g(\varphi y,\varphi z) + \tilde{\tau}\, g(y,\varphi z) - \bar{\tau}\,\bar{g}(\varphi y,\varphi z) - \tilde{\bar{\tau}}\,\bar{g}(y,\varphi z)],$$

where ρ, τ, $\tilde{\tau}$ (respectively $\bar{\rho}$, $\bar{\tau}$, $\tilde{\bar{\tau}}$) are the Ricci tensor and the scalar curvatures of K (respectively $\bar{K}$).

Substituting L into (4.8) and taking into account (4.11), we obtain

(4.13) $$B(\bar{K}) = e^{2u}[\cos 2v\, B(K) + \sin 2v\, B(\tilde{K})].$$

Let $B(K)$ and $B(\bar{K})$ be the corresponding tensors of type (1,3). Then (4.13) and (3.1) imply $B(K) = B(\bar{K})$, i.e. $B(K)$ is an invariant tensor with respect to the group G_{11}^{0}.

Since the group G_0 is a subgroup of G_{11}^{0},then from Theorem 4.2 follows that the Bochner curvature tensor $B(K)$ is a contactly conformal invariant of the group G_0, too.

According to Theorem 3.3 and Theorem 4.2 we get

Corollary 4.2.1. Let (M,φ,ξ,η,g) be an $\mathcal{F}_0$-manifold and $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ be and $\mathcal{F}_{11}{}^{0}$-manifold, which are contactly conformally related by a transformation of the group G_{11}^{0}. If $B(R)$ and $B(\bar{K})$ are the Bochner curvature tensors on (M,φ,ξ,η,g) and $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$, respectively, then $B(R)= B(\bar{K})$.

Since the group G_0 is a subgroup of G_{11}^{0} and the class $\mathcal{F}_0$ is contained in the class $\mathcal{F}_{11}{}^{0}$, then from Proposition 3.3 and Corollary 4.2.1 we obtain that the Bocher curvature tensor $B(R)$ on an $\mathcal{F}_0$-manifold is a contactly conformal invariant of the group G_0..

Let (M,φ,ξ,η,g) and $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ be $\mathcal{F}_{11}{}^{0}$-manifolds contactly conformally related by a transformation of $G_{11}^{0'}$. According to the definition conditions of $G_{11}^{0'}$ $\alpha = du = 0$ from (4.9) we get L=0. Hence (4.8) implies $\bar{K}$=K. So we ascertain the truthfulness of the following important

Theorem 4.3. Let (M,φ,ξ,η,g) and $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ be contactly conformally related $\mathcal{F}_{11}{}^{0}$-manifold by a transformation of the group $G_{11}^{0'}$. Then K is a contactly conformal invariant of the group $G_{11}^{0'}$.

Having in mind Theorem 3.4, we obtain

Corollary 4.3.1. Let (M,φ,ξ,η,g) be an $\mathcal{F}_0$-manifold and $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ be an $\mathcal{F}_{11}{}^0$-manifold, and both are contactly conformally related with respect to a transformation of the group $G_{11}^{0'}$, and R and $\overline{K}$ are the corresponding curvature tensors of ∇ and $\overline{D}$. Then $R=\overline{K}$.

5. The geometric interpretation of the Bochner curvature tensor on an $\mathcal{F}_{11}$-manifold

Let us quote the familiar

Theorem A[5]. Let (M,φ,ξ,η,g) (dim $M \geq 9$) be an $\mathcal{F}_0$-manifold with vanishing Bochner tensor $B(R)$. Then (M,φ,ξ,η,g) is contactly conformally related to an $\mathcal{F}_0$-manifold $(M,\varphi,\xi,\eta,\overline{g})$ by a transformation of C_0, so that the Levi-Civita connection $\overline{\nabla}$ of $(M,\varphi,\xi,\eta,\overline{g})$ is flat.

The analogous theorem for contactly conformally related manifolds by a transformation of the group G_0 is valid, too, because the generalization of C_0 to G_0 does not essentially affect the proof, namely

Theorem A'. Let (M,φ,ξ,η,g) (dim $M \geq 9$) be an $\mathcal{F}_0$-manifold with vanishing Bochner curvature tensor $B(R)$. Then (M,φ,ξ,η,g) is contactly conformally related to an $\mathcal{F}_0$-manifold $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ by a transformation of G_0, so that the Levi-Civita $\overline{\nabla}$ on $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ is flat.

We will prove the following theorem, more general than Theorem A'.

Theorem 5.1. Let (M,φ,ξ,η,g) (dim $M \geq 9$) be an $\mathcal{F}_0$-manifold with vanishing Bochner curvature tensor $B(R)$. Then (M,φ,ξ,η,g) is contactly conformally related to an $\mathcal{F}_{11}{}^0$-manifold $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ by a transformation of G_{11}^{0}, so that the canonical connection $\overline{D}$ of $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ is flat.

Proof. Let (M,φ,ξ,η,g) be an $\mathcal{F}_0$-manifold with $B(R)$=0. According to Theorem A', there exist functions u',v',w', such that transformation f_1 of G_0 gives rise to an $\mathcal{F}_0$-manifold $(M,\varphi,\xi',\eta',g')$ with flat connection ∇' of g', i.e. R'=0. Let us consider the transformation f_2 of $G_{11}^{0'}$, determined by the functions $u''=0$, v'', w''. According to Corollary 4.3.1 the transformation f_2 gives rise to an $\mathcal{F}_{11}{}^0$-manifold $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$, so that $R'=\overline{K}=0$. Then the transformation $f = f_2 \circ f_1$ belongs to G_{11}^{0} and it is determined by the functions $u = u'$,$v=v'+v''$ and $w=w'+w''$.Thus the transformation f gives rise to an $\mathcal{F}_{11}{}^0$-manifold $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ = $f(M,\varphi,\xi,\eta,g)$ with flat canonical connection $\overline{D}$ for $\dim M \geq 9$.

Having in mind the invariance of the Bochner curvature tensor $B(K)$ on an $\mathcal{F}_{11}{}^0$-manifold with respect to the group G_{11}^{0}, the contactly conformal equivalence of $\mathcal{F}_0$ and $\mathcal{F}_{11}{}^0$, as well as Theorems 4.2, 4.3, 5.1 and Corollary 4.3.1, we get the following main

Theorem 5.2. Let (M,φ,ξ,η,g) (dim $M \geq 9$) be an $\mathcal{F}_{11}{}^{0}$-manifold with vanishing Bochner curvature tensor $B(K)$. Then (M,φ,ξ,η,g) is contactly conformally related to an $\mathcal{F}_{11}{}^{0}$-manifold $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ by a transformation of G^{0}_{11}, so that the canonical connection $\overline{D}$ of $(M,\varphi,\overline{\xi},\overline{\eta},\overline{g})$ is flat.

6. An example of an $\mathcal{F}_{11}$ -manifold

Some examples of almost contact manifolds with B-metric of the classes $\mathcal{F}_0$, $\mathcal{F}_4\oplus\mathcal{F}_5$ and $\mathcal{F}_5$ are known from [3], and examples of the manifolds of the classes $\mathcal{F}_1$, $\mathcal{F}_4$, $\mathcal{F}_5$ - from [4]. We are not familiar with the existence in the literature of examples of manifolds of the class $\mathcal{F}_{11}$. Using the results of the previous sections, we shall give an example of an $\mathcal{F}_{11}$-manifold.

Let M be an $\mathcal{F}_0$-manifold. We consider the functions $u,v\in\mathcal{F}M$, which are a φ-holomorphic pair of functions, i.e. $du\circ\varphi = dv\circ\varphi^2$, $du(\xi) = dv(\xi) = 0$. In [4] is given the following example of such functions:

$$(6.1)\qquad u = \ln\sqrt{\prod_{i=1}^{n}\left[\left(x^i\right)^2+\left(y^i\right)^2\right]}, \qquad v = \sum_{i=1}^{n} arctg\frac{x^i}{y^i},$$

where $(x^i;y^i;t)$ ($i=1,\ldots,n$) are the local coordinates of a point p of M.

Let $w_1(x^i;y^i)\in\mathcal{F}M$ such that $dw_1\circ\varphi$ is not closed, i.e. it is not a φ-pluriharmonic function, and let $w_2(t)\in\mathcal{F}M$ satisfies the condition $w_2''(t)\neq 0$.

We consider the function $w = w_1 + w_2$, obtaining $dw(z) = dw_1(z) + dw_2(z) = dw_1(z) + w_2'(z).\eta(z)$, where $z\,(z^i,\tilde{z}^i,\eta(z))\in T_pM$. The last equality implies $dw(\varphi z) = dw_1(\varphi z)$ and $dw(\xi) = w_2'$. It is obvios that $d(dw(\xi))(z) = w_2''.\eta(z)$, whence $d(dw(\xi))(\varphi z) = 0$. Since $dw\circ\varphi$ is not closed, then $dw\circ\varphi \neq 0$. Acording to the definition conditions of the group G^{0}_{11} the transformation f, determined by the triad of functions (u,v,w), belongs to G^{0}_{11}, but does not belong to G_0.

The according to Theorem 3.4, from an $\mathcal{F}_0$-manifold the transformation f gives rise to an $\mathcal{F}_{11}{}^{0}$-manifold.

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