EXAMPLES OF ALMOST CONTACT MANIFOLDS WITH *B*-METRIC, DERIVED FROM ALMOST COMPLEX MANIFOLDS WITH *B*-METRIC

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1. Preliminaries.

Let (M^{2n},J,h) be a $2n$-dimensional almost complex manifold with B-metric, i.e. J is the complex structure and h is the metric on M^{2n} such that:

(1) $$J^2X=-X\,; \qquad h(JX,JY)=-h(X,Y),$$

for all vector fields X,Y on M^{2n}.

Further, X, Y, Z will stand for arbitrary differentiable vector fields on M^{2n} (i.e. $X, Y, Z\in\mathfrak{X}(M^{2n})$) and D — for the Levi-Civita connection of h. The tensor field F of type (0,3) on the manifold M^{2n} is defined by $F(X,Y,Z) = h((D_XJ)Y,Z)$. This tensor has the following symmetries :

(2) $$F(X,Y,Z) = F(X,Z,Y); \qquad F(X,Y,Z) = F(X,JY,JZ).$$

If $\{e_i\}$ ($i= 1, 2, ...,2n$) is an arbitrary basis of T_pM^{2n}, h^{ij} are the components of the inverse matrix of h, then the Lee form θ associated with the tensor F is defined by

(3) $$\theta(x) = h^{ij} F(e_i,e_j,x),$$

for an arbitrary vector x in the tangential space T_pM to M at an arbitrary point p in M.

A classification of the almost compex manifolds with B-metric with respect to the tensor F is given in [5]. The basic classes are defined as following:

(4)
$$W_1:\ F(X,Y,Z) = \{ h(X,Y)\theta(Z)+ h(X,Z)\theta(Y)+ h(X,JY)\theta(JZ)+ h(X,JZ)\theta(JY)\}/2n;$$
$$W_2:\ F(X,Y,JZ)+F(Y,Z,JX)+ F(Z,X,JY)= 0, \qquad \theta=0;$$
$$W_3:\ F(X,Y,Z)+F(Y,Z,X)+ F(Z,X,Y)= 0.$$

The class W_0: $F=0$ of the Kaehler manifolds with B-metric belongs to each of the basic classes W_i $(i=1, 2, 3)$.

Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a $(2n+1)$-dimensional almost contact manifold with B-metric, i.e. (φ,ξ,η,g) is an almost contact structure [2] determined by a tensor field φ of type (1,1), a vector field ξ and 1-form η, and g is a metric on M so that:

(5) $$\varphi^2 = -i\mathrm{d} + \eta\otimes\xi\,;\ \eta(\xi) = 1;\ g(\varphi X,\varphi Y) = - g(X,Y)+ \eta(X)\,\eta(Y),$$

where X,Y are arbitrary differentiable vector fields on M^{2n+1}. It follows immediately $\eta\circ\varphi=0$, $\varphi\xi= 0$, $\eta(X)= g(X,\xi)$. Moreover the endomorphism φ has rank $2n$ [6].

Further, arbitrary differentiable vector fields on M^{2n+1} will be denoted by X, Y, Z, and the Levi-Civita connection of g — by C. The tensor field F of type (0,3) on M^{2n+1} is defined by $F(X,Y,Z) = g((\nabla_X\varphi)Y,Z)$. This tensor has the following properties:

(6)
$$F(X,Y,Z) = F(X,Z,Y);$$
$$F(X,\varphi Y,\varphi Z) = F(X,Y,Z) - \eta(Y)\,F(X,\xi,Z)- \eta(Z)F(X,Y,\xi).$$

The following 1-forms are associated with F:

(7) $$\theta(x) = g^{ij} F(e_i,e_j,x),\ \ \theta^*(x) = g^{ij} F(e_i,\varphi e_j,x),\ \ \omega(x) = F(\xi,\xi,x),$$

where x is an arbitrary vector in the tangent space T_pM^{2n+1} at any point p, $\{e_i,\xi\}$ $(i=1,2,...,2n)$ is a basis of T_pM^{2n+1}, and (g^{ij}) is the inverse matrix of (g_{ij}).

A classification of the almost contact manifolds with B-metric with respect to the tensor F is given in [6], where the basic classes $\mathcal{F}_i$ $(i = 1,2,...,11)$ of almost contact manifolds with B-metric are defined. The classes determined by the main components of the tensor field F are $\mathcal{F}_i$ $(i=1, 4, 5, 11)$. These classes are defined by conditions:

$$\mathcal{F}_1: F(X,Y,Z)=(1/2n)\{g(X,\varphi Y)\theta(\varphi Z)+ g(X,\varphi Z)\theta(\varphi Y)+ g(\varphi X,\varphi Y)\theta(\varphi^2 z)+ g(\varphi X,\varphi Z)\ \theta(\varphi^2 Y)\};$$

(8) $$\mathcal{F}_4: F(X,Y,Z)=-\{\theta(\ \xi\ )/2n\}\{g(\varphi X,\varphi Y)\eta(Z)+ g(\varphi X,\varphi Z)\eta(Y)\};$$

$$\mathcal{F}_5: F(X,Y,Z)=-\{\theta^*(\ \xi\ )/2n\}\{g(X,\varphi Y)\eta(Z)+ g(X,\varphi Z)\eta(Y)\};$$

$$\mathcal{F}_{11}: F(X,Y,Z) = \eta(X)\{\eta(Y)\omega(Z)+ \eta(Z)\omega(Y)\}.$$

The special class $\mathcal{F}_0$: $F=0$ is contained in each of the basic eleven classes.

2. Classes of almost contact manifolds with *B*-metric, derived from almost complex manifolds with *B*-metric.

In this section we shall be constructed an almost contact manifold with B-metric as a direct product of an almost complex manifold with B-metric and a real line R, endowed with a positive definite scalar product.

It is known [1,3] that the product $M^{2n}\times R$ is a $(2n+1)$-dimensional differentiable manifold. A vector field on $M^{2n}\times R$ will be denoted by $(X,a.d/dt)$, where $X\in\mathcal{X}(M^{2n})$, a is a differentiable function on $M^{2n}\times R$ and t is the coordinate of R.

As it is well known [2,4], starting from an almost complex structure J on M^{2n}, an almost contact structure can be defined on $M^{2n}\times R$ by setting

(9) $$\varphi(X,a.d/dt)=(JX,0),\ \xi=(0,d/dt),\ \eta(X,a.d/dt)=a.$$

We define a metric g on $M^{2n}\times R$ by the equality

(10) $$g((X,a.d/dt),(Y,b.d/dt))=h(X,Y)+ab.$$

The equalities (5) are easily verified for so defined structure tensors, hence the structure (φ,ξ,η,g) is an almost contact structure with B-metric.

If $(x_0,t_0)\in M^{2n}\times R$, we consider injections $i: M^{2n}\to M^{2n}\times R$ and $j: R\to M^{2n}\times R$ defined by $i(x)=(x,t_0)$ and $j(t)=(x_0,t)$; if $X\in\mathsf{X}(M^{2n})$ and a is a differentiable function on $M^{2n}\times R$, $X(a\circ i)$ and $d/dt(a\circ j)$ will simply be denoted $X(a)$ and $d/dt(a)$.

Let ∇ (respectively D) be the Levi-Civita connection of the metric g (respectively h). Using the well-known Levi-Civita condition for ∇ and D, we get

(11) $$\nabla_{(X,a.d/dt)}(Y,b.d/dt)=(D_XY,\{X(b)+a.d/dt(b)\}.d/dt),$$

whence

(12) $$(\nabla_{(X,a.d/dt)}\varphi)(Y,b.d/dt)=((D_XJ)Y,0),\ \nabla_{(X,a.d/dt)}\xi=0,\ (\nabla_{(X,a.d/dt)}\eta)(Y,b.d/dt)=0.$$

Hence for the tensor field F and $F^{\times}$ on M^{2n} and $M^{2n}\times R$ and for its associated 1-forms θ and $\theta^{\times}$, respectively, we obtain

(13) $$F^{\times}((X,a.d/dt),(Y,b.d/dt),(Z,c.d/dt))=F(X,Y,Z),\qquad \theta^{\times}(Z,c.d/dt)=\theta(Z),$$

Therefore, we have the following

Theorem 1. If an almost complex manifold M^{2n} belongs to the class W_i, then the almost contact manifold $M^{2n}\times R$ belongs to the class $\mathcal{F}_i$ $(i=0, 1, 2, 3)$.

This result is natural, because the tensor field F is equal to its horizontal component in classification of the almost contact manifolds for the classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$. The classes $\mathcal{F}_2$ and $\mathcal{F}_3$ are defined by conditions [6]:

$$\mathcal{F}_2: F(X,Y,JZ)+F(Y,Z,JX)+ F(Z,X,JY)= 0,\ F(\xi,Y,Z)=F(X,\xi,Z)=0,\ \theta=0;$$

$$\mathcal{F}_3: F(X,Y,Z)+F(Y,Z,X)+ F(Z,X,Y)= 0,\ F(\xi,Y,Z)=F(X,\xi,Z)=0.$$

Lemma 2. Let (M^4, J, h) be an arbitrary almost complex manifold with B-metric. Then (M^4, J, h) cannot belong to the class W_3.

Proof. Let (M^4, J, h) belong to the class W_3 and let $\{e_1, e_2, Je_1, Je_2\}$ be a J-basis of the tangent space T_pM^4 at the point $p$in M^4. Then, according to the definite condition of the class W_3, it may be is verified immediately that the components of F with respect to this basis are zeros. Therefore, the tensor F of this manifold is identically zero, i.e. $M^4\in W_0$.

Corollary 3. The almost contact manifold $M^{2n}\times R$ with B-metric cannot belong to the class $\mathcal{F}_3$.

Let us recall for some subclasses:

(14) $W_1^0=\{ M^{2n}\in W_1 \;/\; d\theta=d(\theta_\circ J)=0\}$ from [9],
(15) $\mathcal{F}_1^0=\{ M^{2n+1}\in\mathcal{F}_1 \;/\; d\theta=d\theta^*=0\}$ from [8]
and
(16). $\mathcal{F}_{11}^0=\{ M^{2n+1}\in\mathcal{F}_1 \;/\; d(\omega_\circ\varphi)=0\}$ from [7].

Having in mind (13), (14) and (15), Theorem 1 (in case for $I=1$) implies the following

Corollary 4. If an almost complex manifold M^{2n} is a W_1^0-manifold, then the almost contact manifold $M^{2n}\times R$ is an $\mathcal{F}_1^0$-manifold.

The contactly conformal transformation of general type of an almost contact structure (φ,ξ,η,g) with B-metric to that sort of structure $(\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g})$ is introduced in [7] so that

$\overline{\varphi}=\varphi;\quad \overline{\xi}=e^{-w}\xi;\quad \overline{\eta}=e^{w}\eta;$

$\overline{g}(X,Y)=e^{2u}\cos 2v\, g(X,Y)+e^{2u}\sin 2v\, g(X,\varphi Y)+(e^{2w}-e^{2u}\cos 2v)\,\eta(X)\eta(Y),$

where u, v, w is arbitrary differentiable functions on M^{2n+1} and $X, Y\in\mathfrak{X}(M^{2n+1})$. Such transformations form a group denoted by G.

Let us note that the contactly conformal transformation c, introduced in [8], where $c(\varphi,\xi,\eta,g)=(\varphi,\xi,\eta,\overline{g})$, is of general type at $w = 0$. It's clear that the group of the contactly conformal transformations C, considered in [8], is a subgroup of G.

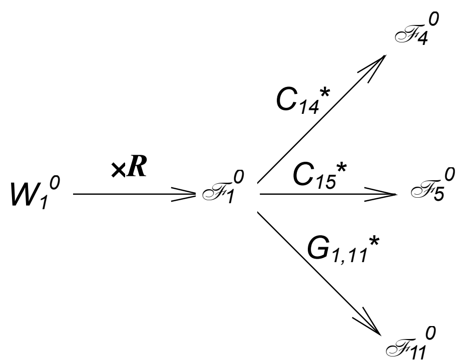
Besides, let us consider some subgroups of C and G:

$C_1=\{ c\in C \mid du(\xi)=dv(\xi)=0\}$,
$C_4=\{c\in C \mid du_o\varphi = dv_o\varphi^2,\ du(\xi)=0\}$
$C_5=\{c\in C \mid du_o\varphi = dv_o\varphi^2,\ dv(\xi)=0\}$,
$C_{15}=\{c\in C \mid dv(\xi)=0\}$,
$C_{14}=\{c\in C \mid du(\xi)=0\}$,
$C_{1j}^*=\{c\in C_{1j} \mid c\notin C_1,\ c\notin C_j\},\ (j=4, 5)$,
$C_1^0=\{c\in C_1 \mid d(du_o\varphi)=d(dv_o\varphi)=du(\xi)=dv(\xi)=0\}$,
$G_{11}^0=\{f\in G \mid du_o\varphi = dv_o\varphi^2,\ du(\xi)=dv(\xi)= d(dw(\xi))_o\varphi=0\}$.

As it is known from [8], the classes $\mathcal{F}_4$ and $\mathcal{F}_5$ are contactly conformally related to the class $\mathcal{F}_1$ by the transformations of the groups C_{14}^* and C_{15}^*, respectively.

In accordance with [8] (respectively [7]) the class $\mathcal{F}_1^0$ (respectively $\mathcal{F}_{11}^0$) is the class of manifolds, which are equivalent to the $\mathcal{F}_0$-manifolds by the transformations of the group C_1^0 (respectively G_{11}^0).

In conclusion, the main classes of the almost contact manifolds $\mathcal{F}_i\,(I=1,4,5,11)$ are derived from the main class of the almost complex manifold W_1. The following diagrams illustrate this derivation:



In this way examples may be constructed of manifolds belonging to the main classes of almost contact manifolds with B-metric.

R e f e r e n c e s

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