J. geom. 00 (2001) 000 – 000
0047–2468/01/000000 – 00 $ 1.50 + 0.00/0
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Journal of Geometry

On almost paracontact Riemannian manifolds of type (n, n)

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Abstract. In this paper we give a classification with eleven basic classes of almost paracontact Riemannian manifolds of type (n, n) with respect to the covariant derivative of the $(1, 1)$-tensor of the almost paracontact structure.

Mathematics Subject Classification (2000): 53C15, 53C25.
Key words: Almost paracontact, Riemannian manifolds.

1. Introduction

In 1976 I. Sato [1] introduced the concepts of almost paracontact manifolds and of almost paracontact Riemannian manifolds as analogues of almost contact manifolds and of almost contact Riemannian manifolds.

After that S. Sasaki [2] defined the notion of an almost paracontact Riemannian manifold of type (p, q) and arbitrary dimension, where p and q are the numbers of the multiplicity of the structural eigenvalues 1 and -1, respectively. In addition, there is a simple eigenvalue 0.

In this paper we consider almost paracontact Riemannian manifolds of type (n, n), i.e. $p = q = n$. We put this fixation in view of reasons of later investigations relevant to $2n$-dimensional Riemannian almost product manifolds (M^{2n}, P, g) with structural group $O(n) \times O(n)$, which are classified in [3]. In this reason the manifolds in our consideration could be construct by natural way as a direct product of (M^{2n}, P, g) and a real line or as a hypersurface of (M^{2n}, P, g).

The method used in the present paper is analogous to the methods of classification in [4] and partly to those in [5] for the almost contact metric manifolds and for the almost contact manifolds with B-metric, respectively.

2. Preliminaries

A $(2n+1)$-dimensional real differentiable manifold M is said to have an almost paracontact structure (ϕ, ξ, η) of type (n, n), if it admits a $(1, 1)$-tensor ϕ, a vector field ξ and a 1-form η satisfying the following conditions:

$$\eta(\xi) = 1, \qquad \phi^2 = I_{2n+1} - \eta \otimes \xi, \qquad tr\phi = 0. \tag{1}$$

A positive definite Riemannian metric g is said to be compatible with the almost paracontact structure if it satisfies the following conditions

$$g(x,\xi) = \eta(x), \qquad g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y) \tag{2}$$

for all vectors x, y in the tangent space T_pM.

The quadruple (ϕ, ξ, η, g) of an almost paracontact structure of type (n, n) and a compatible metric is called an almost paracontact Riemannian structure of type (n, n).

We introduce the tensor $\tilde{g}$ given by the equation

$$\tilde{g}(\cdot, \cdot) = g(\cdot, \phi\cdot) + \eta(\cdot)\eta(\cdot). \tag{3}$$

It is a compatible metric with the almost paracontact structure, too. The associated metric $\tilde{g}$ to the Riemannian metric g is a pseudo-Riemannian metric of signature $(n + 1, n)$.

Let us denote the tensor of type (0, 3) by the equation

$$F(x,y,z) = g((\nabla_x \phi)y,z) \qquad (x,\ y,\ z\ \in\ T_pM). \tag{4}$$

Because of (1) and (2) the tensor F has the following properties

$$\begin{aligned} F(x, y, z) = F(x, z, y) = -F(x, \phi y, \phi z)\\ +\eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi) \end{aligned} \tag{5}$$

for all vectors $x,\ y,\ z$ in T_pM.

If $\{e_i\ , \xi\}$ $(i = 1, 2, \ldots, 2n)$ is a basis of the tangent space T_pM and g^{ij} are the components of the inverse matrix of g, then the following 1-forms are associated with the tensor F

$$\theta(x) = g^{ij}F(e_i, e_j, x), \quad \theta^*(x) = g^{ij}F(e_i, \phi e_j, x), \quad \omega(x) = F(\xi, \xi, x). \tag{6}$$

for an arbitrary vector $x \in T_pM$.

3. The space of covariant derivatives of the (1, 1)-tensor of the almost paracontact structure

Let (V, ϕ, ξ, η, g) be $(2n + 1)$-dimensional vector space with almost paracontact Riemannian structure of type (n, n). We consider the vector space $\mathcal{F}$ of the tensors of type (0, 3) over V having the properties (6). The metric g induces on $\mathcal{F}$ an inner product $\langle\ ,\ \rangle$, defined by

$$\langle F', F''\rangle = g^{ip}\ g^{jq}g^{kr}F'(e_i, e_j, e_k)F''(e_p, e_q, e_r)$$

for arbitrary elements F', F'' in $\mathcal{F}$ and a V's basis $\{e_i, e_{2n+1} = \xi\}$ $(i = 1, 2, \ldots, 2n)$.

The standard representation of the structural group $O(n)\times O(n)\times I$ in V induces a natural representation of the structural group in $\mathcal{F}$:

$$(\lambda(a)F)(x,y,z)=F(a^{-1}x,a^{-1}y,a^{-1}z),$$

where $a\in O(n)\times O(n)\times I$, $F\in\mathcal{F}$, $x,y,z\in V$, so that

$$\langle\lambda(a)F',\lambda(a)F''\rangle=\langle F',F''\rangle.$$

We consider operators h, v, w on V with the properties

$$\begin{gathered} h^2=h,\qquad v^2=v,\qquad w^2=w,\\ h\circ v=v\circ h=h\circ w=w\circ h=v\circ w=w\circ v=0. \end{gathered}\tag{7}$$

The action of these operators on the space $\mathcal{F}$ is represented by the equations

$$\begin{aligned} &hF(x,y,z)=F(\phi^2x,\phi^2y,\phi^2z),\\ &vF(X,Y,Z)=\eta(x)F(\xi,y,z)+\eta(y)F(x,z,\xi)+\eta(z)F(x,y,\xi)\\ &\qquad\qquad -2\eta(x)\left[\eta(y)F(\xi,\xi,z)+\eta(z)F(\xi,\xi,y)\right],\\ &wF(x,y,z)=\eta(x)\left[\eta(y)F(\xi,\xi,z)+\eta(z)F(\xi,\xi,y)\right] \end{aligned}\tag{8}$$

The equality (8) implies

$$F=hF+vF+wF.\tag{9}$$

Now we define basic operators $F_i:\mathcal{F}\to\mathcal{F}$ $(i=1,2,\dots,10)$ by the equations:

$$\begin{aligned} &F_1(F)(x,y,z)=\eta(x)F(\xi,y,z),\\ &F_2(F)(x,y,z)=\eta(y)F(x,z,\xi)+\eta(z)F(x,y,\xi),\\ &F_3(F)(x,y,z)=\eta(x)\left[\eta(y)F(\xi,\xi,z)+\eta(z)F(\xi,\xi,y)\right],\\ &F_4(F)(x,y,z)=\eta(y)F(z,\xi,x)+\eta(z)F(y,\xi,x)\\ &\qquad\qquad -2\eta(y)\eta(z)F(\xi,\xi,x),\\ &F_5(F)(x,y,z)=\eta(y)F(\phi x,\xi,\phi z)+\eta(z)F(\phi x,\xi,\phi y),\\ &F_6(F)(x,y,z)=\eta(y)F(\phi z,\xi,\phi x)+\eta(z)F(\phi y,\xi,\phi x),\\ &F_7(F)(x,y,z)=\tfrac{\theta(F)(\xi)}{2n}\left\{\eta(y)g(\phi x,\phi z)+\eta(z)g(\phi x,\phi y)\right\},\\ &F_8(F)(x,y,z)=\tfrac{\theta^*(F)(\xi)}{2n}\left\{\eta(y)g(x,\phi z)+\eta(z)g(x,\phi y)\right\},\\ &F_9(F)(x,y,z)=\tfrac{1}{2n}\{g(\phi x,\phi y)\theta(\phi^2z)+g(\phi x,\phi z)\theta(\phi^2y)\\ &\qquad\qquad -g(x,\phi y)\theta(\phi z)-g(x,\phi z)\theta(\phi y)\},\\ &F_{10}(F)(x,y,z)=\tfrac{1}{3}\left\{hF(x,y,z)+hF(y,z,x)+hF(z,x,y)\right\}. \end{aligned}\tag{10}$$

It is easy to check that $F_i(F)\in F$ for $F\in\mathcal{F}$ $(i=1,2,\dots,10)$.

By necessity we have to compute the compositions of the basic operators and the associated 1-forms of $F_i(F)$.

LEMMA 1. *Let $F \in \mathcal{F}$ and $F_{i,j}(F) = F_i(F_j(F))$ $(i, j = 1, 2, \dots, 10)$. Then we have:*

a) $F_{i,i}(F) = F_i(F)$ *for* $i = 1, 2, 3, 7, \dots, 10$;
$F_{i,i}(F) = F_2(F) - F_3(F)$ *for* $i = 4, 5, 6$;
$F_{1,i}(F) = F_{i,1}(F) = F_3(F)$ *for* $i = 2, 3$;
$F_{2,i}(F) = F_{i,2}(F) = F_i(F)$ *for* $i = 3, 4, 5, 6, 7, 8$;
$F_{i,j}(F) = F_k(F)$ *for* $i, j, k = 4, 5, 6$;
$F_{i,j}(F) = F_{j,i}(F) = F_j(F)$ *for* $i = 4, 5, 6$ *and* $j = 7, 8$
and the rest of $F_{i,j}(F)$ *are zeros;*

b) $\theta(F_i(F)) = \theta(F)(\xi)\eta$ *for* $i = 2, 4, 5, 6, 7$;
$\theta^*(F_i(F)) = \theta^*(F)(\xi)\eta$ *for* $i = 2, 4, 5, 6, 8$;
$\omega(F_i(F)) = \omega(F)$ *for* $i = 1, 2, 3$
and the rest of the associated 1*-forms are zeros.*

By virtue of the operators F_i we construct new operators L_j $(j = 1, 2, \dots, 9)$. Let

$$L_1 : \mathcal{F} \to \mathcal{F}, \qquad L_1(F) = F - 2F_3(F), \qquad F \in \mathcal{F}.$$

In a straightforward way using Lemma 1 we get that the operator L_1 is an involutive isometry on $\mathcal{F}$ and commutes with the action of $O(n) \times O(n) \times I$. Hence L_1 has two eigenvalues $+1$ and -1, and the corresponding eigenspaces $(L_1\mathcal{F})^+$ and $(L_1\mathcal{F})^-$ are orthogonal and invariant mutually complementary subspaces of $\mathcal{F}$. Besides that, the components of F in the subspaces $(L_1\mathcal{F})^+$ and $(L_1\mathcal{F})^-$ are $\frac{1}{2}\{F + L_1(F)\}$ and $\frac{1}{2}\{F - L_1(F)\}$, respectively.

It is easy to show

LEMMA 2. *If* $F \in \mathcal{F}$ *then*

a) $F_3(F) = 0$ *iff* $\omega(F) = 0$;
b) $F \in (L_1\mathcal{F})^+$, *i.e.* $L_1(F) = F$ *iff* $F_3(F) = 0$;
c) $F \in (L_1\mathcal{F})^-$, *i.e.* $L_1(F) = -F$ *iff* $F = F_3(F)$.

If we denote $\mathcal{F}_{11} = (L_1\mathcal{F})^-$ and $\mathcal{F}_{11}^\perp = (L_1\mathcal{F})^+$, then according Lemma 2 we have

$$\mathcal{F}_{11} = \{F \in \mathcal{F} \mid F = F_3(F)\}, \qquad \mathcal{F}_{11}^\perp = \{F \in \mathcal{F} \mid \omega(F) = 0\}.$$

Thereby we obtain immediately

PROPOSITION 3. *The decomposition* $\mathcal{F} = \mathcal{F}_{11} \oplus \mathcal{F}_{11}^\perp$ *is orthogonal and invariant with respect to the action of* $O(n) \times O(n) \times I$. *The component of an arbitrary element* F *in* $\mathcal{F}_{11}$ *(respectively in* $\mathcal{F}_{11}^\perp$*) is* $F_3(F)$ *(respectively* $F - F_3(F)$*).*

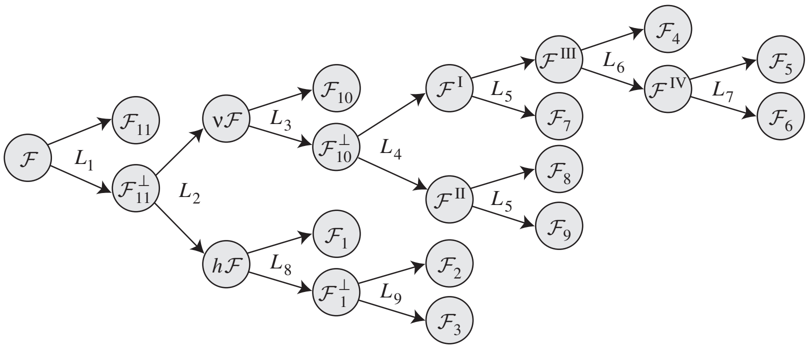

Figure 1

Using the same method we continue to decompose $\mathcal{F}$ orthogonally and invariantly with respect to the action of $O(n) \times O(n) \times I$. Thereby we obtain the complete decomposition into irreducible components by the scheme of Figure 1.

The rest of operators L_j $(j = 2, 3, \dots, 9)$ we define by:

$$\begin{aligned} L_2(F) &= F - 2\{F_1(F) + F_2(F)\}, & L_6(F) &= F - 2F_7(F),\\ L_3(F) &= F_2(F) - F_1(F), & L_7(F) &= F - 2F_8(F),\\ L_4(F) &= -F_5(F), & L_8(F) &= F - 2F_9(F),\\ L_5(F) &= -F_4(F), & L_9(F) &= F - 2F_{10}(F). \end{aligned}$$

The operators L_j $(j = 2, 3, \dots, 9)$ are involutive isometries on the corresponding subspaces and also commute with the action of $O(n) \times O(n) \times I$. According the applying method the corresponding eigenspaces of the eigenvalues $+1$ and -1 are orthogonal and mutually complementary subspaces of the reducible subspace of $\mathcal{F}$.

Since the endomorphism ϕ induces an almost product structure on the orthogonal complement $\{\xi\}^{\perp}$ of the subspace spanned by ξ and the restriction of g on $\{\xi\}^{\perp}$ is a Riemannian metric compatible with the almost product structure, then the decomposition of $h\mathcal{F}$ coincides with the known decomposition of the space of covariant derivatives of the traceless almost product structure [3].

Taking into account the above reasons we obtain

THEOREM 4. *The decomposition* $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_{11}$ *is orthogonal and invariant with respect to the action of* $O(n) \times O(n) \times I$. *The characterization conditions of the factors* $\mathcal{F}_i$ $(i = 1, 2, \dots, 11)$ *for arbitrary vectors* x, y, z *in* V *are:*

$$\begin{aligned}
\mathcal{F}_1:\quad & F(x,y,z)=\tfrac{1}{2n}\{g(\phi x,\phi y)\theta(\phi^2 z)+g(\phi x,\phi z)\theta(\phi^2 y)\\
&\qquad -g(x,\phi y)\theta(\phi z)-g(x,\phi z)\theta(\phi y)\};\\
\mathcal{F}_2:\quad & F(x,y,\phi z)+F(y,z,\phi x)+F(z,x,\phi y)=0,\\
& F(\xi,y,z)=F(x,y,\xi)=0,\quad \theta=0;\\
\mathcal{F}_3:\quad & F(x,y,z)+F(y,z,x)+F(z,x,y)=0,\quad F(\xi,y,z)=F(x,y,\xi)=0;\\
\mathcal{F}_4:\quad & F(x,y,z)=\tfrac{\theta(\xi)}{2n}\{g(\phi x,\phi y)\eta(z)+g(\phi x,\phi z)\eta(y)\};\\
\mathcal{F}_5:\quad & F(x,y,z)=\tfrac{\theta^*(\xi)}{2n}\{g(x,\phi y)\eta(z)+g(x,\phi z)\eta(y)\};\\
\mathcal{F}_6:\quad & F(x,y,z)=\eta(y)F(z,\xi,x)+\eta(z)F(y,\xi,x)-2\eta(y)\eta(z)F(\xi,\xi,x)\\
&\qquad =\eta(y)F(\phi x,\xi,\phi z)+\eta(z)F(\phi x,\xi,\phi y),\quad \theta(\xi)=\theta^*(\xi)=0;\\
\mathcal{F}_7:\quad & F(x,y,z)=-\eta(y)F(z,\xi,x)-\eta(z)F(y,\xi,x)+2\eta(y)\eta(z)F(\xi,\xi,x)\\
&\qquad =\eta(y)F(\phi x,\xi,\phi z)+\eta(z)F(\phi x,\xi,\phi y);\\
\mathcal{F}_8:\quad & F(x,y,z)=\eta(y)F(z,\xi,x)+\eta(z)F(y,\xi,x)-2\eta(y)\eta(z)F(\xi,\xi,x)\\
&\qquad =-\eta(y)F(\phi x,\xi,\phi z)-\eta(z)F(\phi x,\xi,\phi y);\\
\mathcal{F}_9:\quad & F(x,y,z)=-\eta(y)F(z,\xi,x)-\eta(z)F(y,\xi,x)+2\eta(y)\eta(z)F(\xi,\xi,x)\\
&\qquad =-\eta(y)F(\phi x,\xi,\phi z)-\eta(z)F(\phi x,\xi,\phi y);\\
\mathcal{F}_{10}:\quad & F(x,y,z)=\eta(x)F(\xi,y,z),\quad F(x,y,\xi)=0;\\
\mathcal{F}_{11}:\quad & F(x,y,z)=\eta(x)\{\eta(y)\omega(z)+\eta(z)\omega(y)\}.
\end{aligned}$$

The components $p_i(F)$ of an arbitrary element F of $\mathcal{F}$ in $\mathcal{F}_i$ $(i=1,2,\dots,11)$ are

$$\begin{aligned}
& p_1(F)=F_9(F), \qquad p_2(F)=F_{10}(F),\\
& p_3(F)=hF-F_9(F)-F_{10}(F), \quad p_4(F)=F_7(F), \quad p_5(F)=F_8(F),\\
& p_6(F)=\tfrac14\{F_2(F)-F_3(F)+F_4(F)+F_5(F)+F_6(F)\}-F_7(F)-F_8(F),\\
& p_7(F)=\tfrac14\{F_2(F)-F_3(F)-F_4(F)+F_5(F)-F_6(F)\},\\
& p_8(F)=\tfrac14\{F_2(F)-F_3(F)+F_4(F)-F_5(F)-F_6(F)\},\\
& p_9(F)=\tfrac14\{F_2(F)-F_3(F)-F_4(F)-F_5(F)+F_6(F)\},\\
& p_{10}(F)=F_1(F)-F_3(F), \qquad p_{11}(F)=F_3(F).
\end{aligned}$$

4. Basic classes of almost paracontact Riemannian manifolds of type (n, n)

Let (M,ϕ,ξ,η,g) be almost paracontact Riemannian manifolds of type (n,n). The tangent space T_pM at an arbitrary point p in M is the vector space V equipped with an almost paracontact Riemannian structure of type (n,n). Then the corresponding vector space $\mathcal{F}$ considered in the previous section has eleven orthogonal and invariant subspaces $\mathcal{F}_i$. In such a way the conditions for F at every point $p\in M$ give rise to the corresponding class of manifolds under consideration. Namely, an almost paracontact Riemannian manifold of type (n,n) is said to be in the class $\mathcal{F}_i$ if the tensor F belongs to the subspace $\mathcal{F}_i$ $(i=1,2,\dots,11)$ over T_pM at every p in M. Thus the conditions define the eleven basic classes of almost paracontact Riemannian manifolds of type (n,n). Of course, the number of all classes of manifold under conversation is 2^{11} and their defining conditions are easily obtainable by the basic ones. The special class $\mathcal{F}_0$ of almost paracontact Riemannian manifolds of type (n, n) is defined by the condition $F = 0$. This class belongs to each of the defined classes. It is an analogue to the class of cosymplectic almost contact metric manifolds [4] and to the respective class $\mathcal{F}_0$ in the classification of almost contact manifolds with B-metric [5].

The defined class $\mathcal{F}_4$ contains paracontact Riemannian manifolds of type (n, n) [1] and in particular para-Sasakian manifolds of type (n, n) [7].

References

[1] Sato, I., *On a structure similar to the almost contact structure*, Tensor N.S. **30** (1976), 219–224.

[2] Sasaki, S., *On paracontact Riemannian manifolds*, TRU Math. **16–2** (1980), 75–86.

[3] Staikova, M. and Gribachev, K., *Canonical connections and their conformal invariants on Riemannian almost-product manifolds*, Serdica **18** (1992), 150–161.

[4] Alexiev, V. and Ganchev, G., *On the classification of the almost contact metric manifolds*, Math. and Educ. in Math., Proc. 15th Spring Conference of UBM, 1986, 155–161.

[5] Ganchev, G., Mihova, V. and Gribachev, K., *Almost contact manifolds with B-metric*, Math. Balk. **7** (1993), 3–4, 261–276.

[6] Ganchev, G. and Borisov, A., *Note on the almost complex manifolds with a Norden metric*, C.R. Acad. Bulg. Sci. **39** (1986), 3134.

[7] Adati, T. and Miyazawa, T., *On paracontact Riemannian manifolds*, TRU Math. **13–2** (1977), 27–39.

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Received 10 May 2000.

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