ALMOST CONTACT B-METRIC HYPERSURFACES OF KAEHLERIAN MANIFOLDS WITH B-METRIC

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In this paper, we construct two types of almost contact B-metric hypersurfaces of an almost complex manifold with B-metric and we characterize subclasses of these hypersurfaces of a Kaehlerian B-metric manifold with respect to the second fundamental tensor.

Introduction

The geometry of almost complex B-metric manifolds is determined by the action of the almost complex structure as an antiisometry in each tangent fibre. The basic classes of these even dimensional manifolds are given in [1]. The special class $\mathcal{W}_0$ in this classification is the class of the Kaehlerian manifolds with B-metric, where the almost complex structure is parallel with respect to the Levi-Civita connection of the B-metric. This class is contained in each other class. Examples of $\mathcal{W}_0$-manifolds are considered in [2], [3], [4].

The geometry of the almost contact B-metric manifolds is a natural extension of the geometry of the almost complex manifolds with B-metric to the odd dimensional case. A classification of the almost contact manifolds with B-metric is given in [5]. There are studied some examples of manifolds belonging to the basic classes in [5], [6], [7].

In this paper, we construct two types hypersurfaces of an almost complex manifold with B-metric, which are equipped with almost contact B-metric structures. We determine the class of these almost contact B-metric hypersurfaces of a $\mathcal{W}_0$-manifold and we characterize its important subclasses with respect to the second fundamental tensor.

1 Preliminaries

Let (M', J, g') be a $2n'$-dimensional almost complex manifold with B-metric, i.e. J is an almost complex structure and g' is a metric on M' such that:

$$J^2X = -X, \qquad g'(JX, JY) = -g'(X, Y). \tag{1}$$

for all vector fields X, Y on M'. The associated metric $\tilde{g}'$ of the manifold is given by $\tilde{g}'(X,Y) = g'(X,JY)$. Both metrics are necessarily of signature (n',n').

The Levi-Civita connection of g' will be denoted by ∇'. The tensor field F' of type (0,3) on M' is defined by $F'(X,Y,Z) = g'((\nabla'_X J)Y,Z)$ for arbitrary $X,Y,Z \in \mathfrak{X}(M')$ – the Lie algebra of the differentiable vector fields on M'. This tensor has the following symmetries:

$$F'(X,Y,Z) = F'(X,Z,Y), \quad F'(X,JY,JZ) = F'(X,Y,Z).$$

If $\{e_i\}$ $(i = 1,2,\ldots,2n')$ is an arbitrary basis of $T_{p'}M'$ at an arbitrary point p' in M', and g'^{ij} are the components of the inverse matrix of g', then the Lie form θ' associated with the tensor F' is defined by

$$\theta'(x) = g'^{ij} F'(e_i, e_j, x)$$

for an arbitrary vector $x \in T_{p'}M', p' \in M'$.

A classification with three basic classes of the almost complex manifolds with B-metric with respect to F' is given in [1]. Further, we shall consider only the class $\mathcal{W}_0 : \ F' = 0$ of the Kaehlerian manifolds with B-metric belonging to each of the basic classes.

Let (M,φ,ξ,η,g) be a $(2n{+}1)-$dimensional almost contact manifold with B-metric, [5] i.e. (φ,ξ,η) is an almost contact structure determined by a tensor field φ of type (1,1) , a vector field ξ and a 1-form η on M satisfying the conditions:

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{2}$$

and in addition this almost contact manifold (M,φ,ξ,η) admits a metric g such that

$$g(\varphi X,\varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{3}$$

where X,Y are arbitrary differentiable vector fields on M, i.e. $X,Y \in \mathfrak{X}(M)$.

It follows immediately

$$\eta \circ \varphi = 0,\ \varphi\xi = 0,\ \eta(X) = g(X,\xi),\ g(\varphi X,Y) = g(X,\varphi Y). \tag{4}$$

Moreover the endomorphism φ has rank $2n$.

The associated metric $\tilde{g}$ given by $\tilde{g}(X,Y) = g(X,\varphi Y) + \eta(X)\eta(Y)$ is a B-metric, too. Both metrics g and $\tilde{g}$ are indefinite of signature $(n,n+1)$.

Further, X,Y,Z will stand for arbitrary differentiable vector fields on M and x,y,z – arbitrary vectors in tangential space T_pM to M at an arbitrary point p in M.

Let ∇ be the Levi-Civita connection of the metric g. The tensor F of type (0,3) on M is defined by $F(x,y,z) = g((\nabla_x \varphi)y,z)$ and it has the following properties:

$$F(x,y,z) = F(x,z,y) = F(x,\varphi y,\varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi).$$

If $\{e_i,\xi\}$ $(i = 1,2,\ldots,2n)$ is a basis of T_pM, and (g^{ij}) is the inverse matrix of (g_{ij}) then the following 1-forms are associated with F:

$$\theta(\cdot) = g^{ij}F(e_i,e_j,\cdot),\ \ \theta^*(\cdot) = g^{ij}F(e_i,\varphi e_j,\cdot),\ \ \omega(\cdot) = F(\xi,\xi,\cdot).$$

A classification of the almost contact manifolds with B-metric is given in [5], where eleven basic classes $\mathcal{F}_i$ are defined. We use that characterization conditions changing the definitions of the classes $\mathcal{F}_6$,$\mathcal{F}_7$,$\mathcal{F}_8$ and $\mathcal{F}_9$ with equivalent conditions for computing reasons. Let us denote $f(x,y) := F(\varphi^2 x,\varphi^2 x,\xi)$. Then the characteristics of the classes in consideration are:

$$\begin{aligned}
&\mathcal{F}_4: \ F(x,y,z) = -\tfrac{\theta(\xi)}{2n}\{g(\varphi x,\varphi y)\eta(z)+g(\varphi x,\varphi z)\eta(y)\};\\
&\mathcal{F}_5: \ F(x,y,z) = -\tfrac{\theta^*(\xi)}{2n}\{g(x,\varphi y)\eta(z)+g(x,\varphi z)\eta(y)\};\\
&\mathcal{F}_6: \ F(x,y,z) = f(x,y)\eta(z) + f(x,z)\eta(y),\\
&\qquad f(x,y)=f(y,x,), \quad f(\varphi x,\varphi y)=-f(x,y), \quad \theta(\xi) = \theta^*(\xi) = 0;\\
&\mathcal{F}_7: \ F(x,y,z) = f(x,y)\eta(z)+f(x,z)\eta(y),\\
&\qquad f(x,y)=-f(y,x), \quad f(\varphi x,\varphi y)=-f(x,y);\\
&\mathcal{F}_8: \ F(x,y,z) = f(x,y)\eta(z)+f(x,z)\eta(y),\\
&\qquad f(x,y)=f(y,x), \quad f(\varphi x,\varphi y)=f(x,y);\\
&\mathcal{F}_9: \ F(x,y,z) = f(x,y)\eta(z)+f(x,z)\eta(y),\\
&\qquad f(x,y)=-f(y,x), \quad f(\varphi x,\varphi y)=f(x,y);\\
&\mathcal{F}_{11}: \ F(x,y,z) = \eta(x)\{\eta(y)\omega(z)+\eta(z)\omega(y)\}.
\end{aligned} \tag{5}$$

The classes $\mathcal{F}_i \oplus \mathcal{F}_j$, etc., are defined in a natural way by the conditions of the basic classes. There exists 2^{11} classes of almost contact B-metric manifolds. The special class $\mathcal{F}_0 : F = 0$ is contained in each of the defined classes.

2 Time-like hypersurfaces of an almost complex manifold with B-metric

It is known, [9,10] that every differentiable orientiable hypersurface of an almost complex manifold has an almost contact structure. In [5] it is shown that on every real nonisitropic hypersurface of $\mathbb{R}^{2n+2}$ as complex Riemannian manifold with a canonical complex manifold and B-metric there is arised an almost contact structure with B-metric. In a similar way we construct a hypersurface of an almost complex manifold with B-metric.

Let (M', J, g') be a $(2n+2)$-dimensional almost complex manifold with B-metric, and M be a $(2n+1)$-dimensional differentiable hypersurface embedding in M' such that the normal vector field N to M is a time-like unit with respect to g', i.e. $g'(N, N) = -1$. At every point p for $t \in (-\frac{\pi}{2}; \frac{\pi}{2})$ we set

$$g'(N, JN) = \tan t.$$

We define the structural vector field ξ on M by the equalities:

$$\xi = \lambda.N + \mu.JN, \qquad g'(\xi, \xi) = 1, \qquad g'(\xi, N) = 0.$$

Then we have $\xi = \sin t.N + \cos t.JN$, $J\xi = -\cos t.N + \sin t.JN$, whence we receive

$$JN = \frac{1}{\cos t}\xi - \tan t.N, \qquad J\xi = \tan t.\xi - \frac{1}{\cos t}N.$$

From the last equality it is clear that $J\xi$ is transverse to M. The transform vector field JX of X has a tangent component to M denoted by φX and a component with respect to $J\xi$ denoted by $\eta(X)J\xi$, i.e. it is valid the unique decomposition $JX = \varphi X + \eta(X)J\xi$, where η is a differentiable 1-form on M. The last decomposition in tangent and normal components takes the following shape

$$JX = \varphi X + \tan t.\eta(X)\xi - \frac{1}{\cos t}\eta(X)N. \tag{6}$$

By such a way we define the structural (1,1)-tensor φ and the 1-form η in T_pM at an arbitrary point $p \in M$. The restriction of g' on M we denote by g. Then, because of (1), we get immediately (2)–(4). Thus, we obtain that (φ, ξ, η, g) is an almost contact B-metric structure on the hypersurface M. So, we give the following

Definition 2.1 The hypersurface M of an almost complex manifold with B-metric (M', J, g'), determined by the condition the normal unit N to be time-like regarding g', equipped with the almost contact B-metric structure

$$\varphi := J + \cos t.g'(\cdot, JN)\{\cos t.N - \sin t.JN\},$$
$$\xi := \sin t.N + \cos t.JN, \quad \eta := \cos t.g'(\cdot, JN), \quad g := g'|_M,$$

where $t := \arctan\{g'(N, JN)\}$ for $t \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$, will be called *a hypersurface of first type* of (M', J, g').

Let ∇' and ∇ be the Levi-Civita connections of the metrics g' on M' and g on M, respectively. Then the formulas of Gauss and Weingarten in this case are

$$\nabla'_X Y = \nabla_X Y - g(AX,Y)N, \qquad \nabla'_X N = -AX, \tag{7}$$

where A is the second fundamental tensor of M corresponding to N .

Using (6) and (7) we compute $(\nabla'_X J)Y$ and $(\nabla'_X J)N$, whence we obtain

$$\begin{aligned}
F'(X,Y,Z) &= F(X,Y,Z) \\
&\quad + \tan t\,\{F(X,\varphi Y,\xi)\eta(Z) + F(X,\varphi Z,\xi)\eta(Y)\} \\
&\quad + \tfrac{1}{\cos t}\{g(AX,Y)\eta(Z) + g(AX,Z)\eta(Y)\} \\
&\quad + \tfrac{1}{\cos^2 t} dt(X)\eta(Y)\eta(Z), \\
F'(X,Y,N) &= g(AX,\varphi Y) + \tan t.g(AX,Y) \\
&\quad + \tfrac{1}{\cos t}(\nabla_X \eta)Y + \tan t\left\{\tfrac{1}{\cos t} dt(X) + \eta(AX)\right\}\eta(Y), \\
F'(X,N,N) &= \tfrac{1}{\cos t}\left\{\tfrac{1}{\cos t} dt(X) + 2\eta(AX)\right\}.
\end{aligned}$$

In case when (M',J,g') is a Kaehlerian manifold with B-metric there are valid the following conditions for (M,φ,ξ,η,g):

$$\begin{aligned}
(\nabla_X \varphi)Y &= \eta(Y)\{\sin t.\varphi AX - \cos t.\eta(AX)\xi\} \\
&\qquad + \{\sin t.g(AX,\varphi Y) - \cos t.g(AX,Y)\}\xi\,, \\
(\nabla_X \eta)Y &= -\sin t.\{g(AX,Y) - \eta(AX)\eta(Y)\} \\
&\qquad - \cos t.g(AX,\varphi Y)\,, \\
d\eta(X,Y) &= \sin t.\{\eta(AX)\eta(Y) - \eta(AY)\eta(X)\} \\
&\qquad - \cos t.g((\varphi \circ A - A \circ \varphi)X,Y)\,,
\end{aligned}$$

$$\eta(AX) = -\frac{1}{2\cos t} dt(X)\,, \tag{8}$$

$$\nabla_X \xi = -\sin t\{AX - \eta(AX)\xi\} - \cos t.\varphi AX. \tag{9}$$

$$\begin{aligned}
F(X,Y,Z) &= \sin t\,\{g(AX,\varphi Y)\eta(Z) + g(AX,\varphi Z)\eta(Y)\} \\
&\qquad - \cos t\,\{g(AX,Y)\eta(Z) + g(AX,Z)\eta(Y) \\
&\qquad\qquad -2\eta(AX)\eta(Y)\eta(Z)\}\,,
\end{aligned} \tag{10}$$

$$\theta(Z) = \{\sin t.tr(A\circ\varphi) - \cos t.[trA - \eta(A\xi)]\}\eta(Z)\,,$$
$$\theta^*(Z) = \{-\sin t[trA - \eta(A\xi)] - \cos t.tr(A\circ\varphi)\}\eta(Z)\,,$$
$$\omega(Z) = -\tfrac{1}{2}\{dt(\varphi^2 Z) + \tan t.dt(\varphi Z)\}.$$

From (8), (9) and because of the symmetry of A regarding g, we obtain the general shape of the second fundamental tensor of the hypersurface of first type and the corresponding traces:

$$\begin{aligned} AX &= -\tfrac{dt(\xi)}{2\cos t}\eta(X)\xi - \sin t\{\nabla_X\xi + g(\nabla_\xi\xi, X)\xi\} \\ &\qquad\qquad + \cos t\{\varphi\nabla_X\xi + g(\varphi\nabla_\xi\xi, X)\xi\} \\ trA &= -\tfrac{dt(\xi)}{2\cos t} - \cos t.\theta(\xi)) - \sin t.\theta^*(\xi)) \\ tr(A\circ\varphi) &= \sin t.\theta(\xi)) - \cos t.\theta^*(\xi)). \end{aligned} \tag{11}$$

Substituting AX in (10) we get the general shape of F and its associated 1-forms on the hypersurface of first type are

$$F(X,Y,Z) = F(X,Y,\xi)\eta(Z) + F(X,\xi,Z)\eta(Y),$$
$$\theta(Z) = \theta(\xi)\eta(Z), \qquad \theta^*(Z) = \theta^*(\xi)\eta(Z).$$

Hence, according to (5), it is valid the following

Theorem 2.1 *Every hypersurface of first type is an almost contact B-metric manifold belongs to the class* $\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_7\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{11}$.

It is clear that 2^7 subclasses of hypersurfaces under consideration are possible but some subclasses are restricted to $\mathcal{F}_0$.

Now, we will give the characteristics of some subclasses by the second fundamental tensor of the submanifold.

Theorem 2.2 I) *The following classes of hypersurfaces of first type are characterized in terms of the second fundamental tensor A by the conditions:*

$$\begin{aligned} \mathcal{F}_0 &: \quad A = -\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi; \\ \mathcal{F}_4 &: \quad A = -\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi - \tfrac{\theta(\xi)}{2n}\{\sin t.\varphi - \cos t.\varphi^2\}; \\ \mathcal{F}_5 &: \quad A = -\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi + \tfrac{\theta^*(\xi)}{2n}\{\cos t.\varphi + \sin t.\varphi^2\}; \\ \mathcal{F}_6 &: \quad A\circ\varphi = \varphi\circ A\,,\ \ trA - \tfrac{dt(\xi)}{2\cos t} = tr(A\circ\varphi) = 0; \\ \mathcal{F}_{11} &: \quad A = -\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi - \cos t\{\eta\otimes\Omega + \omega\otimes\xi\} \\ &\qquad\qquad\qquad - \sin t\{\eta\otimes\varphi\Omega + (\omega\circ\varphi)\otimes\xi\}, \\ &\quad \textit{where}\quad \omega = g(\cdot,\Omega); \end{aligned}$$

$$\mathcal{F}_4\oplus\mathcal{F}_5 : \quad A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi-\tfrac{1}{2n}\{[\sin t.\theta(\xi)-\cos t.\theta^*(\xi)]\varphi \\ -[\cos t.\theta(\xi)+\sin t.\theta^*(\xi)]\varphi^2\};$$

$$\mathcal{F}_4\oplus\mathcal{F}_6 : \quad A\circ\varphi=\varphi\circ A\,, \quad \sin t\left\{trA-\tfrac{dt(\xi)}{2\cos t}\right\}+\cos t.tr(A\circ\varphi)=0;$$

$$\mathcal{F}_4\oplus\mathcal{F}_{11} : \quad A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi+\cos t\left\{\tfrac{\theta(\xi)}{2n}\varphi^2-\eta\otimes\Omega-\omega\otimes\xi\right\} \\ -\sin t\left\{\tfrac{\theta(\xi)}{2n}\varphi+\eta\otimes\varphi\Omega+(\omega\circ\varphi)\otimes\xi\right\};$$

$$\mathcal{F}_5\oplus\mathcal{F}_6 : \quad A\circ\varphi=\varphi\circ A\,, \quad \cos t\left\{trA-\tfrac{dt(\xi)}{2\cos t}\right\}-\sin t.tr(A\circ\varphi)=0;$$

$$\mathcal{F}_5\oplus\mathcal{F}_{11} : \quad A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi+\cos t\left\{\tfrac{\theta^*(\xi)}{2n}\varphi-\eta\otimes\Omega-\omega\otimes\xi\right\} \\ +\sin t\left\{\tfrac{\theta^*(\xi)}{2n}\varphi^2-\eta\otimes\varphi\Omega-(\omega\circ\varphi)\otimes\xi\right\};$$

$$\mathcal{F}_6\oplus\mathcal{F}_{11} : \quad \varphi^2\circ A\circ\varphi=\varphi\circ A\circ\varphi^2,\ trA-\tfrac{dt(\xi)}{2\cos t}=tr(A\circ\varphi)=0;$$

$$\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6 : \quad A\circ\varphi=\varphi\circ A;$$

$$\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_{11} : \quad A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi \\ +\cos t\left\{\tfrac{\theta(\xi)}{2n}\varphi^2+\tfrac{\theta^*(\xi)}{2n}\varphi-\eta\otimes\Omega-\omega\otimes\xi\right\} \\ -\sin t\left\{\tfrac{\theta(\xi)}{2n}\varphi-\tfrac{\theta^*(\xi)}{2n}\varphi^2+\eta\otimes\varphi\Omega+(\omega\circ\varphi)\otimes\xi\right\};$$

$$\mathcal{F}_4\oplus\mathcal{F}_6\oplus\mathcal{F}_{11} : \quad \varphi^2\circ A\circ\varphi=\varphi\circ A\circ\varphi^2\,, \\ \sin t\left\{trA-\tfrac{dt(\xi)}{2\cos t}\right\}+\cos t.tr(A\circ\varphi)=0;$$

$$\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_{11} : \quad \varphi^2\circ A\circ\varphi=\varphi\circ A\circ\varphi^2\,, \\ \cos t\left\{trA-\tfrac{dt(\xi)}{2\cos t}\right\}-\sin t.tr(A\circ\varphi)=0;$$

$$\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_{11} : \quad \varphi^2\circ A\circ\varphi=\varphi\circ A\circ\varphi^2;$$

II) *A hypersurface of first type can't belong to the classes $\mathcal{F}_7\,,\mathcal{F}_8\,,\mathcal{F}_9$, to their direct sums or to the direct sums of someone of them with someone of $\mathcal{F}_4\,,\mathcal{F}_5$ or $\mathcal{F}_6$.*

Proof. Having in mind (5), the covariant derivative of ξ can be expressed explicite in the subclasses $\mathcal{F}_0,\mathcal{F}_4,\mathcal{F}_5,\mathcal{F}_{11},\mathcal{F}_4\oplus\mathcal{F}_5,\mathcal{F}_4\oplus\mathcal{F}_{11},\mathcal{F}_5\oplus\mathcal{F}_{11}$ and $\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_{11}$. Then A takes the corresponding shape for considered classes.

Now, let us consider the classes $\mathcal{F}_6,\mathcal{F}_7,\mathcal{F}_8,\mathcal{F}_9$ and their direct sums. The conditions $F(X,Y,\xi)=F(Y,X,\xi)$ and $F(\varphi X,\varphi Y,\xi)=-F(X,Y,\xi)$ imply individually $A\circ\varphi=\varphi\circ A$, and the commutation of A and φ implies the properties $F(X,Y,\xi)=F(Y,X,\xi)=-F(\varphi X,\varphi Y,\xi)$.

Besides, each of properties $F(X,Y,\xi)=-F(Y,X,\xi)$ and $F(\varphi X,\varphi Y,\xi)=$

$F(X,Y,\xi)$ follows to $A\circ\varphi=-\varphi\circ A$, and the anticommutation of A and φ implies the properties $F(X,Y,\xi)=-F(Y,X,\xi)=F(\varphi X,\varphi Y,\xi)$.

Obviously, if we suppose that a hypersurface of first type belongs to $\mathcal{F}_7$ or to $\mathcal{F}_8$, we obtain that it is only an $\mathcal{F}_0$-manifold.

The conditions for F on $\mathcal{F}_9$-manifold imply the anticommutation of A and φ. In other hand the 1-form η is clossed on $\mathcal{F}_9$-manifold as well as on a manifold in $\mathcal{F}_4\,,\mathcal{F}_5$ and $\mathcal{F}_6$. Taking into account (8) and (9) we obtain $A\circ\varphi=\varphi\circ A$. Hence a hypersurface of first type can't be in the class $\mathcal{F}_9$ without $\mathcal{F}_0$.

It is easy to get the propositions about $\mathcal{F}_6\,,\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_{11}$ and the remaining classes in I).

The part II) of the theorem is received immediately from the explanations above. Q.E.D.

We shall give geometric interpretation of some of these classes.

We recall, if $A=0$, $trA=0$ or $A=\lambda I$, then the corresponding hypersurface is totally geodesical, minimal or umbilical, respectively.

Having in mind (11) and (5), we get

Proposition 2.3 **(i)** *The totally geodesical hypersurfaces of first type form a subclass of $\mathcal{F}_0$ with the condition $t=const$;*

(ii) *The minimal hypersurfaces of first type form a subclass of the general class of hypersurfaces with the condition* $\cos t.\theta(\xi)+\sin t.\theta^*(\xi)+\frac{dt(\xi)}{2\cos t}=0.$

(iii) *The umbilical hypersurfaces of first type form a subclass of $\mathcal{F}_4\oplus\mathcal{F}_5$ with the condition* $\sin t.\theta(\xi)-\cos t.\theta^*(\xi)=0$. *There can't exist umbilical hypersurfaces of first type in $\mathcal{F}_4$ or $\mathcal{F}_5$. The umbilical $\mathcal{F}_0$-hypersurfaces of first type are totally geodesical.*

Remark. If $M'=\mathbb{R}^{2n+2}$, then it is received the example in [5] of the time-like unit hypersphere of the class $\mathcal{F}_4\oplus\mathcal{F}_5$, which is an umbilic hypersurface of $\mathbb{R}^{2n+2}$.

3 Isotropic hypersurfaces regarding the associated metric of an almost complex manifold with B-metric

In [5] it is defined an real isotropic hypersurface of the associated metric in $\mathbb{R}^{2n+2}$, considered as a complex Riemannian manifold with a canonical complex structure and B-metric. In a similar way we introduce an isotropic hypersurface regarding the associated metric of an almost complex manifold with B-metric.

Let (M',J,g'), $dimM=2n+2$, be an almost complex manifold with B-metric. We determine a $(2n+1)$-dimensional differentiable hypersurface M

embedding in M' by the condition $M:\tilde{g}'(Z,Z)=0$ for a vector field Z on M'. It is clear that Z and its transform vector field JZ by J are orthogonal with respect to the B-metric g', i.e. $g'(Z,JZ)=0$. At every point we put $g'(Z,Z)=\cosh^2 t,\ t>0$ for the sake of the impossibility Z to be a main isotropic direction and in view of definiteness.

We can choose the time-like unit normal $N=\frac{1}{\cosh t}JZ$, i.e. $g'(N,N)=-1$. Hence, JN is a space-like unit tangent vector field on M.

We determine the structural vector field ξ on M by

$$\xi=-JN=\tfrac{1}{\cosh t}Z.$$

Then the vector field $J\xi$ coincides with N.

Thus, in this case the introducing of the structural (1,1)-tensor φ and 1-form η is made by the unique orthogonal decomposition

$$Jx=\varphi x+\eta(x)N. \tag{12}$$

The restriction of g' on M we denote by g. Then, ascertaining (2)–(4), we obtain that (φ,ξ,η,g) is an almost contact B-metric structure on the hypersurface M. In such a way we can formulate

Definition 3.1 The hypersurface M of an almost complex manifold with B-metric (M',J,g'), determined by the condition the normal unit N to be isotropic regarding the associated B-metric $\tilde{g}'$ of g', equipped with the almost contact B-metric structure

$$\varphi:=J+g'(\cdot,JN)N,\quad \xi:=-JN,\quad \eta:=-g'(\cdot,JN),\quad g:=g'|_M$$

will be called *a hypersurface of second type* of (M',J,g').

Now, we shall study a classification of these manifolds with respect to the second fundamental tensor A of the hypersurface.

The formulas of Gauss and Weingarten are

$$\nabla'_X Y=\nabla_X Y-g(AX,Y)N,\qquad \nabla'_X N=-AX. \tag{13}$$

Taking into account (12) and (13) we compute $(\nabla'_X J)Y$ and $(\nabla'_X J)N$. Then we get

$$\begin{aligned}&F'(X,Y,Z)=F(X,Y,Z)-g(AX,Y)\eta(Z)-g(AX,Z)\eta(Y),\\&F'(X,Y,N)=-(\nabla_X\eta)Y+g(AX,\varphi Y),\quad F'(X,N,N)=-2\eta(AX).\end{aligned}$$

In case when (M',J,g') is a Kaehlerian manifold with B-metric, the left hand sites of the last equalities are vanished. Then we obtain the following conditions for $(M, \varphi, \xi, \eta, g)$:

$$(\nabla_X \varphi)Y = \eta(Y)AX + g(AX, Y)\xi,$$
$$(\nabla_X \eta)Y = g(AX, \varphi Y),$$
$$d\eta(X, Y) = g((\varphi \circ A - A \circ \varphi)X, Y),$$
$$\eta(AX) = 0 \ (\iff \ A\xi = 0), \quad \nabla_X \xi = \varphi AX, \tag{14}$$
$$F(X, Y, Z) = g(AX, Y)\eta(Z) + g(AX, Z)\eta(Y), \tag{15}$$
$$\theta(Z) = trA.\eta(Z), \quad \theta^*(Z) = tr(A \circ \varphi)\eta(Z), \quad \omega(Z) = 0.$$

Because of (15) the second fundamental tensor on the hypersurface of second type is $AX = -\varphi \nabla_X \xi$.

From (16) it follows $F(X, Y, \xi) = g(AX, Y)$ and in view of the symmetry of A regarding g we get the symmetry $F(X, Y, \xi) = F(Y, X, \xi)$. Besides, for F, θ , θ^* and ω on the hypersurface of second type we receive

$$F(X, Y, Z) = F(X, Y, \xi)\eta(Z) + F(X, \xi, Z)\eta(Y),$$
$$\theta = \theta(\xi)\eta, \quad \theta^* = \theta^*(\xi)\eta, \quad \omega = 0.$$

Having in mind the classification (5) and the results above, we ascertain the truthfulness of the following

Theorem 3.1 *Every hypersurface of second type is an almost contact B-metric manifold belongs to the class* $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_8$.

Obviously, 16 subclasses of hypersurfaces of second type are possible. Now, we shall write up some of these classes with respect to A.

Theorem 3.2 *Some classes of the hypersurfaces of second type are characterized by the second fundamental tensor A as follows:*

$$\mathcal{F}_0 : \quad A = 0;$$
$$\mathcal{F}_4 : \quad A = -\tfrac{\theta(\xi)}{2n}\varphi^2;$$
$$\mathcal{F}_5 : \quad A = -\tfrac{\theta^*(\xi)}{2n}\varphi;$$
$$\mathcal{F}_6 : \quad A \circ \varphi = \varphi \circ A\,, \quad A\xi = 0\,, \quad trA = tr(A \circ \varphi) = 0;$$
$$\mathcal{F}_8 : \quad A \circ \varphi = -\varphi \circ A\,, \quad A\xi = 0;$$

$$\begin{aligned}
\mathcal{F}_4 \oplus \mathcal{F}_5 &: \quad A = -\tfrac{1}{2n}\{\theta(\xi)\varphi^2 + \theta^*(\xi)\varphi\}; \\
\mathcal{F}_4 \oplus \mathcal{F}_6 &: \quad A\circ\varphi = \varphi\circ A\,, \quad A\xi = 0\,, \quad tr(A\circ\varphi) = 0; \\
\mathcal{F}_5 \oplus \mathcal{F}_6 &: \quad A\circ\varphi = \varphi\circ A\,, \quad A\xi = 0\,, \quad trA = 0\,; \\
\mathcal{F}_6 \oplus \mathcal{F}_8 &: \quad A\xi = 0\,, \quad trA = tr(A\circ\varphi) = 0; \\
\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 &: \quad A\circ\varphi = \varphi\circ A\,, \quad A\xi = 0;
\end{aligned}$$

We give a geometrical interpretation of some of these classes according to the results of the last theorem by the following

Proposition 3.3 **(i)** *The totally geodesical hypersurfaces of second type are the $\mathcal{F}_0$-manifolds;*

(ii) *The minimal hypersurfaces of second type are the $\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_8$-manifolds;*

(iii) *Every umbilical hypersurface of second type is totally geodesical hypersurface, i.e. it belongs to the class $\mathcal{F}_0$.*

Remark. If $M' = \mathbb{R}^{2n+2}$, then it is obtained the example in [5] of the isotropic hypersphere of $\tilde{g}'$ belonging to $\mathcal{F}_5$ and it is a minimal hypersurface of $\mathbb{R}^{2n+2}$.

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