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Classes of real time-like hypersurfaces of a Kaehler manifold with B-metric

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Abstract. There are considered real hypersurfaces of a Kaehler manifold with a time-like normal unit regarding the B-metric and there are obtained four basic classes of such hypersurfaces as almost contact B-metric manifolds. The generated sixteen classes of the considered hypersurfaces are described with respect to the second fundamental form. There is constructed an example of a 3-dimensional manifold of the 11th basic class as a hypersurface of the considered type of a holomorphic sphere in 6-dimensional real space.

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1. Introduction

A manifold over the field of the complex numbers induces an even-dimensional manifold M' over the field of the real numbers and an antiinvolutive endomorphism J of the tangent vector fields on M'. The geometry of the received almost complex manifold (M', J) is governed by the almost complex structure J. If (M', J) is equipped with a metric g' then there are obtained two kinds of manifolds according to the compatibility of g' with J. When J acts as an isometry, then there is said that (M', J, g') is an almost Hermitian manifold. In the other case, when J acts as an antiisometry, then (M', J, g') is an almost complex manifold with B-metric.

On the other hand, the natural extension of J on an odd-dimensional manifold M is the almost contact structure. Thereby, in the first case we receive an almost contact metric manifold, and in the second one – an almost contact manifold with B-metric.

The geometries of the almost Hermitian and the almost contact metric manifolds are well studied. The investigation's beginning in the geometry of the almost complex manifolds with B-metric is put by Norden [18] and the researches have been continued by Ganchev, Borisov, Gribachev, Mihova (e.g. [3]–[5]).

An almost complex B-metric manifold is a Kaehler manifold with B-metric, if the almost complex structure is parallel with respect to the Levi-Civita connection of the B-metric. The class of these manifolds is contained in each other class of the almost complex B-metric manifolds [3]. An example of a Kaehler B-metric manifold is considered in [5]. This is the so-called h-sphere, i.e. a $2n$-dimensional holomorphic hypersurface with constant totally real sectional curvatures in $\mathbb{R}^{2n+2}$.

The geometry of the almost contact B-metric manifolds is a natural extension of the geometry of the almost complex manifolds with B-metric to the odd dimensional case. Ganchev, Mihova and Gribachev have introduced and classified the almost contact manifolds with B-metric in [6]. The development of the geometry of these manifolds is made by Gribachev, Manev and Nakova (e.g. [8]–[17]). There are studied some examples of manifolds from the basic classes obtained as submanifolds [6], [16], [17], [12], [13].

The present paper continues the study of the real time-like hypersurfaces of a Kaehler manifold with B-metric. Our aim is to describe all possible classes of the considered hypersurfaces. In Section 2 we recall different notions needed for later. Our new results are formulated in Section 3 and some explicit constructions are given in Section 4.

2. Preliminaries

Almost complex manifold with B-metric. Let (M', J, g') be a $2n'$-dimensional almost complex manifold with B-metric, i.e. J is an almost complex structure and g' is a metric on M' such that:

$$J^2X = -X, \qquad g'(JX, JY) = -g'(X, Y).$$

for all vector fields X, Y on M'. The associated metric $\tilde{g}$ of the manifold is given by $\tilde{g}(X, Y) = g'(X, JY)$. Both metrics are necessarily of signature (n', n').

The class of the Kaehler manifolds with B-metric is determined by the condition J to be parallel with respect to the Levi-Civita connection of g'.

Almost contact manifold with B-metric. Let $(M, \varphi, \xi, \eta, g, \tilde{g})$ be a $(2n+1)$-dimensional almost contact manifold with B-metric [6], i.e. at first (φ, ξ, η) is an almost contact structure determined by a tensor field φ of type $(1, 1)$, by a vector field ξ and by an 1-form η on M according to the conditions: [2]

$$\varphi^2X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1.$$

In addition this almost contact manifold (M, φ, ξ, η) admits a metric g, called B-metric, such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

where X,Y are arbitrary differentiable vector fields on M, i.e. $X,Y \in \mathfrak{X}(M)$.

Further, X,Y,Z will stand for arbitrary differentiable vector fields on M and x,y,z – arbitrary vectors in the tangential space T_pM to M at an arbitrary point p in M.

We have in mind the following immediate consequences of the above conditions:

$$\eta\circ\varphi=0,\ \varphi\xi=0,\ \operatorname{rank}\varphi=2n,\ \eta(X)=g(X,\xi),$$
$$g(\xi,\xi)=1,\ g(\varphi X,Y)=g(X,\varphi Y).$$

The associated metric $\tilde{g}$ given by $\tilde{g}(X,Y)=g(X,\varphi Y)+\eta(X)\eta(Y)$ is a B-metric, too. Both metrics g and $\tilde{g}$ are indefinite of signature $(n,n+1)$.

Let ∇ be the Levi-Civita connection of the metric g. The tensor F of type $(0,3)$ on M is defined by $F(x,y,z)=g((\nabla_x\varphi)y,z)$ and it has the following properties:

$$F(x,y,z)=F(x,z,y)=F(x,\varphi y,\varphi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi).$$

Each fiber T_pM of the tangent bundle TM is a $(2n+1)$-dimensional vector space with an almost contact B-metric structure.

The decomposition $T_pM=\{D_p=\ker(\eta_p)\}\oplus span\{\xi_p\}$ is orthogonal and invariant with respect to the structural group $(GL(n,\mathbb{C})\cap O(n,n))\times I$. The $2n$-dimensional vector space D_p is equipped with a complex structure φ_p and B-metrics $g_p,\ \tilde{g}_p$.

Let $\{e_i,\xi\}$ $(i=1,2,\ldots,2n)$ be a basis of T_pM, and (g^{ij}) be the inverse matrix of (g_{ij}) then the following 1-forms are associated with F:

$$\theta(\cdot)=g^{ij}F(e_i,e_j,\cdot),\ \ \theta^*(\cdot)=g^{ij}F(e_i,\varphi e_j,\cdot),\ \ \omega(\cdot)=F(\xi,\xi,\cdot).$$

A classification of the almost contact manifolds with B-metric is given in [6]. It contains eleven basic classes $\mathcal{F}_i$ defined with respect to the tensor F. We shall use the following characteristic conditions of the considered classes:

$$\begin{aligned}
\mathcal{F}_4:\quad & F(x,y,z)=-\tfrac{\theta(\xi)}{2n}\{g(\varphi x,\varphi y)\eta(z)+g(\varphi x,\varphi z)\eta(y)\};\\
\mathcal{F}_5:\quad & F(x,y,z)=-\tfrac{\theta^*(\xi)}{2n}\{g(x,\varphi y)\eta(z)+g(x,\varphi z)\eta(y)\};\\
\mathcal{F}_6:\quad & F(x,y,z)=f(x,y)\eta(z)+f(x,z)\eta(y),\\
& f(x,y)=f(y,x),\quad f(\varphi x,\varphi y)=-f(x,y),\quad \theta(\xi)=\theta^*(\xi)=0;\\
\mathcal{F}_7:\quad & F(x,y,z)=f(x,y)\eta(z)+f(x,z)\eta(y),\\
& f(x,y)=-f(y,x),\quad f(\varphi x,\varphi y)=-f(x,y) \qquad (1)\\
\mathcal{F}_8:\quad & F(x,y,z)=f(x,y)\eta(z)+f(x,z)\eta(y),\\
& f(x,y)=f(y,x),\quad f(\varphi x,\varphi y)=f(x,y)\\
\mathcal{F}_9:\quad & F(x,y,z)=f(x,y)\eta(z)+f(x,z)\eta(y),\\
& f(x,y)=-f(y,x),\quad f(\varphi x,\varphi y)=f(x,y)\\
\mathcal{F}_{11}:\quad & F(x,y,z)=\eta(x)\{\eta(y)\omega(z)+\eta(z)\omega(y)\},
\end{aligned}$$

where $f(x,y)=F(\varphi^2x,\varphi^2y,\xi)$, instead of the known ones of [6].

The classes $\mathcal{F}_i \oplus \mathcal{F}_j$, etc., are defined in a natural way by the conditions of the basic classes. There exist 2^{11} classes of almost contact B-metric manifolds. The special class $\mathcal{F}_0$: $F = 0$ is contained in each of the defined classes.

Real hypersurfaces. In [12] two types of real hypersurfaces of a complex manifold with B-metric were introduced. The obtained submanifolds are almost contact B-metric manifolds. Let us recall, that the real time-like hypersurface M of an almost complex manifold with B-metric $(M'^{2n+2}, J, g', \tilde{g}')$ is determined by the condition the normal unit N to be time-like regarding g'. Moreover, M is equipped with the almost contact B-metric structure

$$\begin{aligned} &\varphi := J + \cos t\, g'(\cdot, JN)\{\cos t\, N - \sin t\, JN\}, \quad \xi := \sin t\, N + \cos t\, JN, \\ &\eta := \cos t\, g'(\cdot, JN), \quad g := g'|_M, \quad t := \arctan\{g'(N, JN)\}, \quad t \in \left(-\tfrac{\pi}{2}; \tfrac{\pi}{2}\right). \end{aligned} \tag{2}$$

Let ∇' and ∇ be the Levi-Civita connections of g' on M' and g on M, respectively. If $h(X, Y) = g(AX, Y)$ is the second fundamental form of the hypersurface M, then the formulas of Gauss and Weingarten in this case are:

$$\nabla'_X Y = \nabla_X Y - h(X, Y)N, \qquad \nabla'_X N = -AX.$$

In [12] we find the classification tensor of the time-like hypersurface of a Kaehler manifold with B-metric and in consequence we receive the following type of F:

$$\begin{aligned} F(X, Y, Z) &= f(X, Y)\eta(Z) + f(X, Z)\eta(Y) \\ &+ \eta(X)\{\eta(Y)\omega(Z) + \eta(Z)\omega(Y)\}, \\ f(X, Y) &= -\sin t\, h(\varphi^2 X, \varphi Y) + \cos t\, h(\varphi X, \varphi Y). \end{aligned} \tag{3}$$

This result means that the considered hypersurface belongs to the class $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \cdots \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}$. In the same work there are given the characteristic conditions in terms of A only for the classes $\mathcal{F}_0$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_{11}$ and for the their direct sums.

We shall consider the orthogonal and invariant decomposition of h with respect to the structural group. Using this decomposition we get four basic components and the corresponding classes which generate all sixteen possible classes. Thereby we give conditions for h which characterize those real time-like hypersurfaces belonging to each of the sixteen classes.

3. The sixteen classes

The tensor $f(X, Y)$ can satisfy the property of symmetry or the property of antisymmetry. On the other hand, $f(X, Y)$ can be pure or hybrid with respect to the action of φ, i.e. $f(\varphi X, \varphi Y) = f(X, Y)$ or $f(\varphi X, \varphi Y) = -f(X, Y)$, respectively. The combination of these two kinds of properties of f implies the possibility f to be only symmetric and hybrid. The equivalent condition in terms of h is the following

$$h(\varphi^2 X, \varphi^2 Y) + h(\varphi X, \varphi Y) = 0. \tag{4}$$

The remaining cases imply $f = 0$.

In this way we obtain a more precise determination of the class of the considered hypersurfaces stated in the next theorem.

THEOREM 1. *The class* $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{11}$ *is the class of the real time-like hypersurfaces of a Kaehler manifold with* B*-metric. There are* 16 *classes of these hypersurfaces in all.*

REMARK 2. When $n = 1$ the class $\mathcal{F}_6$ is restricted to $\mathcal{F}_0$. Therefore, for a 4-dimensional Kaehler manifold with B-metric there are only 8 classes of the considered hypersurfaces.

Now we characterize these classes in terms of their second fundamental form h. Since $X = -\varphi^2 X + \eta(X)\xi$, $Y = -\varphi^2 Y + \eta(Y)\xi$, then

$$\begin{aligned} h(X,Y) = h(\varphi^2 X, \varphi^2 Y) - \eta(X)h(\xi, \varphi^2 Y) \\ -\eta(Y)h(\varphi^2 X, \xi) + \eta(X)\eta(Y)h(\xi,\xi). \end{aligned}$$

Having in mind $dt(X) = -2\cos t.h(X,\xi)$, $\omega(X) = \frac{1}{2}[dt(\varphi^2 X) + \tan t.dt(\varphi X)]$ from [12] and Equations (3) and (4), we denote the following symmetric tensors:

$$\begin{aligned} h_0(X,Y) &= \eta(X)\eta(Y)h(\xi,\xi) = -\tfrac{dt(\xi)}{2\cos t}\eta(X)\eta(Y), \\ h_{11}(X,Y) &= -\sin t[\eta(X)\omega(\varphi Y) + \eta(Y)\omega(\varphi X)] \\ &\quad - \cos t[\eta(X)\omega(Y) + \eta(Y)\omega(X)]. \end{aligned} \tag{5}$$

The component $h(\varphi^2\cdot, \varphi^2\cdot)$ of h is a $(0,2)$-tensor over the complex B-metric vector space $(D_p, \varphi_p, g_p, \tilde{g}_p)$.

Let $(V, J, g', \tilde{g}')$ be a $2n$-dimensional vector space with a complex structure J and B-metrics g' and $\tilde{g}'$. Let V^* denote the dual space of V. We consider the space $V^* \otimes V^*$, i.e. the vector space of the tensors of type $(0,2)$ over V. The metric g' induces an inner product $\langle\, , \,\rangle$ on $V^* \otimes V^*$, given by

$$\langle f_1, f_2 \rangle = g^{ij} g^{ks} f_1(e_i, e_k) f_2(e_j, e_s)$$

for f_1, f_2 in $V^* \otimes V^*$ and $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ – a basis of V. With every f in $V^* \otimes V^*$ we associate the functions: $\mathrm{tr} f = g^{ij} f(e_i, e_j)$, $\mathrm{tr}^* f = g^{ij} f(e_i, Je_j)$. We denote $G = GL(n, \mathbb{C}) \cap O(n,n)$. The standard representation of G on V induces a natural representation λ of G on $V^* \otimes V^*$ and

$$\langle (\lambda a) f_1, (\lambda a) f_2 \rangle = \langle f_1, f_2 \rangle;\ a \in G;\ f_1, f_2 \in V^* \otimes V^*.$$

Let us consider the vector subspace W of $V^* \otimes V^*$ of the symmetric and J-hybrid $(0,2)$-tensors over V. We remark that g and $\tilde{g}$ are elements of W. Then it is easy to check the truthfulness of the following

LEMMA 3. *Every symmetric and J-hybrid $(0,2)$-tensor $f(x,y)$ over $(V,J,g',\tilde{g}')$, $\dim V = 2n$, has three orthogonal and invariant components with respect to the action of G on W:*

$$f_1(x,y) = \frac{\operatorname{tr} f}{2n} g(x,y), \qquad f_2(x,y) = -\frac{\operatorname{tr}^* f}{2n} g(x,Jy),$$
$$f_3(x,y) = -f(Jx,Jy) - \frac{\operatorname{tr} f}{2n} g(x,y) + \frac{\operatorname{tr}^* f}{2n} g(x,Jy).$$

Since h is a symmetric and J-hybrid $(0,2)$-tensor over $(D_p, \varphi_p, g_p, \tilde{g}_p)$, we can apply the last lemma for the corresponding component of h, having in mind the interconnection

$$h(X,Y) = -\cos t\, f(X,Y) - \sin t\, f(X,\varphi Y).$$

We denote the following symmetric tensors:

$$\begin{aligned}
h_4(X,Y) &= \frac{\theta(\xi)}{2n}\{\cos t\, g(\varphi X,\varphi Y) - \sin t\, g(X,\varphi Y)\},\\
h_5(X,Y) &= \frac{\theta^*(\xi)}{2n}\{\sin t\, g(\varphi X,\varphi Y) + \cos t\, g(X,\varphi Y)\},\\
h_6(X,Y) &= \frac{1}{2}[\sin t\,(\mathcal{L}_\xi g)(\varphi X,\varphi Y) - \cos t\,(\mathcal{L}_\xi g)(\varphi X,\varphi^2 Y)]\\
&\quad - \frac{1}{2n}\{[\theta(\xi)\cos t + \theta^*(\xi)\sin t]g(\varphi X,\varphi Y)\\
&\quad - [\theta(\xi)\sin t - \theta^*(\xi)\cos t]g(X,\varphi Y)\},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
(\mathcal{L}_\xi g)(X,Y) &= (\nabla_X \eta)Y + (\nabla_Y \eta)X\\
&= \sin t[h(X,\varphi^2 Y) + h(Y,\varphi^2 X)] - \cos t[h(X,\varphi Y) + h(Y,\varphi X)].
\end{aligned}$$

Therefore the second fundamental form of the considered hypersurface has the form

$$h = h_0 + h_4 + h_5 + h_6 + h_{11}.$$

Taking into account (1) and (3), we describe the mentioned sixteen classes in terms of h. This is our main result in the present paper.

THEOREM 4. *The sixteen classes of real time-like hypersurfaces of a Kaehler manifold with B-metric are characterized in terms of their second fundamental form h as follows:*

$$\begin{array}{llll}
\mathcal{F}_0: & h = h_0; & \mathcal{F}_i \oplus \mathcal{F}_j \oplus \mathcal{F}_k: & h = h_0 + h_i + h_j + h_k;\\
\mathcal{F}_i: & h = h_0 + h_i; & (i,j,k = 4,5,6,11; & i \neq j \neq k \neq i)\\
\mathcal{F}_i \oplus \mathcal{F}_j: & h = h_0 + h_i + h_j; & \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{11}: & h = \sum_s h_s,
\end{array}$$

where the components h_s $(s = 0,4,5,6,11)$ are given in (5) *and* (6).

4. An example of an $\mathcal{F}_{11}$-manifold

In this section we construct an example of a 3-dimensional real time-like hypersurface of an h-sphere. We show that it belongs to the eleventh basic class. This example is the first one of an $\mathcal{F}_{11}$-manifold obtained as a submanifold.

Let $(\mathbb{C}^m, G)$ is a complex Euclidean space, where

$$G(z, w) = z^1 w^1 + \cdots + z^m w^m, \quad z = (z^1, \ldots, z^m),\ w = (w^1, \ldots, w^m) \in \mathbb{C}^m.$$

The decomplexification $r : \mathbb{C}^m \longrightarrow \mathbb{R}^{2m}$ is determined by

$$r : z = (z^1, \ldots, z^m) \longrightarrow r(z) = x = (x^1, \ldots, x^m, x^{m+1}, \ldots, x^{2m}),$$

where $z^k = x^k + ix^{m+k}$, $k = 1, 2, \ldots, m$. The complex structure induces a canonical complex structure J on $\mathbb{R}^{2m}$ by the following way:

$$iz \longrightarrow Jz = (-x^{m+1}, \ldots, -x^{2m}, x^1, \ldots, x^m).$$

There are induced two metrics on $\mathbb{R}^{2m}$ by G:

$$\bar{g}(x, y) = -\mathrm{Re}G(z, w) = -x^1 y^1 - \cdots - x^m y^m + x^{m+1} y^{m+1} + \cdots + x^{2m} y^{2m},$$
$$\tilde{\bar{g}}(x, y) = \mathrm{Im}G(z, w) = x^1 y^{m+1} + \cdots + x^m y^{2m} + x^{m+1} y^1 + \cdots + x^{2m} y^m.$$

Then $(\mathbb{R}^{2m}, J, \bar{g}, \tilde{\bar{g}})$ is a Kaehler manifold with B-metric. There is defined an h-sphere S^{2n} in $\mathbb{R}^{2n+2}$ at the origin with parameters the real numbers a and b:

$$S^{2n} : \bar{g}(x, x) = a,\ \tilde{\bar{g}}(x, x) = b, \quad (a, b) \neq (0, 0).$$

The h-sphere S^{2n} is a holomorphic hypersurface of $\mathbb{R}^{2n+2}$ and a $2n$-dimensional Kaehler manifold with B-metric and constant totally real sectional curvatures. [5]
Let us consider a central sphere with a real radius a in $(\mathbb{C}^3, G)$

$$\bar{S}^2:\ z^1 = a\cos(u^1 + iu^2)\cos(u^3 + iu^4),\ z^2 = a\cos(u^1 + iu^2)\sin(u^3 + iu^4),$$
$$z^3 = a\sin(u^1 + iu^2),$$

where $u^1, u^2, u^3, u^4 \in \mathbb{R}$, $(u^1 + iu^2) \neq \frac{\pi}{2}$.

The real interpretation of $\bar{S}^2$ in $\mathbb{R}^6$ is the h-sphere S^4:

$$\begin{aligned}
x^1 &= \tfrac{a}{2}[\cos(u^1 - u^3)\cosh(u^2 - u^4) + \cos(u^1 + u^3)\cosh(u^2 + u^4)],\\
x^2 &= \tfrac{a}{2}[\sin(u^1 + u^3)\cosh(u^2 + u^4) - \sin(u^1 - u^3)\cosh(u^2 - u^4)],\\
x^3 &= a\sin u^1 \cosh u^2,\\
x^4 &= -\tfrac{a}{2}[\sin(u^1 + u^3)\sinh(u^2 + u^4) + \sin(u^1 - u^3)\sinh(u^2 - u^4)],\\
x^5 &= -\tfrac{a}{2}[\cos(u^1 - u^3)\sinh(u^2 - u^4) - \cos(u^1 + u^3)\sinh(u^2 + u^4)],\\
x^6 &= a\cos u^1 \sinh u^2, \qquad (u^1, u^2) \neq (\tfrac{\pi}{2}, 0).
\end{aligned} \tag{7}$$

In this case we get $g'_{ij} = g'(\frac{\partial x}{\partial u^i}, \frac{\partial x}{\partial u^j})$ for $\frac{\partial x}{\partial u^i}(\frac{\partial x^1}{\partial u^i}, \dots, \frac{\partial x^6}{\partial u^i})$ $(i, j = 1, 2, 3, 4)$:

$$g'_{11} = -g'_{22} = -a^2,\ g'_{33} = -g'_{44} = -\frac{a}{2}(1 + \cos 2u^1 \mathrm{ch} 2u^2),$$
$$g'_{34} = -\frac{a}{2}\sin 2u^1 \mathrm{sh} 2u^2$$

and the rest of g'_{ij} are 0. Hence the components of the connection $\Gamma'^k_{ij} = \frac{1}{2} g'^{ks}(\frac{\partial}{\partial u^j} g'_{is} + \frac{\partial}{\partial u^i} g'_{js} - \frac{\partial}{\partial u^s} g'_{ij}),\ i, j, k, s \in \{1, 2, 3, 4\}$ are:

$$\Gamma'^3_{13} = \Gamma'^4_{14} = \Gamma'^4_{23} = -\Gamma'^3_{24} = -\frac{\sin 2u^1}{\cos 2u^1 + \cosh 2u^2},$$
$$\Gamma'^1_{33} = \Gamma'^2_{34} = -\Gamma'^1_{44} = \tfrac{1}{2}\sin 2u^1 \cosh 2u^2,$$
$$\Gamma'^4_{13} = -\Gamma'^3_{14} = -\Gamma'^3_{23} = -\Gamma'^4_{24} = -\frac{\sinh 2u^2}{\cos 2u^1 + \cosh 2u^2},$$
$$\Gamma'^2_{33} = -\Gamma'^1_{34} = -\Gamma'^2_{44} = \tfrac{1}{2}\cos 2u^1 \sinh 2u^2.$$

By substituting $e_k = \frac{1}{\sqrt{|g'_{kk}|}} \frac{\partial x}{\partial u^k}$ we obtain the basis $\{e_k\}$, $k \in \{1, 2, 3, 4\}$, for which

$$g'(e_k, e_k) = (-1)^k,\ g'(e_1, e_2) = g'(e_1, e_3) = g'(e_1, e_4) = g'(e_2, e_3)$$
$$= g'(e_2, e_4) = 0 \text{ and } Je_1 = e_2,\ Je_2 = -e_1,\ Je_3 = e_4,\ Je_4 = -e_3.$$

Let us determine a real time-like hypersurface of the Kaehler manifold with B-metric (S^4, J, g')

$$M^3 : g'(N, N) = -1$$

having in mind (2).

We choose an adapted φ-basis $\{e_1, e_2, e_4\}$ of the tangent space $T_p M^3$ and the metric g has signature $(1, 2)$ on M^3. The normal unit N has to be in plane $\{e_3, e_4\}$, i.e.

$$N = \lambda e_3 + \mu e_4. \tag{8}$$

Therefore according to $g'(N, N) = -1$, $g'(N, e_4) = 0$, we determine λ and μ:

$$\lambda = \pm \frac{\sqrt{2}\sqrt{1 + \cos 2u^1 \cosh 2u^2}}{a(\cos 2u^1 + \cosh 2u^2)},\ \mu$$
$$= \pm \frac{\sqrt{2}\sin 2u^1 \sinh 2u^2}{a(\cos 2u^1 + \cosh 2u^2)\sqrt{1 + \cos 2u^1 \cosh 2u^2}}.$$

Because of $g'(N, JN) = \tan t$, we obtain

$$t = \arctan \frac{\sin 2u^1 \sinh 2u^2}{1 + \cos 2u^1 \cosh 2u^2} \tag{9}$$

and according to (2) we get

$$\xi = e_4,\ \varphi e_1 = e_2,\ \varphi e_2 = -e_1,\ \varphi e_4 = \varphi\xi = 0,$$
$$\eta(e_1) = \eta(e_2) = 0,\ \eta(e_4) = \eta(\xi) = 1.$$

By such a way we receive an almost contact B-metric manifold $(M^3, \varphi, \xi, \eta, g, \tilde{g})$ as a hypersurface of an h-sphere. Since $\omega(x) = \frac{1}{2}\{dt(x) - \tan t\, dt(\varphi x)\}$, where $dt(x) = x^1 \frac{\partial t}{\partial u^1} + x^2 \frac{\partial t}{\partial u^2}$, $\frac{\partial t}{\partial u^4} = 0$, and using (9), (5), we compute that

$$h_0 = 0, \quad \omega(x) = \frac{1}{1 + \cos 2u^1 \cosh 2u^2}\{x^1 \cos 2u^1 \sinh 2u^2 + x^2 \sin 2u^1 \cosh 2u^2\},$$

$$\begin{aligned} h_{11} = -\frac{a}{\sqrt{2}\sqrt{1 + \cos 2u^1 \cosh 2u^2}} &\{\sinh 2u^2 (x^1 y^4 + x^4 y^1) \\ &+ \sin 2u^1 (x^2 y^4 + x^4 y^2)\}. \end{aligned} \tag{10}$$

On the other hand, from the formulas of Gauss and Weingarten it follows that $h_{ij} = -g'\left(\nabla_{e_i} N, e_j\right)$ and because of (8) and $\frac{\partial \lambda}{\partial u^i} g'_{3j} = \frac{\partial \mu}{\partial u^i} g'_{4j} = 0$ $(i, j = 1, 2, 4)$, we receive $h_{ij} = -\lambda \Gamma'^k_{i3} g'_{kj} - \mu \Gamma'^k_{i4} g'_{kj}$. Therefore we compute that

$$h_{11} = h_{12} = h_{22} = h_{44} = 0,$$
$$h_{14} = \frac{a}{\sqrt{2}} \frac{\sinh 2u^2}{\sqrt{1 + \cos 2u^1 \cosh 2u^2}} \neq 0,$$
$$h_{24} = \frac{a}{\sqrt{2}} \frac{\sin 2u^1}{\sqrt{1 + \cos 2u^1 \cosh 2u^2}} \neq 0.$$

Hence, according to the decomposition of the second fundamental form $h(x, y) = x^i y^j h_{ij}$ $(i, j = 1, 2, 4)$ and (10) we get

$$h(x, y) = h_{11}(x, y).$$

Having in mind Theorem 4, we conclude that the constructed manifold $(M^3, \varphi, \xi, \eta, g, \tilde{g})$ is an almost contact B-metric manifold belonging to the basic class $\mathcal{F}_{11}$.

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