Classes of Real Isotropic Hypersurfaces of a Kaehler Manifold with B-Metric

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1 Introduction

We shall consider almost complex manifolds (M, J) equipped with two different kinds of metrics. When J acts as an isometry we receive the notion of almost Hermitian manifold. But in the case when J acts as an antiisometry we have the notion of almost complex manifold with B-metric.

It is well known how to extend J to an almost contact structure on an odd-dimensional manifold. In the first case the Hermitian metric of (M, J) induces an almost contact metric manifold. In the second – we receive an almost contact manifold with B-metric.

We shall study the case of B-metric.

Let (M', J, g') be a $2n'$-dimensional almost complex manifold with B-metric, i.e. J is an almost complex structure and g' is a metric on M' such that: [2]

$$J^2X = -X, \qquad g'(JX, JY) = -g'(X, Y).$$

for all vector fields X, Y on M'. The associated metric $\tilde{g}'$ of the manifold is defined as follows $\tilde{g}'(X, Y) := g'(X, JY)$. Both metrics are necessarily of signature (n', n').

The class of the Kaehler manifolds with B-metric is determined by the condition that J be parallel with respect to the Levi-Civita connection of g'.

Let $(M, \varphi, \xi, \eta, g, \tilde{g})$ be a $(2n + 1)$-dimensional almost contact manifold with B-metric [4], i.e. at first (φ, ξ, η) is an almost contact structure determined by a tensor field φ of type $(1, 1)$, by a vector field ξ and by an 1-form η on M according the conditions: [1]

$$\varphi^2X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1.$$

In addition this almost contact manifold (M, φ, ξ, η) admits a metric g, called B-metric, such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

where X,Y are arbitrary differentiable vector fields on M, i.e. X,$Y \in \mathfrak{X}(M)$.

The associated metric $\tilde{g}$ given by $\tilde{g}(X, Y) := g(X, \varphi Y) + \eta(X)\eta(Y)$ $(X,Y \in \mathfrak{X}(M))$ is a B-metric, too. Both metrics g and $\tilde{g}$ are indefinite of signature $(n, n+1)$.

We have in mind the following immediate consequences of the above conditions:

$$\eta \circ \varphi = 0,\ \varphi\xi = 0,\ \mathrm{rank}\varphi = 2n,\ \eta(X) = g(X,\xi),\ g(\xi,\xi) = 1,\ g(\varphi X,Y) = g(X,\varphi Y).$$

Further, X, Y, Z will stand for arbitrary differentiable vector fields on M, and x, y, z – for arbitrary vectors in the tangent space T_pM to M at an arbitrary point p in M.

Let ∇ be the Levi-Civita connection of the metric g. The tensor F of type $(0,3)$ on M is defined by $F(x,y,z) = g((\nabla_x\varphi)y,z)$. It has the following properties:

$$F(x,y,z) = F(x,z,y) = F(x,\varphi y,\varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi).$$

Each fiber T_pM of the tangent bundle TM is a $(2n+1)$-dimensional vector space with the structure $(\varphi_p, \xi_p, \eta_p, g_p)$.

The decomposition $T_pM = \{D_p = \ker(\eta_p)\} \oplus span\{\xi_p\}$ is orthogonal and invariant with respect to the structural group $\big(GL(n,\mathbb{C}) \cap O(n,n)\big) \times I$. The $2n$-dimensional vector space D_p is equipped with a complex structure φ_p and B-metrics g_p, $\tilde{g}_p$.

Let $\{e_i, \xi\}$ $(i = 1, 2, \dots, 2n)$ be a basis of T_pM, and (g^{ij}) be the inverse matrix of the matrix (g_{ij}) of g. Then the following 1-forms are associated with F:

$$\theta(\cdot) = g^{ij}F(e_i, e_j, \cdot),\ \ \theta^*(\cdot) = g^{ij}F(e_i, \varphi e_j, \cdot),\ \ \omega(\cdot) = F(\xi, \xi, \cdot). \tag{1}$$

A classification of the almost contact manifolds with B-metric is given in [4]. It contains eleven basic classes $\mathcal{F}_i$ defined with respect to the tensor F. We shall use the following characteristic conditions of the considered classes:

$$\begin{array}{ll}
\mathcal{F}_4: & F(x,y,z) = -\frac{\theta(\xi)}{2n}\{g(\varphi x,\varphi y)\eta(z) + g(\varphi x,\varphi z)\eta(y)\};\\
\mathcal{F}_5: & F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\{g(x,\varphi y)\eta(z) + g(x,\varphi z)\eta(y)\};\\
\mathcal{F}_6: & F(x,y,z) = f(x,y)\eta(z) + f(x,z)\eta(y),\\
 & \quad f(x,y) = f(y,x), \quad f(\varphi x,\varphi y) = -f(x,y), \quad \theta(\xi) = \theta^*(\xi) = 0;\\
\mathcal{F}_8: & F(x,y,z) = f(x,y)\eta(z) + f(x,z)\eta(y),\\
 & \quad f(x,y) = f(y,x), \quad f(\varphi x,\varphi y) = f(x,y),
\end{array} \tag{2}$$

where $f(x,y) = F(\varphi^2 x,\varphi^2 y,\xi)$, instead of the known ones of [4].

The classes $\mathcal{F}_i \otimes \mathcal{F}_j$, etc., are defined in a natural way by the conditions of the basic classes. There exist 2^{11} classes of almost contact B-metric manifolds.

In [5] two types of real hypersurfaces of a complex manifold with B-metric were introduced. The obtained submanifolds are almost contact B-metric manifolds. Let us recall, the real isotropic hypersurface M of an almost complex manifold with B-metric $(M'^{2n+2}, J, g', \tilde{g}')$ is determined by the condition the normal unit N to be isotropic regarding the associated B-metric $\tilde{g}'$ of g'. Moreover, M is equipped with the almost contact B-metric structure

$$\varphi := J + g'(\cdot, JN)N, \quad \xi := -JN, \quad \eta := -g'(\cdot, JN), \quad g := g'|_M. \tag{3}$$

Let ∇' and ∇ be the Levi-Civita connections of g' on M' and g on M, respectively. If $h(X,Y) = g(AX,Y)$ is the second fundamental form of the hypersurface M, then the formulas of Gauss and Weingarten seem as follows:

$$\nabla'_X Y = \nabla_X Y - h(X,Y)N, \qquad \nabla'_X N = -AX.$$

In [5] we found out the classification tensor of the real isotropic hypersurface of a Kaehler manifold with B-metric

$$(4) \quad F(X,Y,Z) = f(X,Y)\eta(Z) + f(X,Z)\eta(Y) = h(X,Y)\eta(Z) + h(X,Z)\eta(Y).$$

In consequence it is obtained that every real isotropic hypersurface of a Kaehler manifold with B-metric is an almost contact B-metric manifold belonging to the class $\mathcal{F}_4 \otimes \mathcal{F}_5 \otimes \mathcal{F}_6 \otimes \mathcal{F}_8$. In the same work are given the characteristic conditions in terms of the second fundamental tensor A only for the basic classes $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_8$ and for some of their direct sums.

The present paper continues the studies of the real isotropic hypersurfaces of a Kaehler manifold with B-metric. Our aim is to describe all possible classes of the considered hypersurfaces with respect to their second fundamental form. Using an orthogonal and invariant with respect to the structural group decomposition of h we get four basic components and we show that the four basic corresponding classes generate all sixteen possible classes. Thereby we give conditions for h which characterize when a real isotropic hypersurface belongs to everyone of the sixteen classes.

2 The sixteen classes

It follows from (4) that the tensor $f(X,Y)$ is symmetric on the considered hypersurface. On the other hand, f can be pure tensor with respect to φ, i.e. $f(\varphi X, \varphi Y) = f(X,Y)$. The equivalent condition in terms of h is the following $h(\varphi^2 X, \varphi^2 Y) - h(\varphi X, \varphi Y) = 0$. Another case is the case of hybrid tensor with respect to φ, i.e. $f(\varphi X, \varphi Y) = -f(X,Y)$. Equivalently, we have $h(\varphi^2 X, \varphi^2 Y) + h(\varphi X, \varphi Y) = 0$. So we receive the following result.

Theorem 1 *The class of the real isotropic hypersurfaces of a Kaehler manifold with B-metric is the class $\mathcal{F}_4 \otimes \mathcal{F}_5 \otimes \mathcal{F}_6 \otimes \mathcal{F}_8$. There are 16 classes of these hypersurfaces in all.*

Remark 2 *When $n = 1$ the class $\mathcal{F}_6$ is restricted to $\mathcal{F}_0$. Therefore, for 4-dimensional Kaehler manifold with B-metric there are only 8 classes of considered 3-dimensional hypersurfaces.*

Now we characterize these classes in terms of their second fundamental form h. Since $X = -\varphi^2 X + \eta(X)\xi$, $Y = -\varphi^2 Y + \eta(Y)\xi$, we have

$$h(X,Y) = h(\varphi^2 X, \varphi^2 Y) - \eta(X)h(\xi, \varphi^2 Y) - \eta(Y)h(\varphi^2 X, \xi) + \eta(X)\eta(Y)h(\xi,\xi).$$

Having in mind $h(X,\xi)=h(\xi,Y)=0$, it follows $h(X,Y)=h(\varphi^2X,\varphi^2Y)$, i.e. the tensor h has components only over the complex B-metric vector space $(D,\varphi,g,\tilde{g})$.

Let $(V,J,g',\tilde{g}')$ be a $2n$-dimensional vector space with a complex structure J and B-metrics g' and $\tilde{g}'$. Let V^* denote the dual space of V. We consider the space $V^*\otimes V^*$, i.e. the vector space of the tensors of type $(0,2)$ over V. The metric g' induces an inner product $\langle\,,\,\rangle$ on $V^*\otimes V^*$, given by

$$\langle f_1,f_2\rangle = g^{ij}g^{ks}f_1(e_i,e_k)f_2(e_j,e_s)$$

for f_1, f_2 in $V^*\otimes V^*$ and $\{e_i\}$ $(i=1,2,\dots,2n)$ – a basis of V. With every f in $V^*\otimes V^*$ we associate the functions: $\mathrm{tr}f=g^{ij}f(e_i,e_j)$, $\mathrm{tr}^*f=g^{ij}f(e_i,Je_j)$. We denote $G=GL(n,\mathbb{C})\cap O(n,n)$. The standard representation of G on V induces a natural representation λ of G on $V^*\otimes V^*$ and

$$\langle(\lambda a)f_1,(\lambda a)f_2\rangle=\langle f_1,f_2\rangle;\ a\in G;\ f_1,f_2\in V^*\otimes V^*.$$

Let us consider the vector subspace W of $V^*\otimes V^*$ of the symmetric $(0,2)$-tensors over V. We remark that g and $\tilde{g}$ are symmetric and hybrid tensors with respect to the almost complex structure J and, besides, every symmetric J-pure tensor is traceless. Then it is easy to check the truthfulness of the following

Lemma 3 *Every symmetric* $(0,2)$*-tensor* $f(x,y)$ *over* $(V,J,g',\tilde{g}')$, $\dim V=2n$, *has four orthogonal and invariant components with respect to the action of* G *on* W*:*

$$f_1(x,y)=\tfrac{1}{2}[f(x,y)+f(Jx,Jy)],\ f_2(x,y)=\tfrac{\mathrm{tr}f}{2n}g(x,y),\ f_3(x,y)=-\tfrac{\mathrm{tr}^*f}{2n}g(x,Jy),$$
$$f_4(x,y)=\tfrac{1}{2}[f(x,y)-f(Jx,Jy)]-\tfrac{\mathrm{tr}f}{2n}g(x,y)+\tfrac{\mathrm{tr}^*f}{2n}g(x,Jy).$$

As h is a symmetric $(0,2)$-tensor over (D_p,φ_p,g_p), we can apply the last lemma for h . We denote the following symmetric tensors:

$$\begin{aligned}
&h_4(X,Y)=-\tfrac{\theta(\xi)}{2n}g(\varphi X,\varphi Y), \qquad h_5(X,Y)=-\tfrac{\theta^*(\xi)}{2n}g(X,\varphi Y),\\
&h_6(X,Y)=\tfrac{1}{2}\left\{h(\varphi^2X,\varphi^2Y)-h(\varphi X,\varphi Y)\right\}+\tfrac{\theta(\xi)}{2n}g(\varphi X,\varphi Y)+\tfrac{\theta^*(\xi)}{2n}g(X,\varphi Y)\\
&\qquad=\tfrac{1}{2}(\mathcal{L}_\xi g)(\varphi X,\varphi^2Y)+\tfrac{\theta(\xi)}{2n}g(\varphi X,\varphi Y)+\tfrac{\theta^*(\xi)}{2n}g(X,\varphi Y),\\
&h_8(X,Y)=\tfrac{1}{2}\left\{h(\varphi^2X,\varphi^2Y)+h(\varphi X,\varphi Y)\right\}=-\tfrac{1}{2}d\eta(\varphi X,\varphi^2Y),
\end{aligned}\tag{5}$$

where

$$\begin{aligned}
d\eta(X,Y)&=(\nabla_X\eta)Y-(\nabla_Y\eta)X=h(X,\varphi Y)-h(Y,\varphi X),\\
(\mathcal{L}_\xi g)(X,Y)&=(\nabla_X\eta)Y+(\nabla_Y\eta)X=h(X,\varphi Y)-h(Y,\varphi X).
\end{aligned}$$

Therefore, h has the form

$$h=h_4+h_5+h_6+h_8.$$

Taking into account (2) and (4), we describe the mentioned sixteen classes in terms of h. This is our main result.

Theorem 4 *The sixteen classes of real isotropic hypersurfaces of a Kaehler manifold with B-metric are characterized in terms of their second fundamental form h as follows:*

$$
(6)\qquad
\begin{array}{rlrl}
\mathcal{F}_0: & h=0; & \mathcal{F}_i\otimes\mathcal{F}_j\otimes\mathcal{F}_k: & h=h_i+h_j+h_k;\\
\mathcal{F}_i: & h=h_i; & (i,j,k=4,5,6,8; & i\neq j\neq k\neq i)\\
\mathcal{F}_i\otimes\mathcal{F}_j: & h=h_i+h_j; & \mathcal{F}_4\otimes\mathcal{F}_5\otimes\mathcal{F}_6\otimes\mathcal{F}_8: & h=h_4+h_5+h_6+h_8,
\end{array}
$$

where the components h_i $(i=4,5,6,8)$ are given in (5).

3 An example

In this section we construct an example of a 3-dimensional real isotropic hypersurface of a holomorphic sphere. We show that it belongs to the basic class $\mathcal{F}_5$.

A surface S^{2n} of a Kaehler manifold with B-metric $(\mathbb{R}^{2n+2},J,\bar{g})$, called h-sphere, is defined by $\bar{g}(x,x)=a$, $\tilde{\bar{g}}(x,x)=b$, $a,b\in\mathbb{R}$, $(a,b)\neq(0,0)$ in [3]. It is shown there that the h-sphere is a Kaehler manifold with B-metric.

We propose the following explicit example of an h-sphere S^4 in $\mathbb{R}^6$:

$$
(7)\qquad
\begin{aligned}
x^1&=\tfrac{a}{2}[\cos(u^1-u^3)\cosh(u^2-u^4)+\cos(u^1+u^3)\cosh(u^2+u^4)],\\
x^2&=\tfrac{a}{2}[\sin(u^1+u^3)\cosh(u^2+u^4)-\sin(u^1-u^3)\cosh(u^2-u^4)],\\
x^3&=a\sin u^1\cosh u^2,\\
x^4&=-\tfrac{a}{2}[\sin(u^1+u^3)\sinh(u^2+u^4)+\sin(u^1-u^3)\sinh(u^2-u^4)],\\
x^5&=-\tfrac{a}{2}[\cos(u^1-u^3)\sinh(u^2-u^4)-\cos(u^1+u^3)\sinh(u^2+u^4)],\\
x^6&=a\cos u^1\sinh u^2,
\end{aligned}
$$

omitting the set of points $(u^1=\frac{\pi}{2},u^2=0)$. The almost complex structure J is defined by

$$
J\tfrac{\partial x}{\partial u^1}=\tfrac{\partial x}{\partial u^2},\ J\tfrac{\partial x}{\partial u^2}=-\tfrac{\partial x}{\partial u^1},\ J\tfrac{\partial x}{\partial u^3}=\tfrac{\partial x}{\partial u^4},\ J\tfrac{\partial x}{\partial u^4}=-\tfrac{\partial x}{\partial u^3}.
$$

We find the non-zero components of the B-metric $g'_{ij}=g'\left(\frac{\partial x}{\partial u^i},\frac{\partial x}{\partial u^j}\right)$ for $\frac{\partial x}{\partial u^i}\left(\frac{\partial x^1}{\partial u^i},\dots,\frac{\partial x^6}{\partial u^i}\right)$:

$$
\begin{aligned}
&g'_{11}=-g'_{22}=-a^2,\quad g'_{33}=-g'_{44}=-\tfrac{a}{2}\left(1+\cos 2u^1\cosh 2u^2\right),\\
&g'_{34}=-\tfrac{a}{2}\sin 2u^1\sinh 2u^2
\end{aligned}
$$

and the non-zero components Γ'^k_{ij} $(i,j,k\in\{1,2,3,4\})$ of the Levi-Civita connection:

$$
\begin{aligned}
&\Gamma'^3_{13}=\Gamma'^4_{14}=\Gamma'^4_{23}=-\Gamma'^3_{24}=-\tfrac{\sin 2u^1}{\cos 2u^1+\cosh 2u^2},\\
&\Gamma'^1_{33}=\Gamma'^2_{34}=-\Gamma'^1_{44}=\tfrac{1}{2}\sin 2u^1\cosh 2u^2,\\
&\Gamma'^4_{13}=-\Gamma'^3_{14}=-\Gamma'^3_{23}=-\Gamma'^4_{24}=-\tfrac{\sinh 2u^2}{\cos 2u^1+\cosh 2u^2},\\
&\Gamma'^2_{33}=-\Gamma'^1_{34}=-\Gamma'^2_{44}=\tfrac{1}{2}\cos 2u^1\sinh 2u^2.
\end{aligned}
$$

We obtain the basis $\{e'_k = \frac{\partial x}{\partial u^k}\}$, $k \in \{1,2,3,4\}$ of the tangent space T_pS^4. Since $g'_{1k} = g'_{2s} = 0$, $k = 2,3,4$, $s = 3,4$ we can substitute $N = \frac{1}{a}e'_1$ and $\xi = -\frac{1}{a}e'_2$. For the corresponding 1-form η of ξ we have $\eta(x) = -ax^2$, $x = x^i e'_i$, $i = 2,3,4$. The action of φ on the orthogonal distribution of N is determined by the following way: $\varphi e'_2 = 0$, $\varphi e'_3 = e'_4$, $\varphi e'_4 = -e'_3$.

Hence we construct an isotropic hypersurface $M^3 : \tilde{g}'(N,N) = 0$ (alternatively $M^3 : u^1 = 0$) of the given h-sphere and we equip M^3 with the almost contact B-metric structure $(\varphi,\xi,\eta,g,\tilde{g})$ according to (3) .

We choose a basis $\{e_1,e_2,e_3\}$ of the tangent space T_pM^3 by $e_1 = e'_3$, $e_2 = e'_4$, $e_3 = e'_2$ and then we have to substitute u^3 for u^2 .

Therefore, the non-zero components g_{ij} and Γ^k_{ij} are:

$$\begin{aligned}&g_{11} = -g_{22} = \tfrac{a^2}{2}\left(1+\cosh 2u^3\right), \qquad g_{33} = a^2\\ &\Gamma^3_{11} = -\Gamma^3_{22} = -\tfrac{1}{2}\sinh 2u^3, \quad \Gamma^1_{13} = \Gamma^2_{23} = \tfrac{\sinh 2u^3}{1+\cosh 2u^3}.\end{aligned} \tag{8}$$

According to (1) and the above results, we compute for $x = x^i e_i$, $i = 1,2,3$:

$$\theta(x) = 0, \quad \theta^*(x) = 2x^3\frac{\sinh 2u^3}{1+\cosh 2u^3}, \quad \omega(x) = 0. \tag{9}$$

It is known [5], that for an isotropic hypersurface of a Kaehler manifold with B-metric $h(x,y) = -g(\nabla_x\xi,\varphi y)$. Then we obtain

$$h(x,y) = \frac{\sinh 2u^3}{a(1+\cosh 2u^3)}g(x,\varphi y)$$

by direct computation in the basis $\{e_1,e_2,e_3\}$ using (8).

Hence, according to (9) and (5), we get $h(x,y) = h_5(x,y)$. Having in mind Theorem 4, we establish that the constructed manifold $(M^3,\varphi,\xi,\eta,g,\tilde{g})$ is an almost contact B-metric manifold belonging to the basic class $\mathcal{F}_5$.

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