ON HYPERCOMPLEX PSEUDO-HERMITIAN MANIFOLDS

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The class of the hypercomplex pseudo-Hermitian manifolds is considered. The flatness of the considered manifolds with the 3 parallel complex structures is proved. Conformal transformations of the metrics are introduced. The conformal invariance and the conformal equivalence of the basic types manifolds are studied. A known example is characterized in relation to the obtained results.

Introduction

This paper is a continuation of the same authors's paper [4] which is inspired by the seminal work [1] of D. V. Alekseevsky and S. Marchiafava. We follow a parallel direction including skew-Hermitian metrics with respect to the almost hypercomplex structure.

In the first section we give some necessary facts concerning the almost hypercomplex pseudo-Hermitian manifolds introduced in [4].

In the second one we consider the special class of (integrable) hypercomplex pseudo-Hermitian manifolds, namely pseudo-hyper-Kähler manifolds. Here we expose the proof of the mentioned in [4] statement that each pseudo-hyper-Kähler manifold is flat.

The third section is fundamental for this work. A study of the group of conformal transformations of the metric is initiated here. The conformal invariant classes and the conformal equivalent class to the class of the pseudo-hyper-Kähler manifolds are found.

Finally, we characterize a known example in terms of the conformal transformations.

1 Preliminaries

1.1 Hypercomplex pseudo-Hermitian structures in a real vector space

Let V be a real $4n$-dimensional vector space. By $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^i}\right\}$, $i = 1, 2, \ldots, n$, is denoted a (local) basis on V. Each vector x of V is represented in the mentioned basis as follows

$$x = x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + u^i \frac{\partial}{\partial u^i} + v^i \frac{\partial}{\partial v^i}. \tag{1}$$

A standard complex structure on V is defined as in [5]:

$$\begin{array}{llll} J_1 \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, & J_1 \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}, & J_1 \frac{\partial}{\partial u^i} = -\frac{\partial}{\partial v^i}, & J_1 \frac{\partial}{\partial v^i} = \frac{\partial}{\partial u^i}; \\ J_2 \frac{\partial}{\partial x^i} = \frac{\partial}{\partial u^i}, & J_2 \frac{\partial}{\partial y^i} = \frac{\partial}{\partial v^i}, & J_2 \frac{\partial}{\partial u^i} = -\frac{\partial}{\partial x^i}, & J_2 \frac{\partial}{\partial v^i} = -\frac{\partial}{\partial y^i}; \\ J_3 \frac{\partial}{\partial x^i} = -\frac{\partial}{\partial v^i}, & J_3 \frac{\partial}{\partial y^i} = \frac{\partial}{\partial u^i}, & J_3 \frac{\partial}{\partial u^i} = -\frac{\partial}{\partial y^i}, & J_3 \frac{\partial}{\partial v^i} = \frac{\partial}{\partial x^i}. \end{array} \tag{2}$$

The following properties about J_α are direct consequences of (2)

$$\begin{array}{l} J_1^2 = J_2^2 = J_3^2 = -Id, \\ J_1 J_2 = -J_2 J_1 = J_3, \quad J_2 J_3 = -J_3 J_2 = J_1, \quad J_3 J_1 = -J_1 J_3 = J_2. \end{array} \tag{3}$$

If $x \in V$, i.e. $x(x^i, y^i, u^i, v^i)$ then according to (2) and (3) we have

$$J_1 x(-y^i, x^i, v^i, -u^i), \quad J_2 x(-u^i, -v^i, x^i, y^i), \quad J_3 x(v^i, -u^i, y^i, -x^i). \tag{4}$$

Definition 1.1 ([1]) *A triple $H = (J_1, J_2, J_3)$ of anticommuting complex structures on V with $J_3 = J_1 J_2$ is called* a hypercomplex structure *on V;*

A bilinear form f on V is defined as ordinary, $f : V \times V \to \mathbb{R}$. We denote by $\mathcal{B}(V)$ the set of all bilinear forms on V. Each f is a tensor of type $(0, 2)$, and $\mathcal{B}(V)$ is a vector space of dimension $16n^2$.

Let J be a given complex structure on V. A bilinear form f on V is called *Hermitian* (respectively, *skew-Hermitian*) with respect to J if the identity $f(Jx, Jy) = f(x, y)$ (respectively, $f(Jx, Jy) = -f(x, y)$ holds true.

Definition 1.2 ([1]) *A bilinear form f on V is called* an Hermitian bilinear form *with respect to $H = (J_\alpha)$ if it is Hermitian with respect to any complex structure $J_\alpha, \alpha = 1, 2, 3$, i.e.*

$$f(J_\alpha x, J_\alpha y) = f(x, y) \qquad \forall\, x, y \in V. \tag{5}$$

We denote by $L_0 = \mathcal{B}_H(V)$ the set of all Hermitian bilinear forms on V. The notion of pseudo-Hermitian bilinear forms is introduced by the following

Definition 1.3 ([4]) *A bilinear form f on V is called* a pseudo-Hermitian bilinear form with respect to $H = (J_1, J_2, J_3)$, *if it is Hermitian with respect to J_α and skew-Hermitian with respect to J_β and J_γ, i.e.*

$$f(J_\alpha x, J_\alpha y) = -f(J_\beta x, J_\beta y) = -f(J_\gamma x, J_\gamma y) = f(x, y) \quad \forall\, x, y \in V, \tag{6}$$

where (α, β, γ) is a circular permutation of $(1, 2, 3)$.

We denote $f \in L_\alpha \subset \mathcal{B}(V)$ $(\alpha = 0, 1, 2, 3)$ when f satisfies the conditions (5) and (6), respectively.

In [1] is introduced a pseudo-Euclidian metric g with signature $(2n, 2n)$ as follows

$$g(x, y) := \sum_{i=1}^{n} \left(-x^i a^i - y^i b^i + u^i c^i + v^i d^i \right), \tag{7}$$

where $x(x^i, y^i, u^i, v^i)$, $y(a^i, b^i, c^i, d^i) \in V$, $i = 1, 2, \dots, n$. This metric satisfies the following properties

$$g(J_1 x, J_1 y) = -g(J_2 x, J_2 y) = -g(J_3 x, J_3 y) = g(x, y). \tag{8}$$

This means that the pseudo-Euclidean metric g belongs to L_1.

The form $g_1 : g_1(x, y) = g(J_1 x, y)$ coincides with the Kähler form Φ which is Hermitian with respect to J_α, i.e.

$$\Phi(J_\alpha x, J_\alpha y) = \Phi(x, y), \quad \alpha = 1, 2, 3, \qquad \Phi \in L_0.$$

The attached to g associated bilinear forms $g_2 : g_2(x, y) = g(J_2 x, y)$ and $g_3 : g_3(x, y) = g(J_3 x, y)$ are symmetric forms with the properties

$$\begin{array}{c} -g_2(J_1 x, J_1 y) = -g_2(J_2 x, J_2 y) = g_2(J_3 x, J_3 y) = g_2(x, y), \\ -g_3(J_1 x, J_1 y) = g_3(J_2 x, J_2 y) = -g_3(J_3 x, J_3 y) = g_3(x, y), \end{array} \tag{9}$$

i.e. $g_2 \in L_3$, $g_3 \in L_2$.

It follows that the Kähler form Φ is Hermitian regarding H and the metrics g, g_2, g_3 are pseudo-Hermitian of different types with signature $(2n, 2n)$.

Now we recall the following notion:

Definition 1.4 ([4]) *The structure $(H, G) := (J_1, J_2, J_3, g, \Phi, g_2, g_3)$ is called* a hypercomplex pseudo-Hermitian structure *on V.*

1.2 Structural tensors on an almost (H, G)-manifold

Let (M, H) be an almost hypercomplex manifold [1]. We suppose that g is a symmetric tensor field of type $(0, 2)$. If it induces a pseudo-Hermitian inner product in T_pM, $p \in M$, then g is called *a pseudo-Hermitian metric on M.*

The structure $(H, G) := (J_1, J_2, J_3, g, \Phi, g_2, g_3)$ is called *an almost hypercomplex pseudo-Hermitian structure on* M or in short *an almost* (H, G)*-structure on* M. The manifold M equipped with H and G, i.e. (M, H, G), is called *an almost hypercomplex pseudo-Hermitian manifold*, or in short *an almost* (H, G)*-manifold*. [4]

The 3 tensors of type $(0, 3)$ $F_\alpha : F_\alpha(x, y, z) = g\big((\nabla_x J_\alpha)\, y, z\big), \alpha = 1, 2, 3,$ where ∇ is the Levi-Civita connection generated by g, is called *structural tensors of the almost* (H, G)*-manifold*. [4]

The structural tensors satisfy the following properties:

$$\begin{array}{l} F_1(x, y, z) = F_2(x, J_3y, z) + F_3(x, y, J_2z), \\ F_2(x, y, z) = F_3(x, J_1y, z) + F_1(x, y, J_3z), \\ F_3(x, y, z) = F_1(x, J_2y, z) - F_2(x, y, J_1z); \end{array} \tag{10}$$

$$\begin{array}{l} F_1(x, y, z) = -F_1(x, z, y) = -F_1(x, J_1y, J_1z), \\ F_2(x, y, z) = F_2(x, z, y) = F_2(x, J_2y, J_2z), \\ F_3(x, y, z) = F_3(x, z, y) = F_3(x, J_3y, J_3z). \end{array} \tag{11}$$

Let us recall the Nijenhuis tensors $N_\alpha(X, Y) = \frac{1}{2}\,[[J_\alpha, J_\alpha]]\,(X, Y)$ for almost complex structures J_α and $X, Y \in \mathfrak{X}(M)$, where

$$[[J_\alpha, J_\alpha]]\,(X, Y) = 2\big\{[J_\alpha X, J_\alpha Y] - J_\alpha\,[J_\alpha X, Y] - J_\alpha\,[X, J_\alpha Y] - [X, Y]\big\}.$$

It is well known that the almost hypercomplex structure $H = (J_\alpha)$ is a hypercomplex structure if $[[J_\alpha, J_\alpha]]$ vanishes for each $\alpha = 1, 2, 3$. Moreover it is known that one almost hypercomplex structure H is hypercomplex if and only if two of the structures J_α $(\alpha = 1, 2, 3)$ are integrable. This means that two of the tensors N_α vanish. [1]

We recall also the following definitions. Since g is Hermitian metric with respect to J_1, according to [3] the class $\mathcal{W}_4$ is a subclass of the class of Hermitian manifolds. If (H, G)-manifold belongs to $\mathcal{W}_4$, with respect to J_1, then the almost complex structure J_1 is integrable and

$$\begin{array}{rl} F_1(x, y, z) = \frac{1}{2(2n-1)} & [g(x, y)\theta_1(z) - g(x, z)\theta_1(y) \\ & -g(x, J_1y)\theta_1(J_1z) + g(x, J_1z)\theta_1(J_1y)]\,, \end{array} \tag{12}$$

where $\theta_1(\cdot) = g^{ij}F_1(e_i, e_j, \cdot) = \delta\Phi(\cdot)$ for the basis $\{e_i\}_{i=1}^{4n}$, and δ – the coderivative.

On other side the metric g is a skew-Hermitian with respect to J_2 and J_3, i.e. $g(J_2x, J_2y) = g(J_3x, J_3y) = -g(x, y)$. A classification of all almost complex manifolds with skew-Hermitian metric (Norden metric or B-metric) is given in [2]. One of the basic classes of integrable almost complex manifolds with skew-Hermitian metric is $\mathcal{W}_1$. It is known that if an almost (H,G)-manifold belongs to $\mathcal{W}_1(J_\alpha), \alpha = 2,3$, then J_α is integrable and the following equality holds

$$\begin{aligned} F_\alpha(x,y,z) = \tfrac{1}{4n}\, [g(x,y)\theta_\alpha(z) + g(x,z)\theta_\alpha(y) \\ +g(x,J_\alpha y)\theta_\alpha(J_\alpha z) + g(x,J_\alpha z)\theta_\alpha(J_\alpha y)]\,, \end{aligned} \tag{13}$$

where $\theta_\alpha(z) = g^{ij}F_\alpha(e_i,e_j,z)$, $\alpha = 2,3$, for an arbitrary basis $\{e_i\}_{i=1}^{4n}$.

When (12) is satisfied for (M,H,G), we say that $(M,H,G) \in \mathcal{W}(J_1)$. In the case, (M,H,G) satisfies (13) for $\alpha = 2$ or $\alpha = 3$, we say $(M,H,G) \in \mathcal{W}(J_2)$ or $(M,H,G) \in \mathcal{W}(J_3)$. Let us denote the class $\mathcal{W} := \bigcap_{\alpha=1}^{3} \mathcal{W}(J_\alpha)$.

The next theorem gives a sufficient condition an almost (H,G)-manifold to be integrable.

Theorem 1.1 ([4]) *Let (M,H,G) belongs to the class $\mathcal{W}(J_\alpha)\bigcap\mathcal{W}(J_\beta)$. Then (M,H,G) is of class $\mathcal{W}(J_\gamma)$ for all cyclic permutations (α,β,γ) of $(1,2,3)$.*

Let us remark that necessary and sufficient conditions (M,H,G) to be in $\mathcal{W}$ are

$$\theta_\alpha \circ J_\alpha = -\frac{2n}{2n-1}\theta_1 \circ J_1, \qquad \alpha = 2,3. \tag{14}$$

2 Pseudo-hyper-Kähler manifolds

Definition 2.1 ([4]) *A pseudo-Hermitian manifold is called a* pseudo-hyper-Kähler manifold, *if $\nabla J_\alpha = 0$ $(\alpha = 1,2,3)$ with respect to the Levi-Civita connection generated by g.*

It is clear, then $F_\alpha = 0\,(\alpha = 1,2,3)$ holds or the manifold is Kählerian with respect to J_α, i.e. $(M,H,G) \in \mathcal{K}(J_\alpha)$.

Immediately we obtain that if (M,H,G) belongs to $\mathcal{K}(J_\alpha)\bigcap\mathcal{W}(J_\beta)$ then $(M,H,G) \in \mathcal{K}(J_\gamma)$ for all cyclic permutations (α,β,γ) of $(1,2,3)$.

Then the following sufficient condition for a $\mathcal{K}$-manifold is valid.

Theorem 2.1 ([4]) *If $(M,H,G) \in \mathcal{K}(J_\alpha)\bigcap\mathcal{W}(J_\beta)$ then M is a pseudo-hyper-Kähler manifold $(\alpha \neq \beta \in \{1,2,3\})$.*

Let (M^{4n},H,G) be a pseudo-hyper-Kähler manifold and ∇ be the Levi-Civita connection generated by g. The curvature tensor seems as follows

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{15}$$

and the corresponding tensor of type $(0,4)$ is

$$R(X,Y,Z,W) = g\left(R(X,Y)Z,W\right), \quad \forall\, X,Y,Z,W \in \mathfrak{X}(M). \tag{16}$$

Lemma 2.2 *The curvature tensor of a pseudo-hyper-Kähler manifold has the following properties:*

$$
\begin{aligned}
R(X,Y,Z,W) &= R(X,Y,J_1Z,J_1W) = R(J_1X,J_1Y,Z,W) \\
&= -R(X,Y,J_2Z,J_2W) = -R(J_2X,J_2Y,Z,W) \\
&= -R(X,Y,J_3Z,J_3W) = -R(J_3X,J_3Y,Z,W),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
R(X,Y,Z,W) &= R(X,J_1Y,J_1Z,W) \\
&= -R(X,J_2Y,J_2Z,W) = -R(X,J_3Y,J_3Z,W).
\end{aligned} \tag{18}
$$

Proof. The equality (17) is valid, because of (15), (16), the condition $\nabla J_\alpha = 0$ $(\alpha = 1,2,3)$, the equality (8) and the properties of the curvature $(0,4)$-tensor.

To prove (18), we will show at first that the property $R(X,J_2Y,J_2Z,W) = -R(X,Y,Z,W)$ holds. Indeed, from (17) we get

$$
R(J_2X,Y,Z,W) = R(X,J_2Y,Z,W), \quad R(X,Y,J_2Z,W) = R(X,Y,Z,J_2W)
$$

and $\mathfrak{S}_{X,Y,Z} R(X,Y,J_2Z,J_2W) = 0$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum regarding X,Y,Z. In the last equality we replace Y by J_2Y and W by J_2W. We get

$$
-R(X,J_2Y,J_2Z,W) - R(J_2Y,Z,J_2X,W) + R(Z,X,Y,W) = 0. \tag{19}
$$

Replacing Y by Z, and inversely, we get

$$
-R(X,J_2Z,J_2Y,W) - R(J_2Z,Y,J_2X,W) + R(Y,X,Z,W) = 0. \tag{20}
$$

As we have

$$
-R(J_2Z,Y,J_2X,W) = -R(Z,J_2Y,J_2X,W) = R(J_2Y,Z,X,W),
$$

with the help of (19) and (20) we obtain

$$
\begin{aligned}
&-R(X,J_2Y,J_2Z,W) - R(X,J_2Z,J_2Y,W) \\
&+R(Z,X,Y,W) + R(Y,X,Z,W) = 0.
\end{aligned} \tag{21}
$$

According to the first Bianchi identity and (17), we obtain

$$
\begin{aligned}
-R(X,J_2Z,J_2Y,W) &= R(J_2Z,J_2Y,X,W) + R(J_2Y,X,J_2Z,W) \\
&= -R(Z,Y,X,W) - R(X,J_2Y,J_2Z,W).
\end{aligned}
$$

Then the equality (21) seem as follows

$$
-2R(X,J_2Y,J_2Z,W) + R(Z,X,Y,W) - R(X,Y,Z,W) + R(Y,Z,X,W) = 0
$$

By the first Bianchi identity the equality is transformed in the following

$$
-2R(X,J_2Y,J_2Z,W) - 2R(X,Y,Z,W) = 0,
$$

which is equivalent to

$$R(X, J_2Y, J_2Z, W) = -R(X, Y, Z, W). \tag{22}$$

As the tensor R has the same properties with respect to J_3, and to J_2, it follows that the next equality holds, too.

$$R(X, J_3Y, J_3Z, W) = -R(X, Y, Z, W). \tag{23}$$

Using (22) and (23) for $J_1 = J_2J_3$ we get successively that

$$\begin{aligned} R(X, Y, Z, W) &= R(X, J_1Y, J_1Z, W) \\ &= R(X, J_2(J_3Y), J_2(J_3Z), W) = -R(X, J_3Y, J_3Z, W), \end{aligned}$$

which completes the proof of (18).

Now we will prove a theorem which gives us a geometric characteristic of the pseudo-hyper-Kähler manifolds.

Theorem 2.3 *Each pseudo-hyper-Kähler manifold is a flat pseudo-Riemannian manifold with signature* $(2n, 2n)$.

Proof. Lemma 2.2 implies the properties

$$\begin{aligned} -R(X, Y, Z, W) &= R(X, J_1Y, Z, J_1W) \\ &= R(X, J_2Y, Z, J_2W) = R(X, J_3Y, Z, J_3W). \end{aligned} \tag{24}$$

As $J_1 = J_2J_3$, we also have the following

$$\begin{aligned} R(X, J_1Y, Z, J_1W) &= R(X, J_2(J_3Y), Z, J_2(J_3W)) \\ &= -R(X, J_3Y, Z, J_3W) = R(X, Y, Z, W). \end{aligned}$$

Comparing (24) with the last equality we receive

$$-R(X, Y, Z, W) = R(X, J_1Y, Z, J_1W) = R(X, Y, Z, W),$$

or $R \equiv 0$.

3 Conformal transformations of the pseudo-Hermitian metric

The usual conformal transformation $c : \bar{g} = e^{2u}g$, where u is a differential function on M^{4n}, is known. Since $g_\alpha(\cdot, \cdot) = g(J_\alpha\cdot, \cdot)$, the conformal transformation of g causes the same changes of the pseudo-Hermitian metrics g_2, g_3 and the Kähler form $\Phi \equiv g_1$. Then we say that it is given a conformal transformation c of G to $\bar{G}$ determined by $u \in \mathcal{F}(M)$. These conformal transformations form a group denoted by C. The hypercomplex pseudo-Hermitian manifolds (M, H, G) and $(M, H, \bar{G})$ we call C-equivalent manifolds or conformal-equivalent manifolds.

Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections determined by the metrics g and $\bar{g}$, respectively. The known condition for a Levi-Civita connection implies the following relation

$$\bar{\nabla}_X Y = \nabla_X Y + \mathrm{d}u(X)Y + \mathrm{d}u(Y)X - g(X,Y)\mathrm{grad}(u). \tag{25}$$

Using (25) and the definitions of structural tensors for ∇ and $\bar{\nabla}$ we obtain

$$\begin{aligned}\bar{F}_1(X,Y,Z) = e^{2u}\,[F_1(X,Y,Z) - g(X,Y)\mathrm{d}u(J_1Z) + g(X,Z)\mathrm{d}u(J_1Y)\\ +g(J_1X,Y)\mathrm{d}u(Z) - g(J_1X,Z)\mathrm{d}u(Y)]\,,\end{aligned} \tag{26}$$

$$\begin{aligned}\bar{F}_\alpha(X,Y,Z) = e^{2u}\,[F_\alpha(X,Y,Z) + g(X,Y)\mathrm{d}u(J_\alpha Z) + g(X,Z)\mathrm{d}u(J_\alpha Y)\\ -g(J_\alpha X,Y)\mathrm{d}u(Z) - g(J_\alpha X,Z)\mathrm{d}u(Y)]\end{aligned} \tag{27}$$

for $\alpha = 2,3$. The last two equalities imply the following relations for the corresponding structural 1-forms

$$\bar{\theta}_1 = \theta_1 - 2(2n-1)\mathrm{d}u\circ J_1, \qquad \bar{\theta}_\alpha = \theta_\alpha + 4n\mathrm{d}u\circ J_\alpha, \quad \alpha = 2,3. \tag{28}$$

Let us denote the following (0,3)-tensors.

$$\begin{aligned}P_1(x,y,z) = F_1(x,y,z)\\ -\tfrac{1}{2(2n-1)}\,[g(x,y)\theta_1(z) - g(x,z)\theta_1(y)\\ -g(x,J_1y)\theta_1(J_1z) + g(x,J_1z)\theta_1(J_1y)]\,,\end{aligned} \tag{29}$$

$$\begin{aligned}P_\alpha(x,y,z) = F_\alpha(x,y,z)\\ -\tfrac{1}{4n}\,[g(x,y)\theta_\alpha(z) + g(x,z)\theta_\alpha(y)\\ +g(x,J_\alpha y)\theta_\alpha(J_\alpha z) + g(x,J_\alpha z)\theta_\alpha(J_\alpha y)]\,, \quad \alpha = 2,3.\end{aligned} \tag{30}$$

According to (12) and (13) it is clear that

$$(M,H,G)\in\mathcal{W}(J_\alpha) \iff P_\alpha = 0 \quad (\alpha = 1,2,3).$$

The equalities (26)–(28) imply the following two interconnections

$$\bar{P}_\alpha = e^{2u}P_\alpha, \quad \alpha = 1,2,3; \tag{31}$$

$$\bar{\theta}_\alpha\circ J_\alpha + \frac{2n}{2n-1}\bar{\theta}_1\circ J_1 = \theta_\alpha\circ J_\alpha + \frac{2n}{2n-1}\theta_1\circ J_1, \quad \alpha = 2,3. \tag{32}$$

From (31) we receive that each of $\mathcal{W}(J_\alpha)\,(\alpha = 1,2,3)$ is invariant with respect to the conformal transformations of C, i.e. they are C-invariant classes. Having in mind also (32), we state the validity of the following

Theorem 3.1 *The class $\mathcal{W}$ of hypercomplex pseudo-Hermitian manifolds is C-invariant.*

Now we will determine the class of the (locally) C-equivalent $\mathcal{K}$-manifolds. Let us denote the following subclass $\mathcal{W}^0 := \{\mathcal{W} \mid d\,(\theta_1\circ J_1) = 0\}$.

Theorem 3.2 *A hypercomplex pseudo-Hermitian manifold belongs to $\mathcal{W}^0$ if and only if it is C-equivalent to a pseudo-hyper-Kähler manifold.*

Proof. Let (M,H,G) be a pseudo-hyper-Kähler manifold, i.e. $(M,H,G)\in\mathcal{K}$. Then $F_\alpha=\theta_\alpha=0$ $(\alpha=1,2,3)$. Hence (28) has the form

$$\bar\theta_1=-2(2n-1)\mathrm{d}u\circ J_1,\qquad \bar\theta_\alpha=4n\mathrm{d}u\circ J_\alpha,\quad \alpha=2,3. \tag{33}$$

From (26), (27) and (33) and having in mind (12) and (13) we obtain that $(M,H,\bar G)$ is a $\mathcal{W}$-manifold. According to (33) the 1-forms $\bar\theta_\alpha\circ J_\alpha$ $(\alpha=1,2,3)$ are closed. Because of (14) the condition $\mathrm{d}(\bar\theta_1\circ J_1)=0$ is sufficient.

Conversely, let $(M,H,\bar G)$ be a $\mathcal{W}$-manifold with closed $\bar\theta_1\circ J_1$. Because of (14) the 1-forms $\bar\theta_\alpha\circ J_\alpha$ $(\alpha=2,3)$ are closed, too. We determine the function u as a solution of the differential equation $\mathrm{d}u=-\frac{1}{2(2n-1)}\bar\theta_1\circ J_1$. Then by an immediate verification we state that the transformation $c^{-1}:g=e^{-2u}\bar g$ converts $(M,H,\bar G)$ into $(M,H,G)\in\mathcal{K}$. This completes the proof.

Let us remark the following inclusions

$$\mathcal{K}\subset\mathcal{W}^0\subset\mathcal{W}\subset\mathcal{W}(J_\alpha),\quad \alpha=1,2,3.$$

Let R,ρ,τ and $\bar R,\bar\rho,\bar\tau$ be the curvature tensors, the Ricci tensors, the scalar curvatures corresponding to ∇ and $\bar\nabla$, respectively. The following tensor is curvature-like, i.e. it has the same properties as R.

$$\begin{aligned}\psi_1(S)(X,Y,Z,U)=\;&g(Y,Z)S(X,U)-g(X,Z)S(Y,U)\\&+g(X,U)S(Y,Z)-g(Y,U)S(X,Z),\end{aligned}$$

where S is a symmetric tensor.

Having in mind (25) and (15), we obtain

Proposition 3.3 *The following relations hold for the C-equivalent (H,G)-manifolds*

$$\begin{gathered}\bar R=e^{2u}\{R-\psi_1(S)\},\\ \bar\rho=\rho-\mathrm{tr}Sg-2(2n-1)S,\qquad \bar\tau=e^{-2u}\{\tau-2(4n-1)\mathrm{tr}S\},\end{gathered} \tag{34}$$

where

$$S(Y,Z)=S(Z,Y)=(\nabla_Y\mathrm{d}u)\,Z+\mathrm{d}u(Y)\mathrm{d}u(Z)-\frac{1}{2}\mathrm{d}u(\mathrm{grad}(\mathrm{d}u))g(Y,Z). \tag{35}$$

If (M,H,G) is a C-equivalent $\mathcal{W}$-manifold to a $\mathcal{K}$-manifold, i.e. $(M,H,G)\in\mathcal{W}^0$, then Proposition 3.3 implies

Corollary 3.4 *A $\mathcal{W}^0$-manifold has the following curvature characteristic*

$$R=\frac{1}{2(2n-1)}\left\{\psi_1(\rho)-\frac{\tau}{4n-1}\pi_1\right\},$$

where $\pi_1(X,Y,Z,U)=\frac{1}{2}\psi_1(g)=g(Y,Z)g(X,U)-g(X,Z)g(Y,U)$.

It is well known that the C-invariant tensor of each pseudo-Riemannian manifold is the so-called Weil tensor W. From (34) we receive immediately

$$\bar{W} = e^{2u}W, \qquad W = R - \frac{1}{2(2n-1)}\left\{\psi_1(\rho) - \frac{\tau}{4n-1}\pi_1\right\}. \tag{36}$$

Let us remark that the vanishing of the Weil tensor is a necessary and sufficient condition a pseudo-Riemannian manifold to be conformal equivalent to a flat manifold with dimension greater than 3.

This is confirmed by the combining of Theorem 2.3, Theorem 3.2 and Corollary 3.4, i.e. $(M,H,G) \in \mathcal{W}^0$ iff $W = 0$ on (M,H,G).

Since each conformal transformation determines uniquely a symmetric tensor S by (35) then it takes an interest in the consideration S as a bilinear form on T_pM belonging to each of the components L_α, $(\alpha = 0,1,2,3)$.

Let $S \in L_0$. In view of (5) $\mathrm{tr}S = 0$ holds and according to (34) we receive $\bar{\tau} = e^{-2u}\tau$ and an invariant tensor $W_0 = R - \frac{1}{2(2n-1)}\psi_1(\rho)$. When W_0 vanishes on (M,H,G) then the curvature tensor has the form $R = \frac{1}{2(2n-1)}\psi_1(\rho)$.

In the cases when $S \in L_\alpha$ $(\alpha = 1,2,3)$ we consider (M,H,G) as an $\mathcal{W}^0$-manifold. Then according to Theorem 2.3 and Theorem 3.2 we have $\bar{R} = 0$ on the C-equivalent $\mathcal{K}$-manifold of (M,H,G).

Now let $S \in L_1$. By reason of $g \in L_1$ we have a cause for the consideration of the possibility $S = \lambda g$. Hence $\lambda = \frac{\mathrm{tr}S}{4n} = \frac{\tau}{8n(4n-1)}$. Then having in mind (34) $R = \frac{\tau}{4n(4n-1)}\pi_1$ holds true. From here it is clear that if $S \in L_1$ then (M,H,G) is an Einstein manifold.

Let us consider the case when $S \in L_2$. Then according to (6) $\mathrm{tr}S$ vanishes, and from (34) τ vanishes, too. Because of $g_3 \in L_2$ we consider $S = \lambda g_3$, whence $\lambda = -\frac{\mathrm{tr}(S \circ J_3)}{4n}$. Then (34) implies $R = \frac{\mathrm{tr}(S \circ J_3)}{4n}\pi_3^{J_3}$, where $\pi_3^{J_3}$ is the following tensor π_3 with respect to the complex structure $J = J_3$

$$\pi_3(X,Y,Z,U) = -\pi_1(X,Y,JZ,U) - \pi_1(X,Y,Z,JU).$$

It is known [2] that π_3 is a Kähler curvature-like tensor, i.e. it satisfies the property $\pi_3(X,Y,JZ,JU) = -\pi_3(X,Y,Z,U)$. Therefore in this case R is Kählerian with respect to J_3 and the tensor $R^{*J_3} : R^{*J_3}(X,Y,Z,U) = R(X,Y,Z,J_3U)$ is curvature-like. Then we obtain immediately

$$R = \frac{\tau(R^{*J_3})}{8n(2n-1)}\pi_3^{J_3}, \qquad \rho = -\frac{\tau(R^{*J_3})}{4n}g_3.$$

Hence if $S \in L_2$ then (M,H,G) is a *-Einstein manifold with respect to J_3.

By an analogous way, in the case when $S \in L_3$ we receive that (M,H,G) is a *-Einstein manifold with respect to J_2.

4 A 4-dimensional pseudo-Riemannian spherical manifold with (H, G)-structure

In [4] is considered a hypersurface S^4_2 in $\mathbb{R}^5_2$ by the equation

$$-\left(z^1\right)^2 - \left(z^2\right)^2 + \left(z^3\right)^2 + \left(z^4\right)^2 + \left(z^5\right)^2 = 1, \tag{37}$$

where $Z\left(z^1, z^2, z^3, z^4, z^5\right)$ is the positional vector of $p \in S^4_2$.

Let $\left(u^1, u^2, u^3, u^4\right)$ be local coordinates of p on S^4_2. The hypersurface S^4_2 is defined by the scalar parametric equations:

$$\begin{gathered} z^1 = \sinh u^1 \cos u^2, \quad z^2 = \sinh u^1 \sin u^2, \quad z^3 = \cosh u^1 \cos u^3 \cos u^4, \\ z^4 = \cosh u^1 \cos u^3 \sin u^4, \quad z^5 = \cosh u^1 \sin u^3. \end{gathered} \tag{38}$$

Further we consider the manifold on $\tilde{S}^4_2 = S^4_2 \backslash \{(0,0,0,0,\pm 1)\}$, i.e. we omit two points for which $\{u^1 \neq 0\} \cap \{u^3 \neq (2k+1)\pi/2, k \in \mathbb{Z}\}$. The tangent space $T_p\tilde{S}^4_2$ of $\tilde{S}^4_2$ in the point $p \in \tilde{S}^4_2$ is determined by the vectors $z_i = \frac{\partial Z}{\partial u^i} (i = 1,2,3,4)$. The vectors z_i are linearly independent on $\tilde{S}^4_2$, defined by (38), and $T_p\tilde{S}^4_2$ has a basis (z_1, z_2, z_3, z_4) in every point $p \in \tilde{S}^4_2$.

The restriction of $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^5_2$ to S^4_2 is a pseudo-Riemannian metric g on S^4_2 with signature $(2,2)$. The non-zero components $g_{ij} = \langle z_i, z_j \rangle$ are

$$g_{11} = -1,\ g_{22} = -\sinh^2 u^1,\ g_{33} = \cosh^2 u^1,\ g_{44} = \cosh^2 u^1 \cos^2 u^3. \tag{39}$$

The hypersurface S^4_2 is equipped with an almost hypercomplex structure $H = (J_\alpha), (\alpha = 1,2,3)$, where the non-zero components of the matrix of J_α with respect to the local basis $\left\{\frac{\partial}{\partial u^i}\right\}_{i=1}^4$ are

$$\begin{array}{ll} (J_1)_2^1 = -\frac{1}{(J_1)_1^2} = -\sinh u^1, & (J_1)_4^3 = -\frac{1}{(J_1)_3^4} = \cos u^3, \\ (J_2)_3^1 = -\frac{1}{(J_2)_1^3} = -\cosh u^1, & (J_2)_4^2 = -\frac{1}{(J_2)_2^4} = -\coth u^1 \cos u^3, \\ (J_3)_4^1 = -\frac{1}{(J_3)_1^4} = \cosh u^1 \cos u^3, & (J_3)_2^3 = -\frac{1}{(J_3)_3^2} = \tanh u^1. \end{array} \tag{40}$$

Theorem 4.1 ([4]) *The spherical pseudo-Riemannian 4-dimensional manifold, defined by (38), admits a hypercomplex pseudo-Hermitian structure on $\tilde{S}^4_2$, determined by (40) and (39), with respect to which it is of the class $\mathcal{W}(J_1)$ but it does not belong to $\mathcal{W}$ and it has a constant sectional curvature $k = 1$.*

Let us consider a conformal transformation determined by the function u which is a solution of the equation $\mathrm{d}u = -\frac{1}{2(2n-1)}(\theta_1 \circ J_1)$, where the nonzero component of θ_1 with respect to the local basis $\left\{\frac{\partial}{\partial u^i}\right\}$ $(i = 1,2,3,4)$ is $\theta_1\left(\frac{\partial}{\partial u^2}\right) = \frac{2\sinh^2 u^1}{\cosh u^1}$.

Since $\tilde{S}^4_2$ has a constant sectional curvature then the Weil tensor is vanishes, i.e. $\tilde{S}^4_2$ is C-equivalent to a flat $\mathcal{K}(J_1)$-manifold. If we admit that it is in

$\mathcal{K}$, then according to Theorem 3.2 we obtain that the manifold $(\tilde{S}_2^4, H, G) \in \mathcal{W}$ which is a contradiction. Therefore the considered manifold is C-equivalent to a flat $\mathcal{K}(J_1)$-manifold, but it is not a pseudo-hyper-Kähler manifold. By direct verification we state that the tensor S of this conformal transformation belongs to L_1. Therefore $(\tilde{S}_2^4, H, G)$ is an Einstein manifold.

References

1. D. V. Alekseevsky and S. Marchiafava, *Quaternionic structures on a manifold and subordinated structures*, Ann. Mat. Pura Appl. (IV), CLXXI (1996), 205–273.
2. G. Ganchev and A. Borisov, *Note on the almost complex manifolds with a Norden metric*, Compt. rend. Acad. bulg. Sci., **39** (1986), no. 5, 31–34.
3. A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (IV), CXXIII (1980), 35–58.
4. K. Gribachev, M. Manev and S. Dimiev, *Almost hypercomplex pseudo-Hermitian manifolds*, Boll. Unione Mat. Ital. Sez. A, (to appear)
5. A. Sommese, *Quaternionic manifolds*, Math. Ann. **212** (1975), 191–214.
6. J. Wolf, *Spaces of constant curvature*, University of California, Berkley, California, 1972.